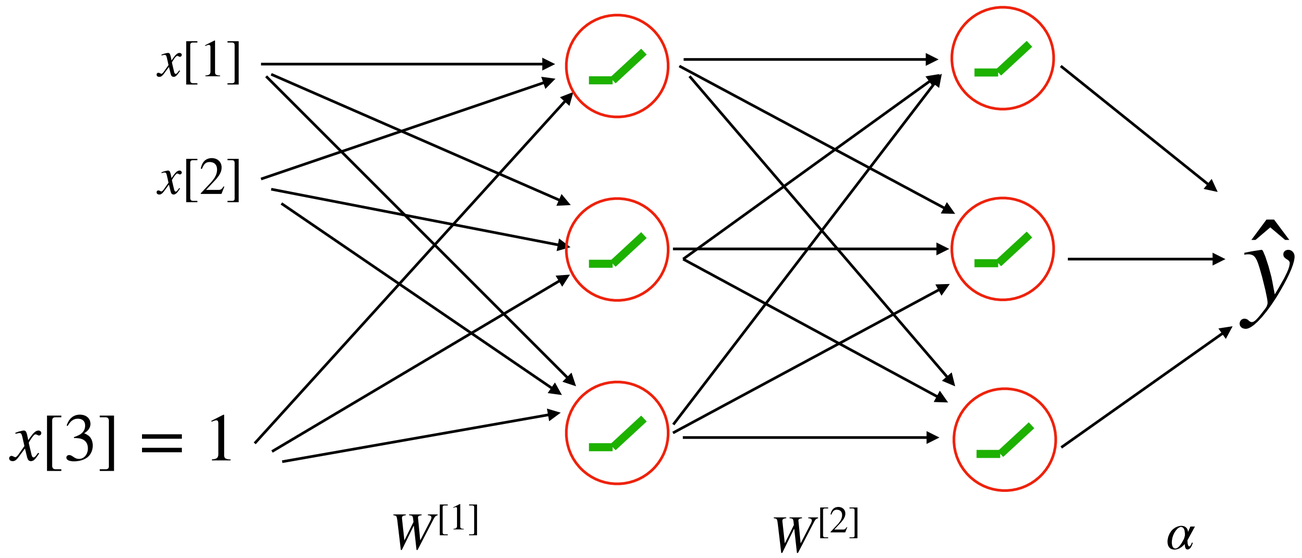
Neural Network:
Training & Backpropagation

Announcements

Recap

A two layer fully connected feedforward NN:



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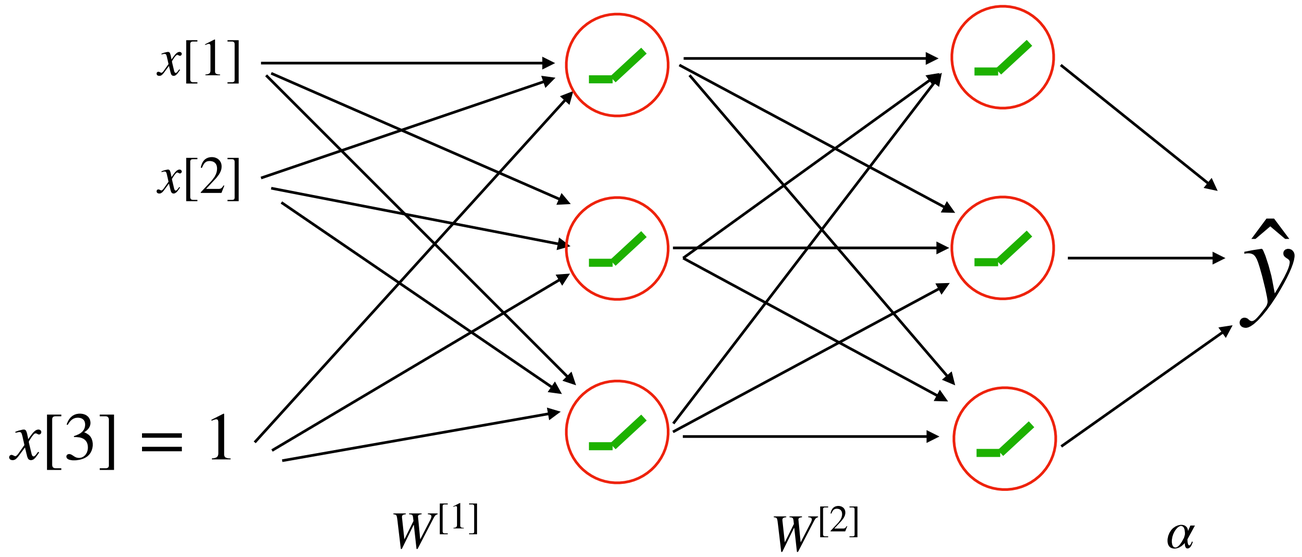

$$h(x) := \alpha^\top \mathsf{ReLU}\left(W^{[2]}\mathsf{ReLU}\left(W^{[1]}x\right)\right) + b$$

Outline of Today

1. Training NNs via SGD

2. A Naive approach of computing gradients

3. Backpropagation: efficient way of computing gradients

Training neural network via SGD

Square loss on training example (x, y)

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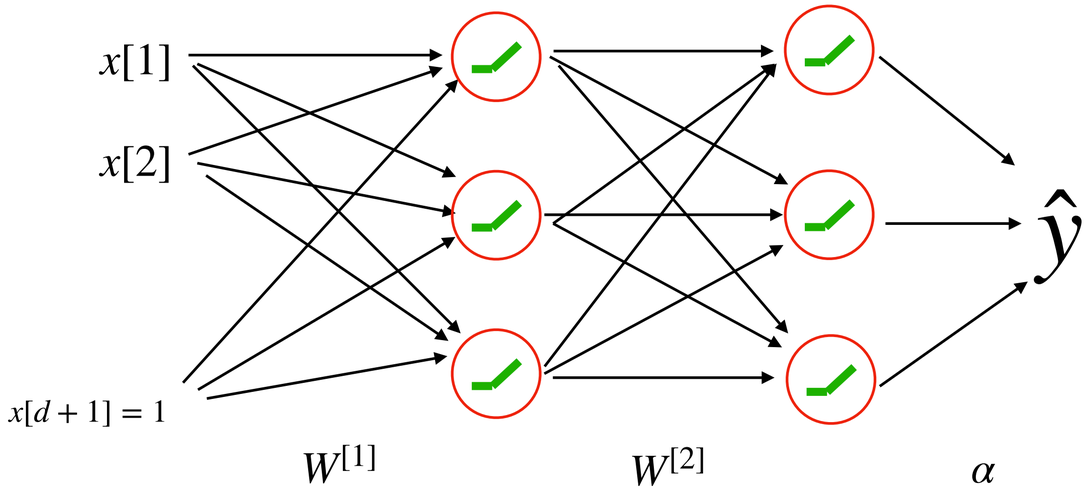

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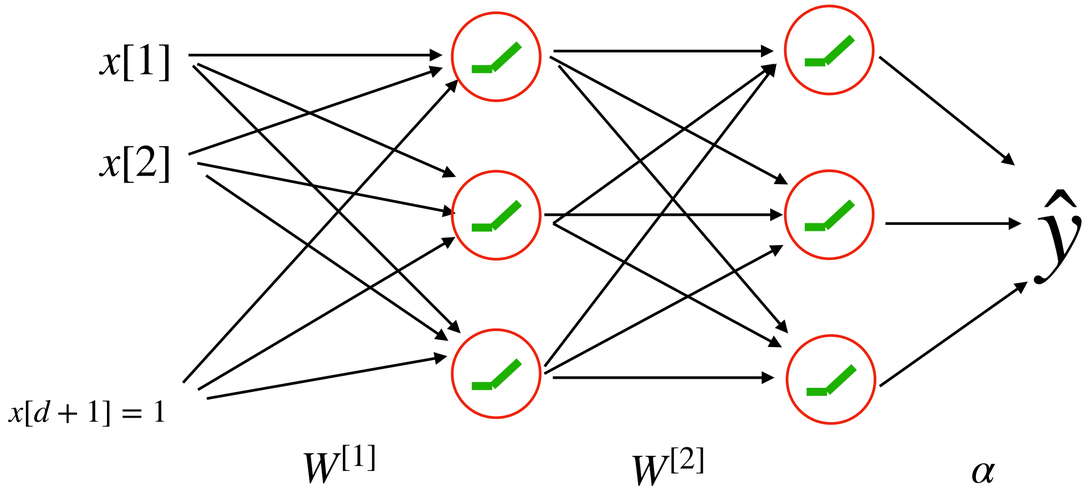

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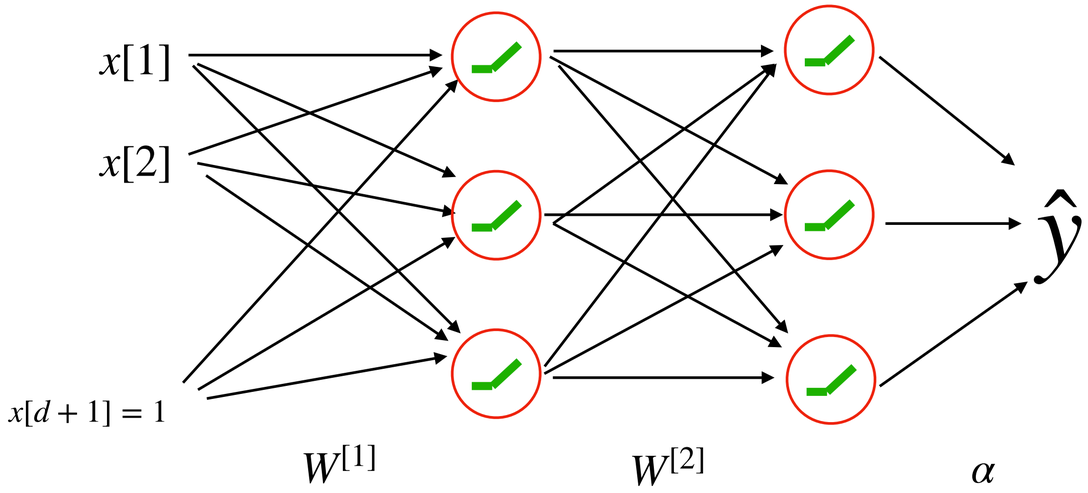

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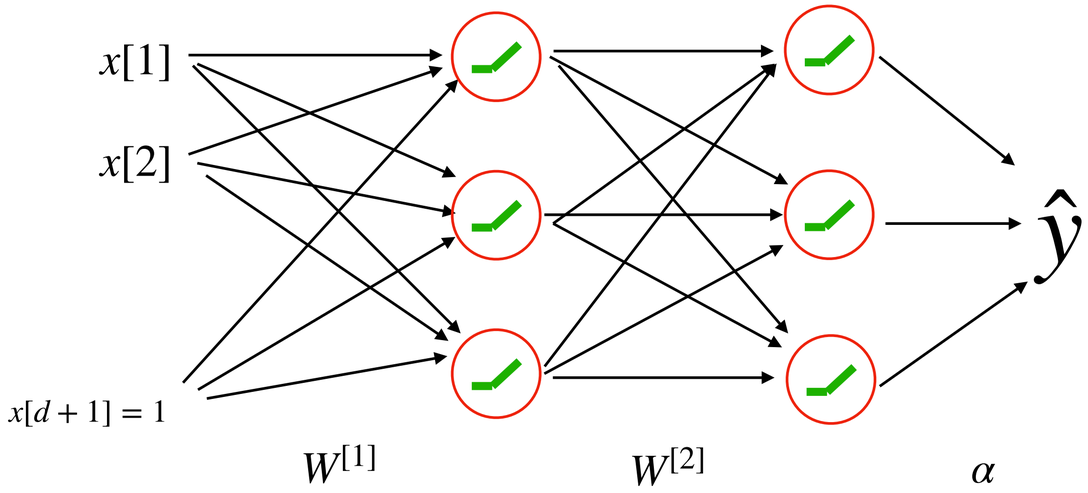

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Compute gradients:

$$\frac{\partial \ell(h(x), y)}{\partial W^{[1]}} \qquad \frac{\partial \ell(h(x), y)}{\partial W^{[2]}}$$

$$\frac{\partial \ell(h(x), y)}{\partial \alpha} \qquad \frac{\partial \ell(h(x), y)}{\partial b}$$

Training neural network via SGD

Mini-batch Stochastic gradient descent

$\theta = [W^{[1]}, W^{[2]}, \alpha, b]$

For epoc $t = 1$ to T:

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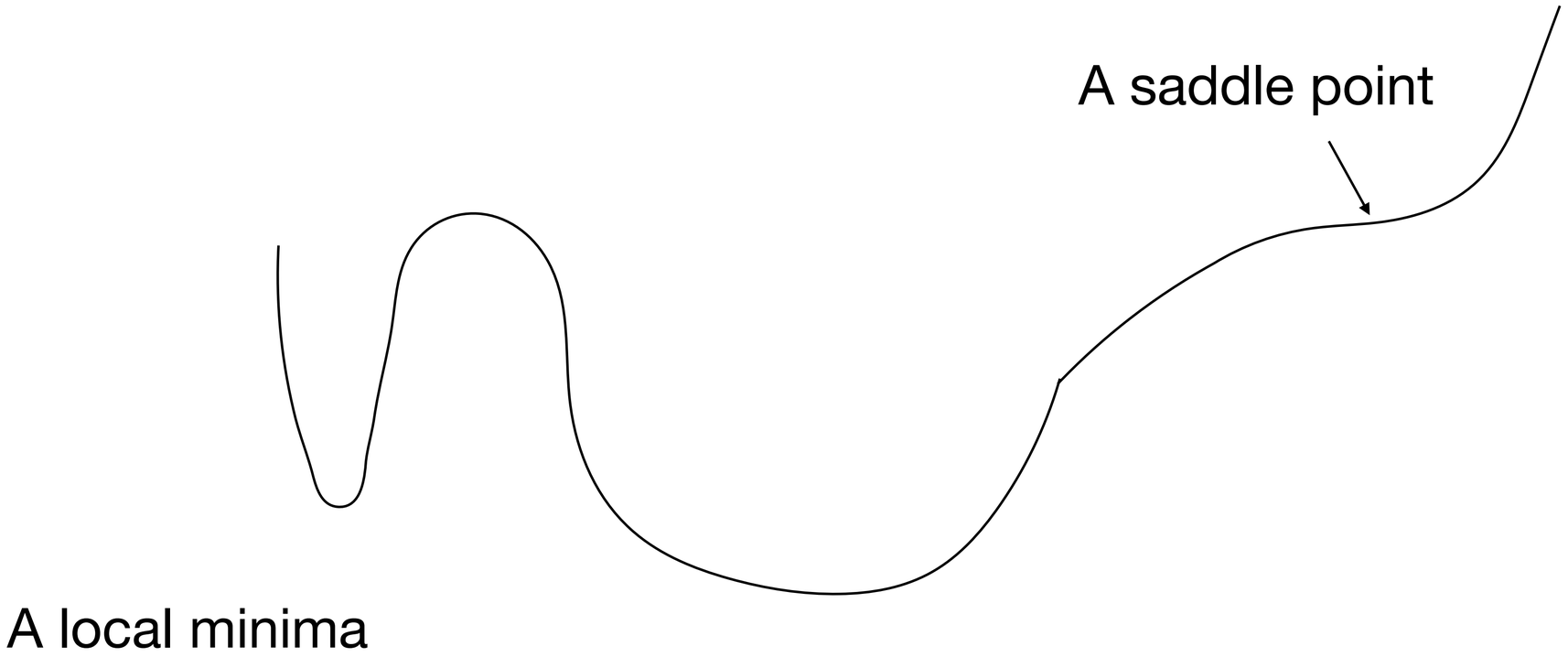
    $\theta = \theta - \eta g$

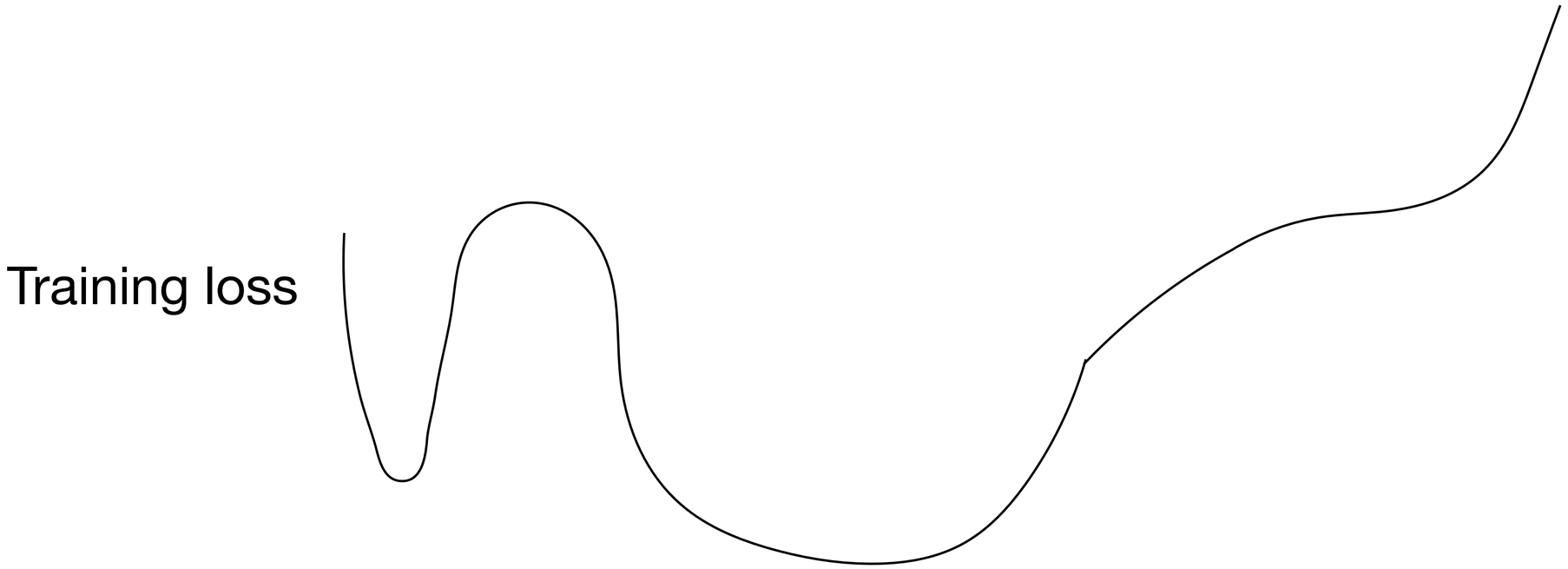
Training neural network via SGD

SGD helps avoiding local minima and saddle point



Training neural network via SGD

SGD tends to converge to a flat region



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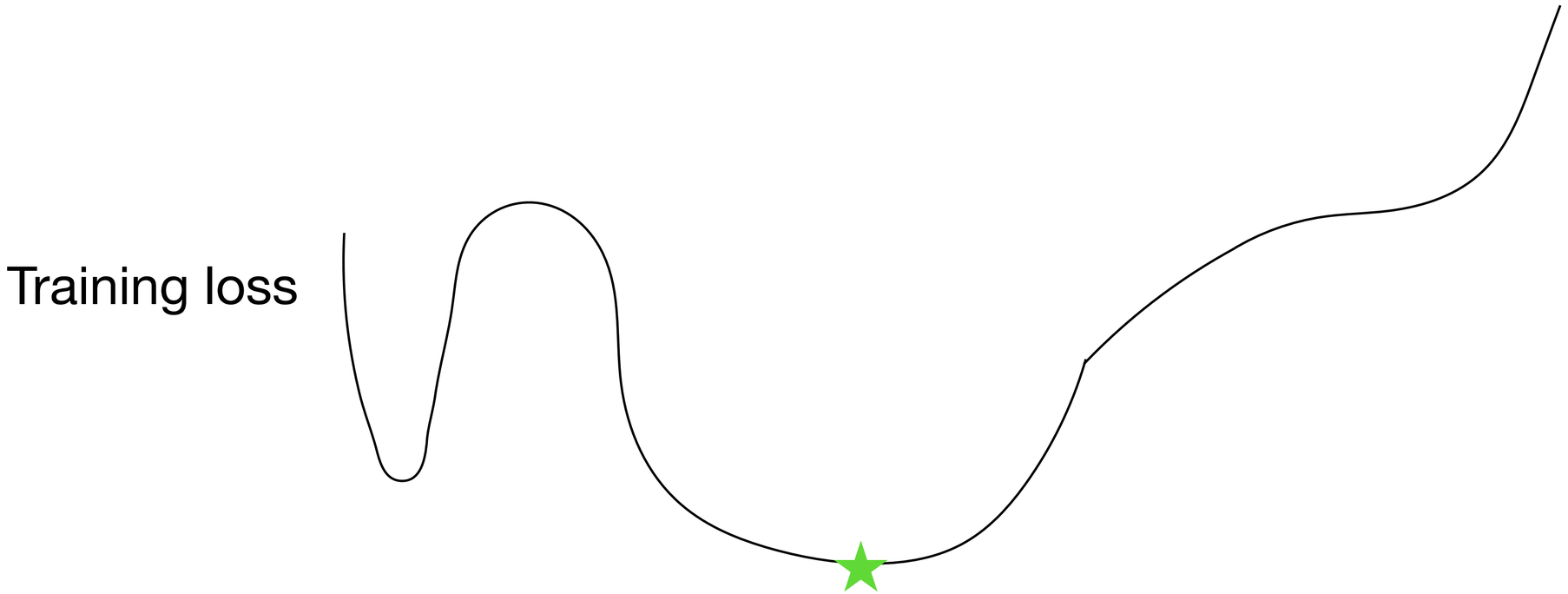

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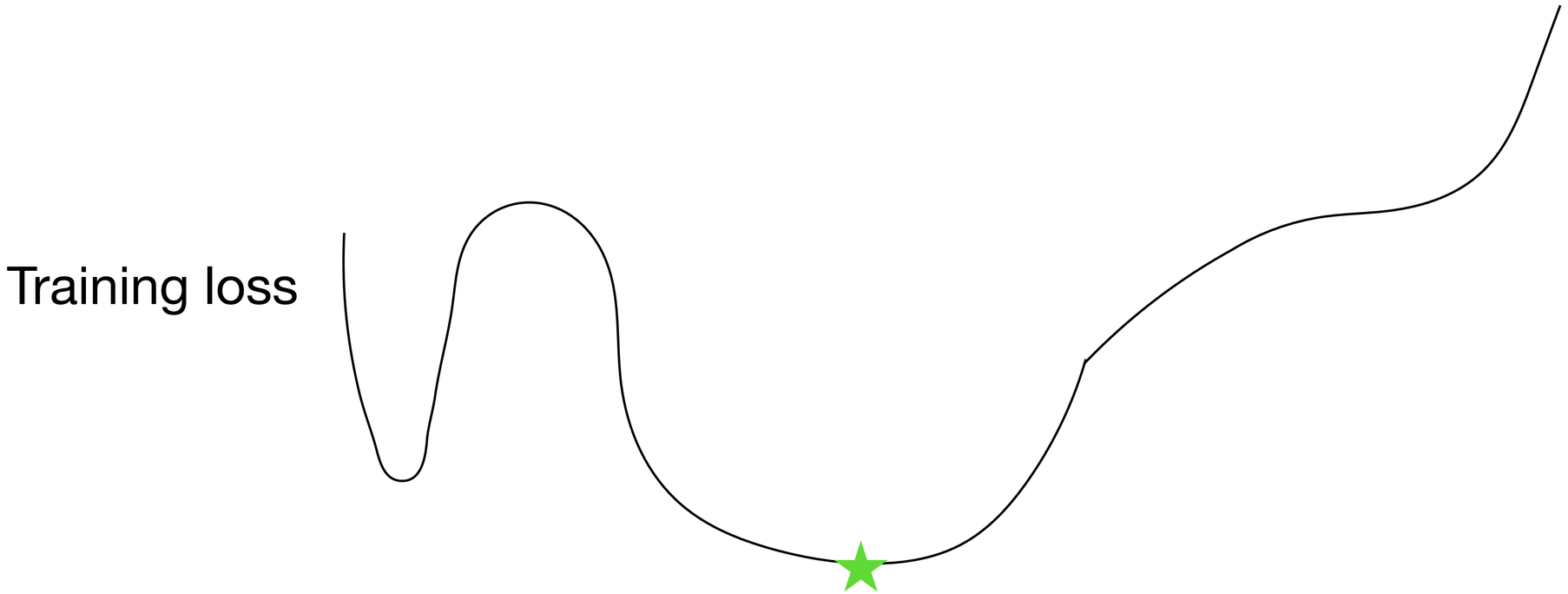

A flat local minima solution can help generalizes better to test data

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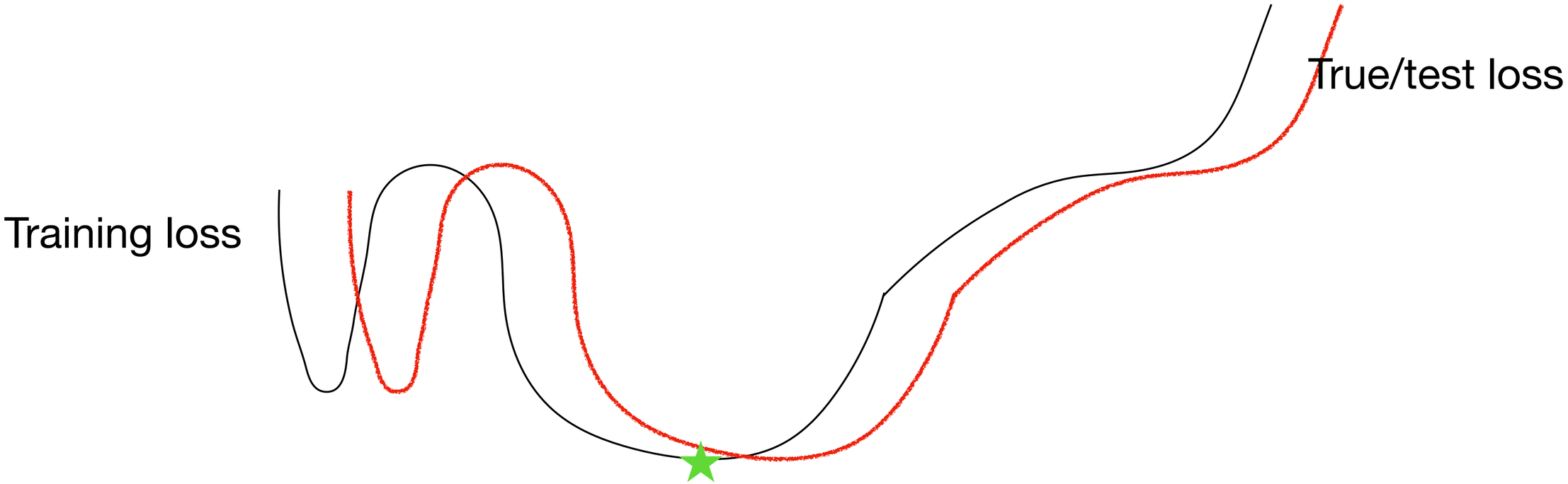

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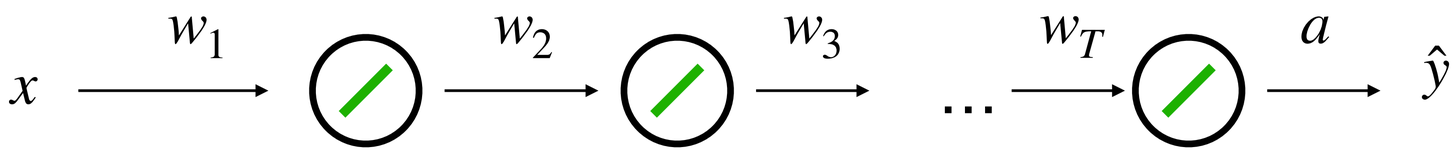

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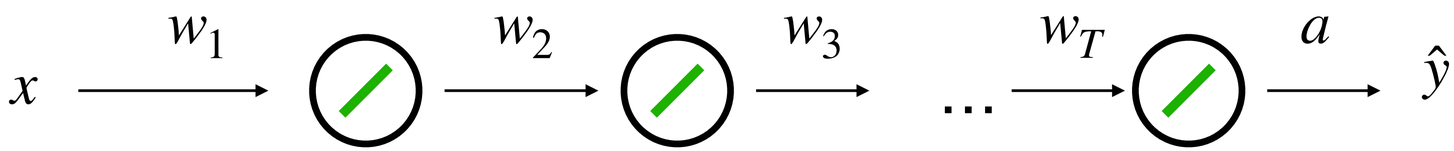

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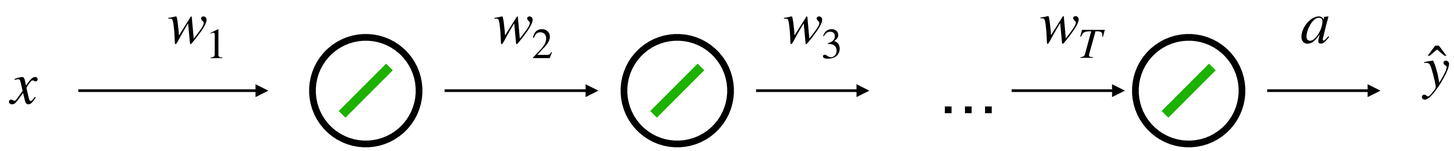

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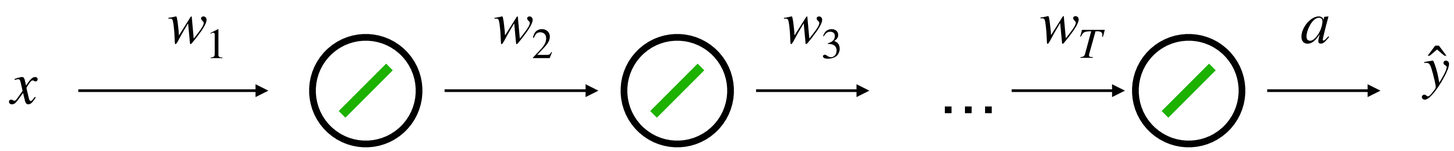

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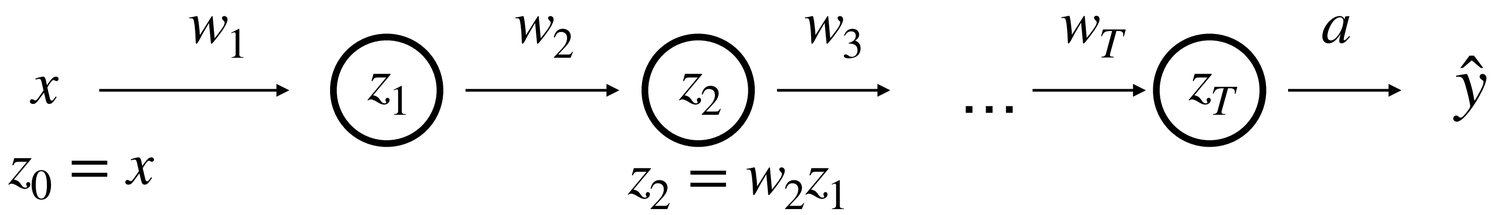

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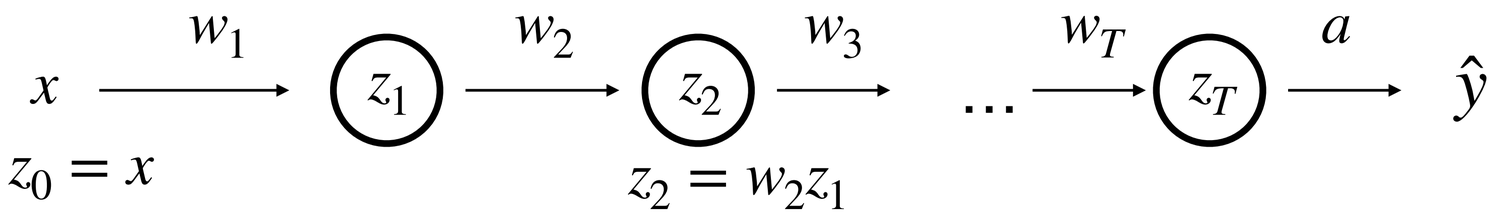

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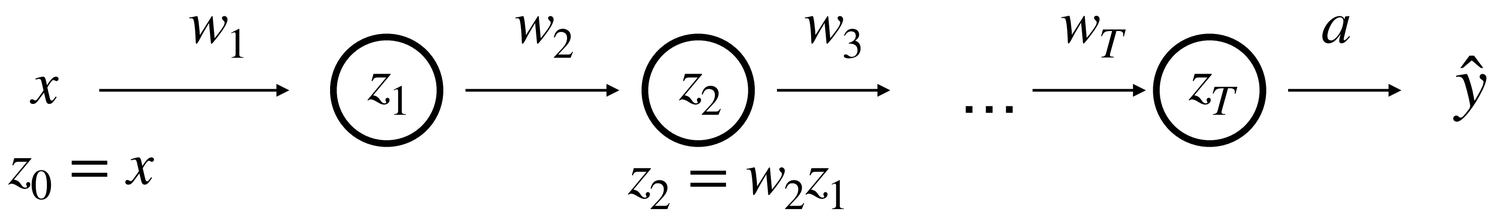

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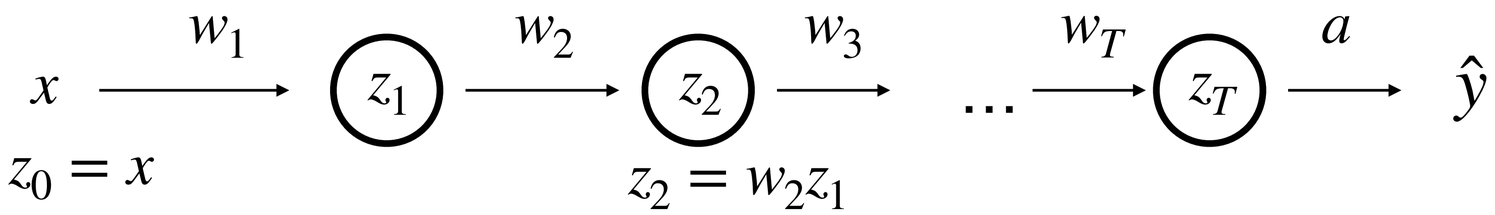

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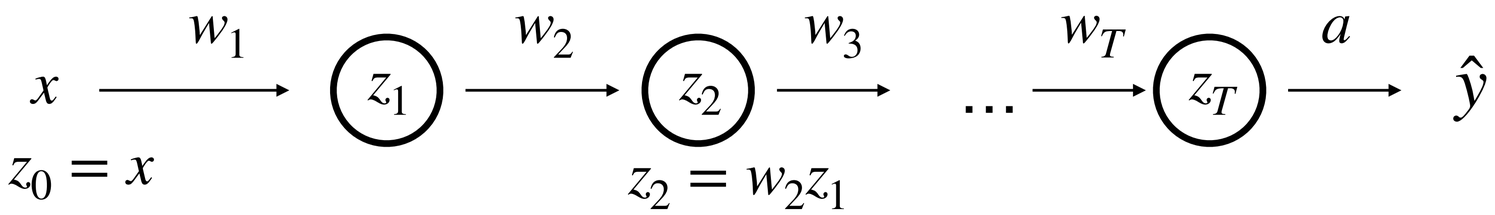

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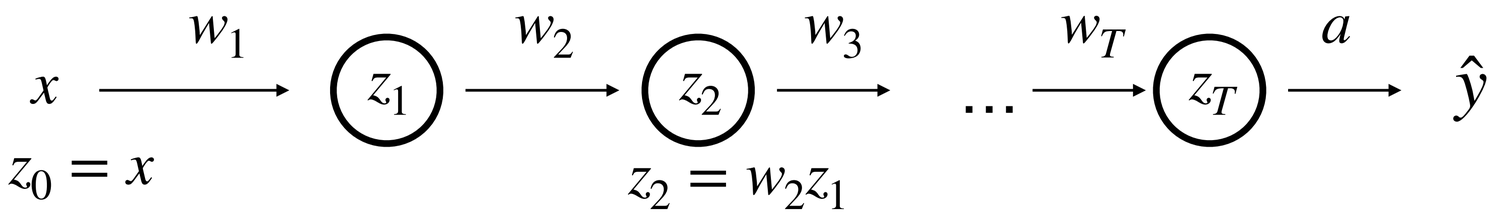

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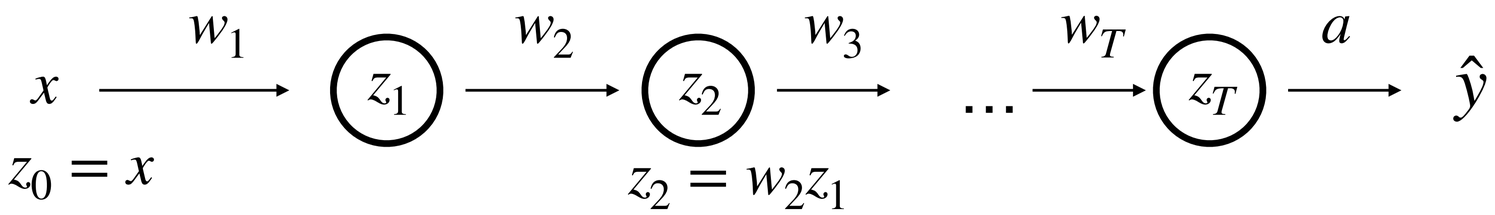

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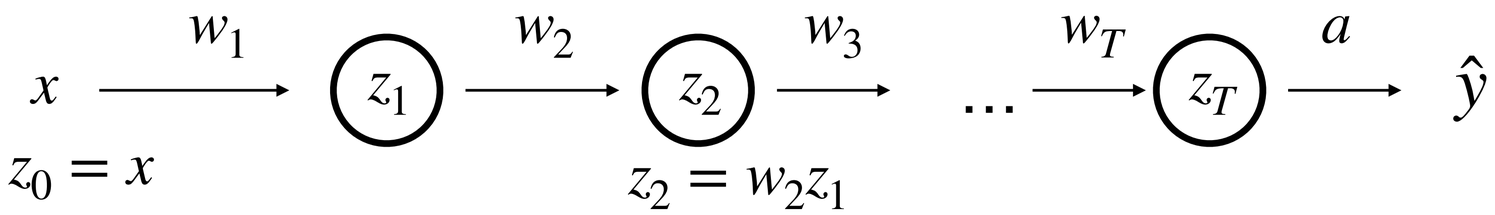

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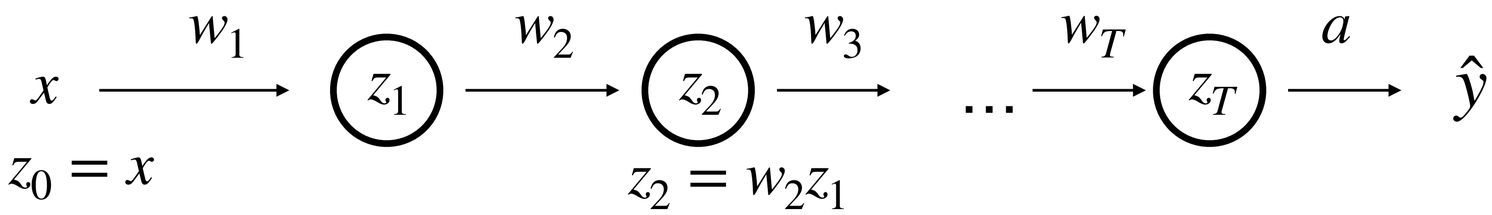

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Total complexity:

$$1 + 2 + \ldots + T = O(T^2)$$

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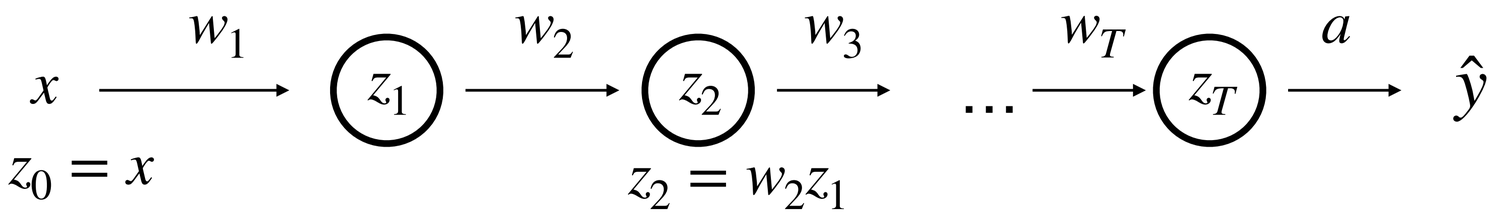

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Quadratic in size of the graph!

Summary so far

What we did:

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Can we do better in running time?

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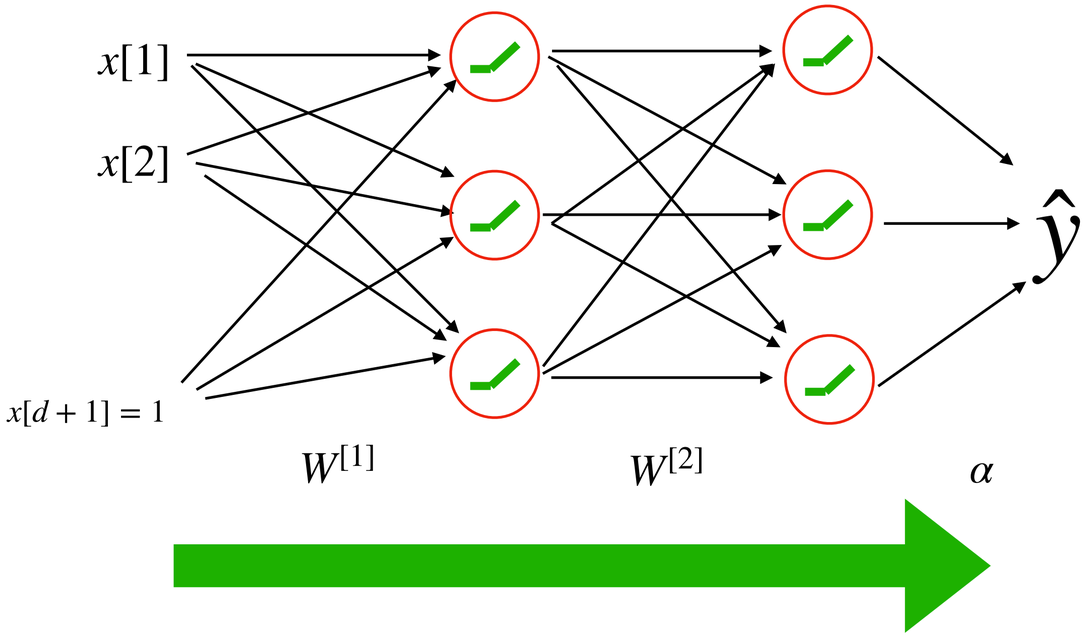
This allowed networks with intermediate “hidden” neurons between
input and output layers to **learn efficiently,** overcoming the
limitations noted by Minsky and Papert.

Overview of backpropagation

Forward pass followed by a backward pass

Forward pass:



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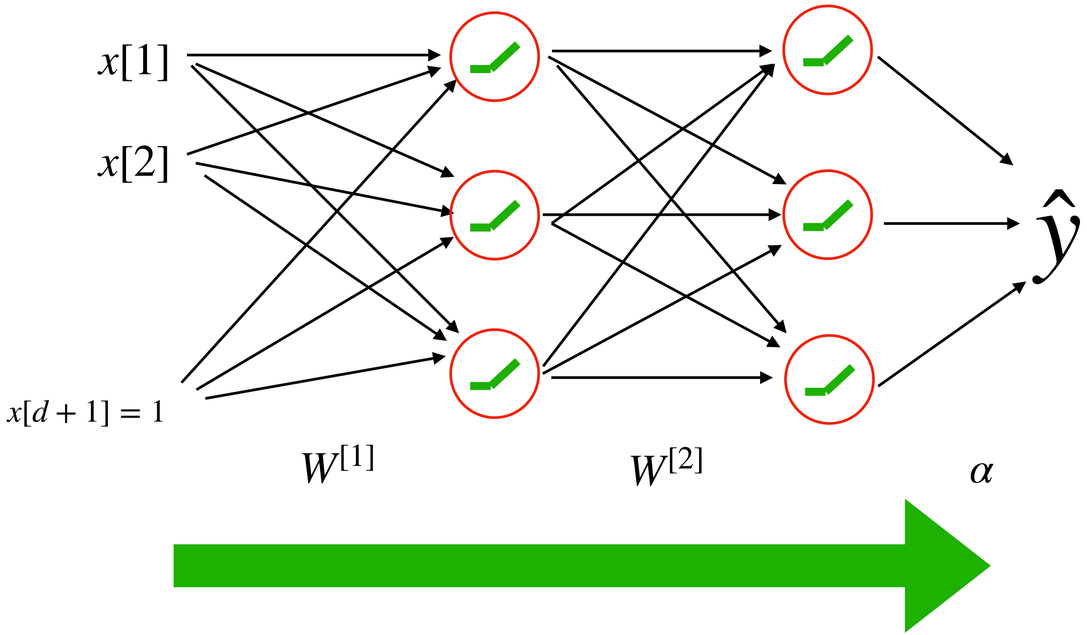

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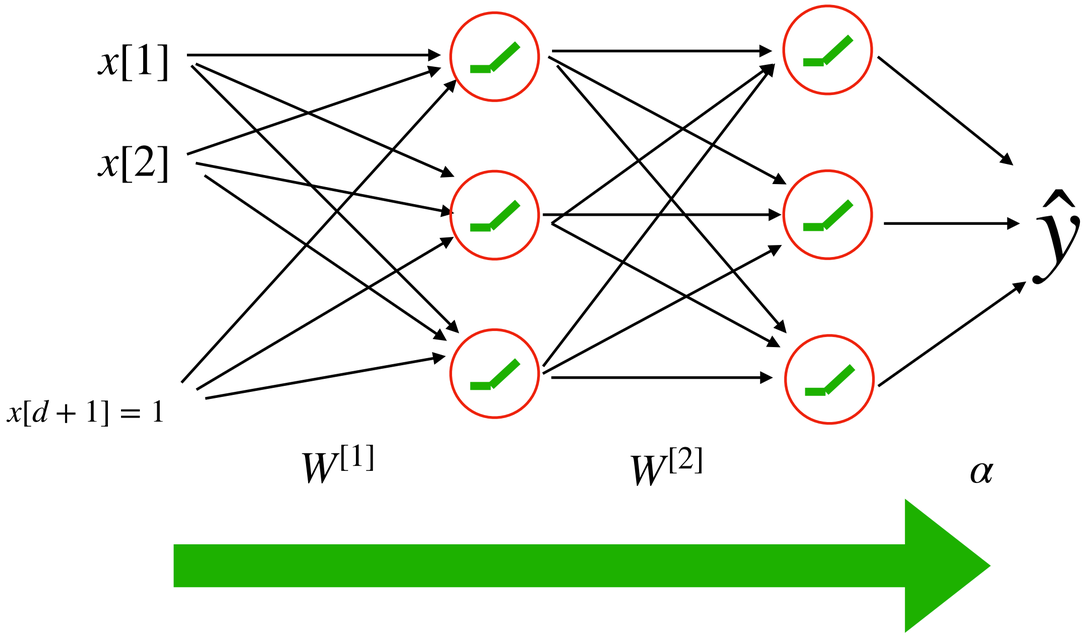

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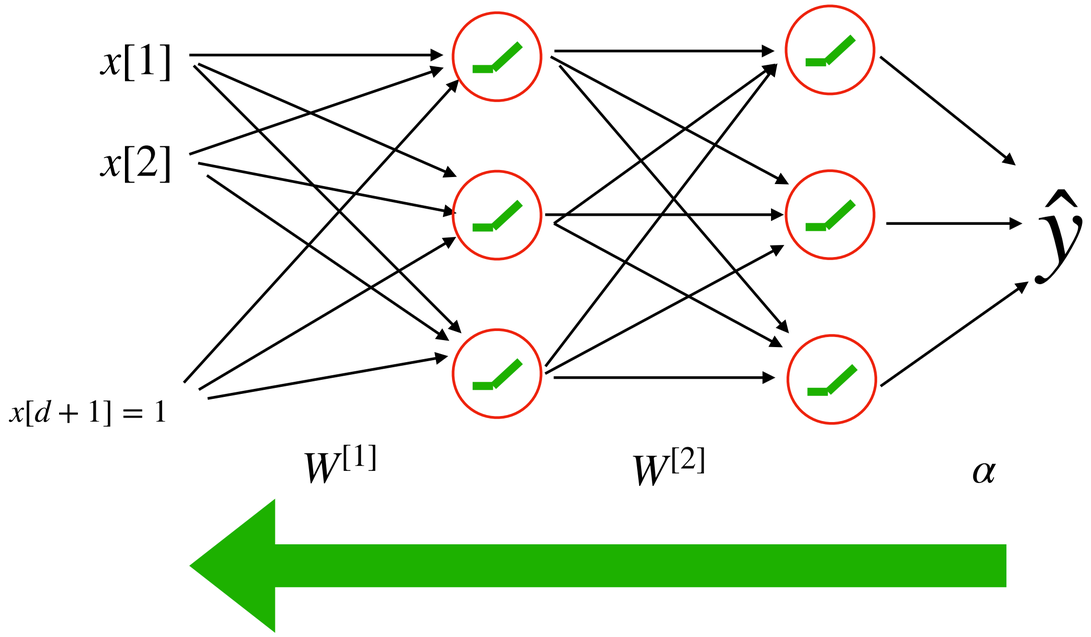

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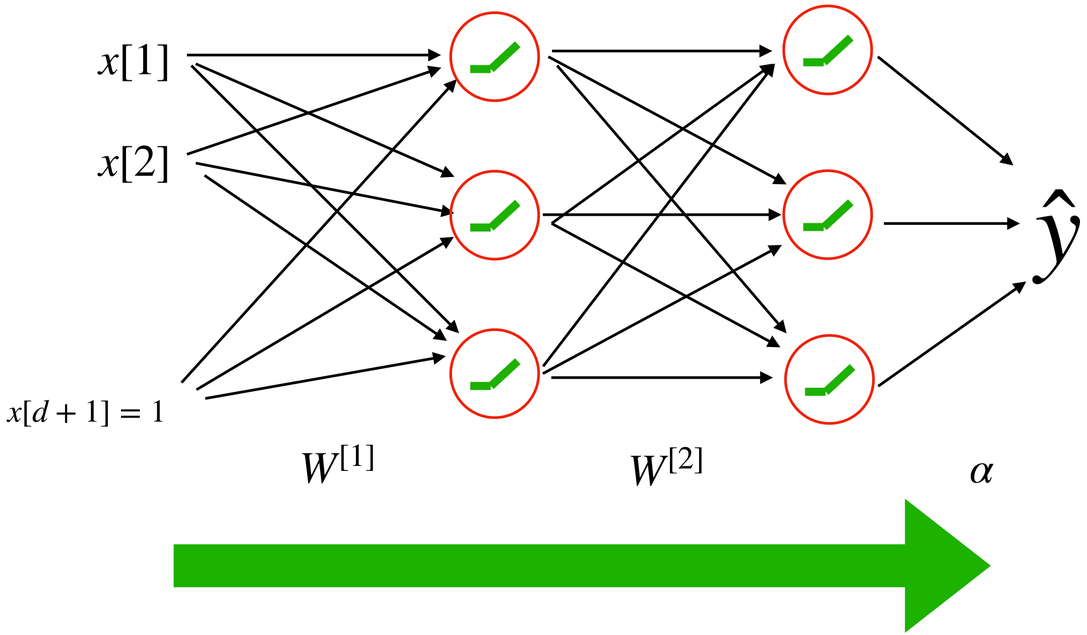

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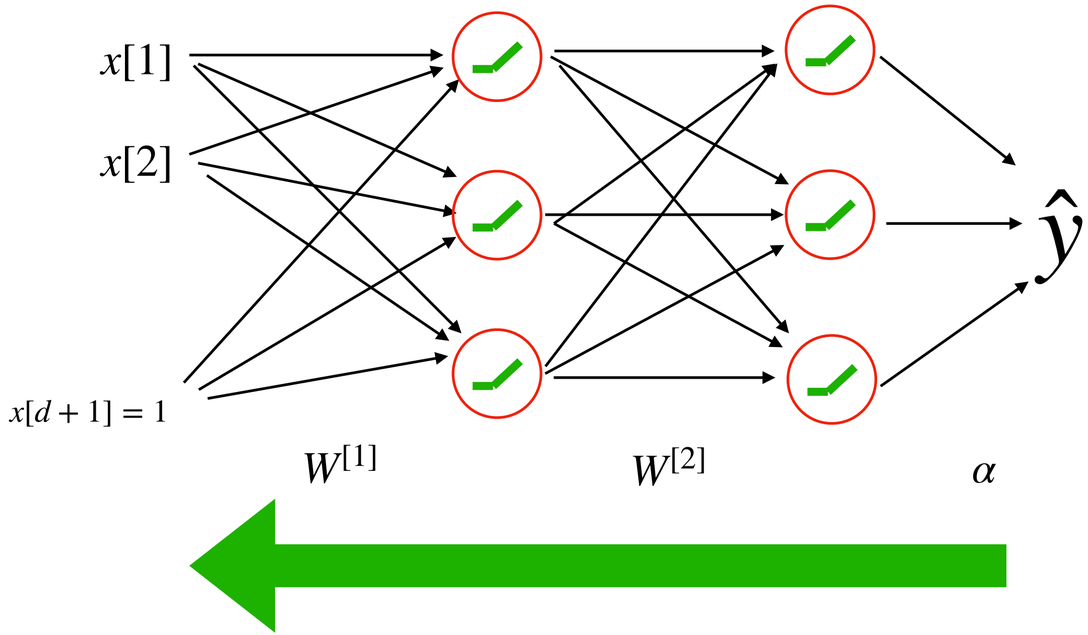
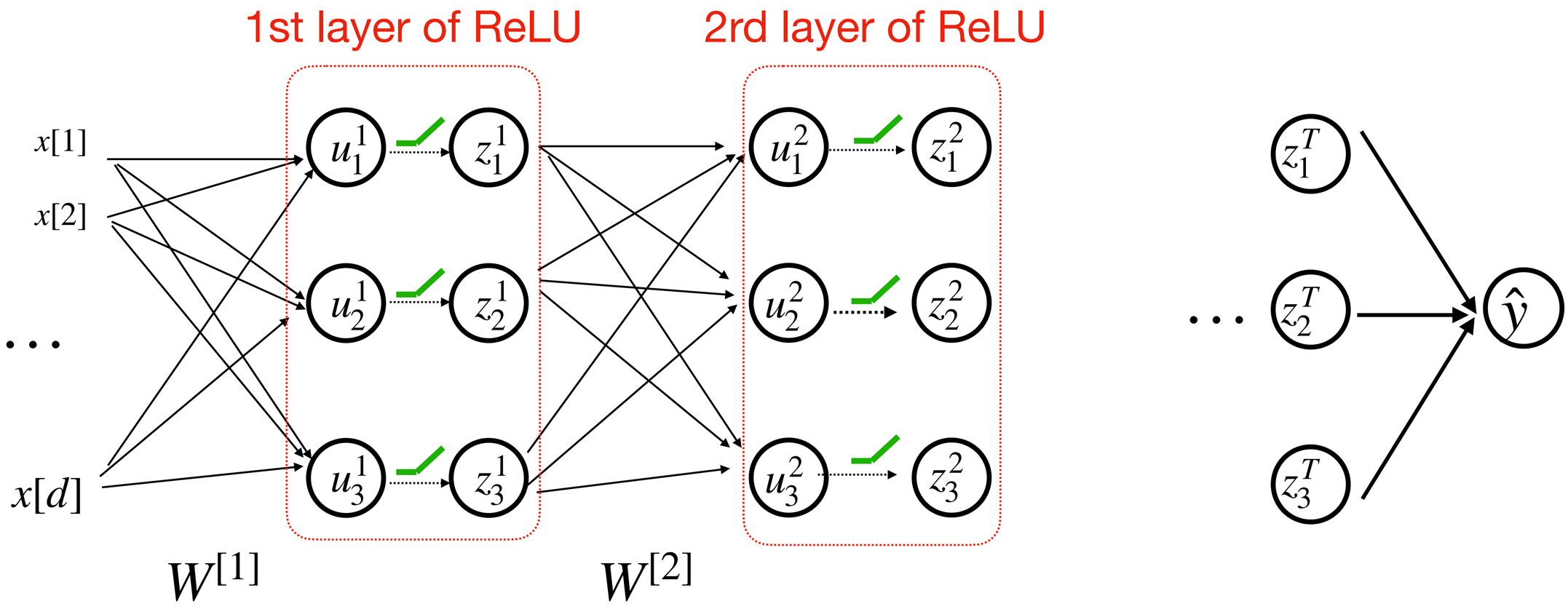


Compute derivatives

A Forward Pass: from $t = 0$ to T



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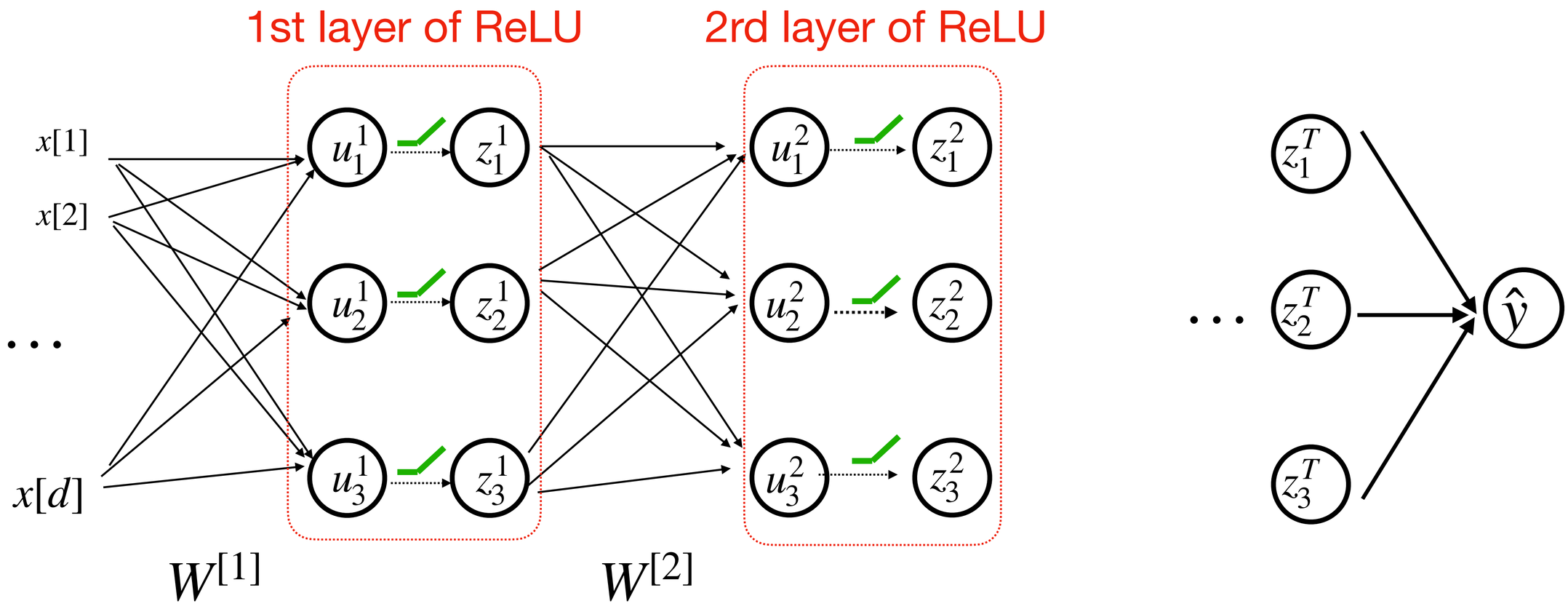
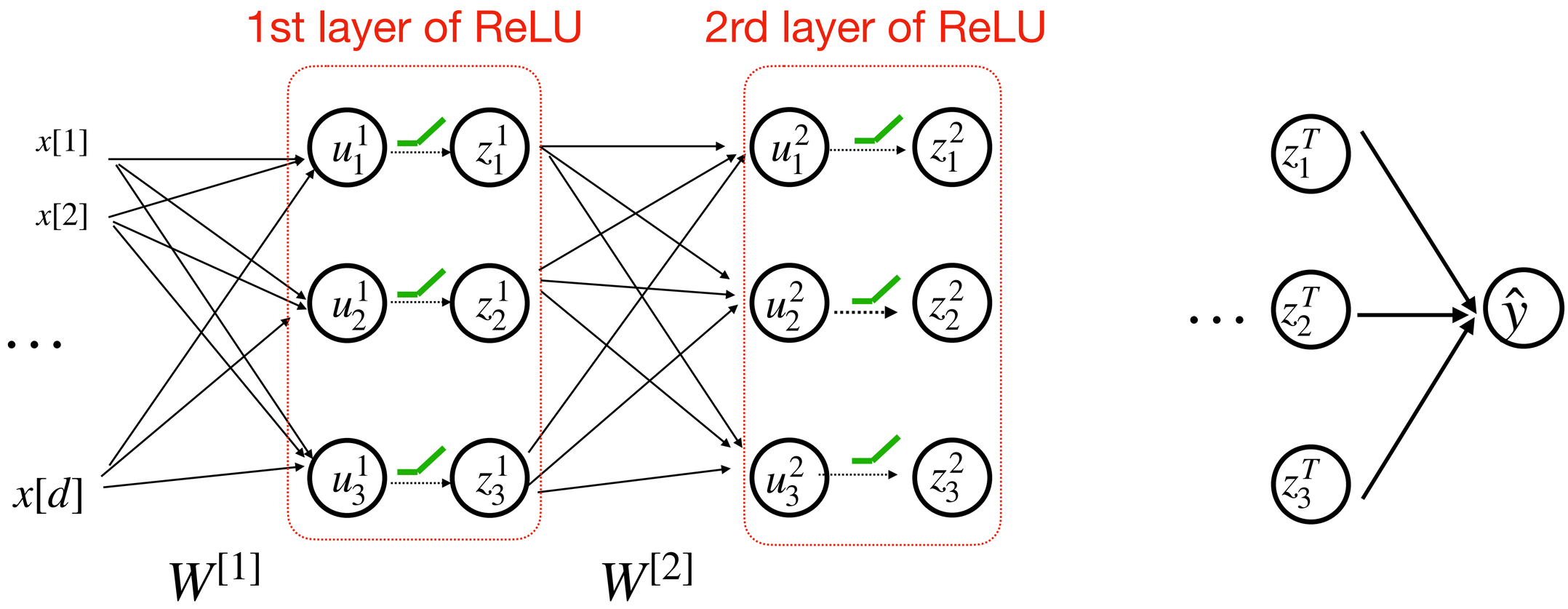


$z^0 = x$

A Forward Pass: from $t = 0$ to T



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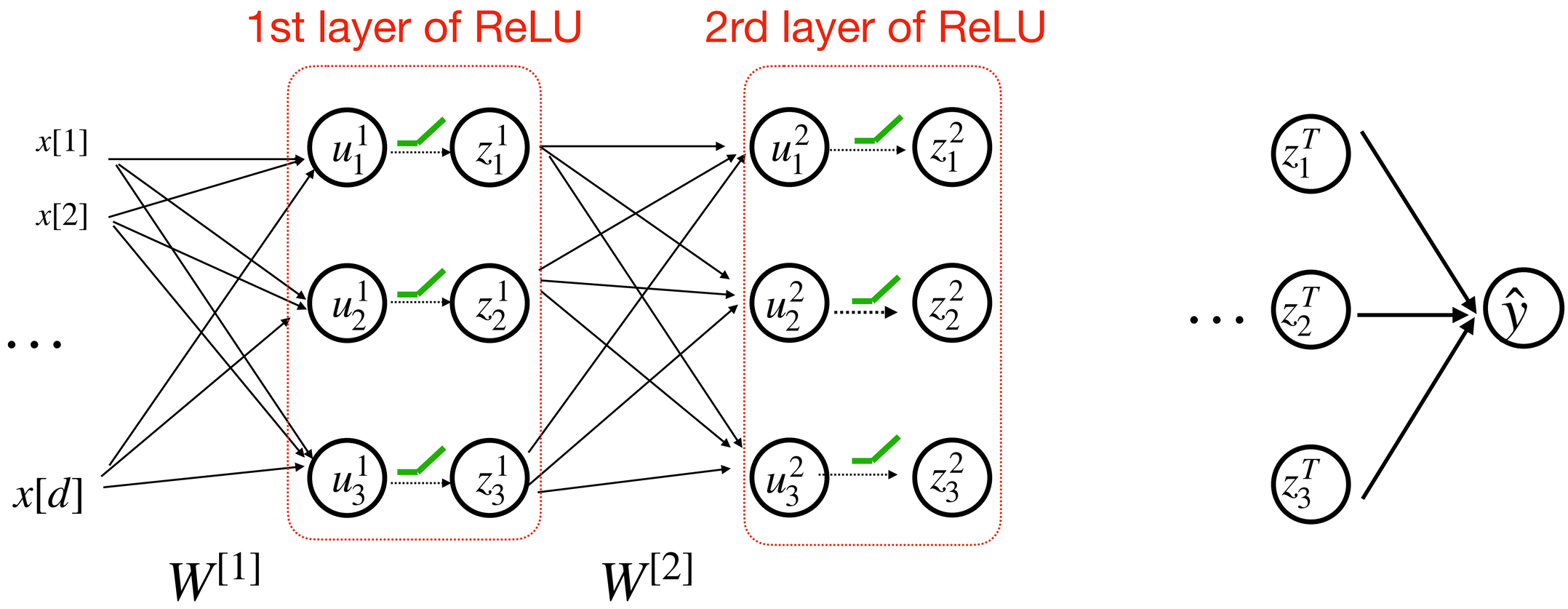

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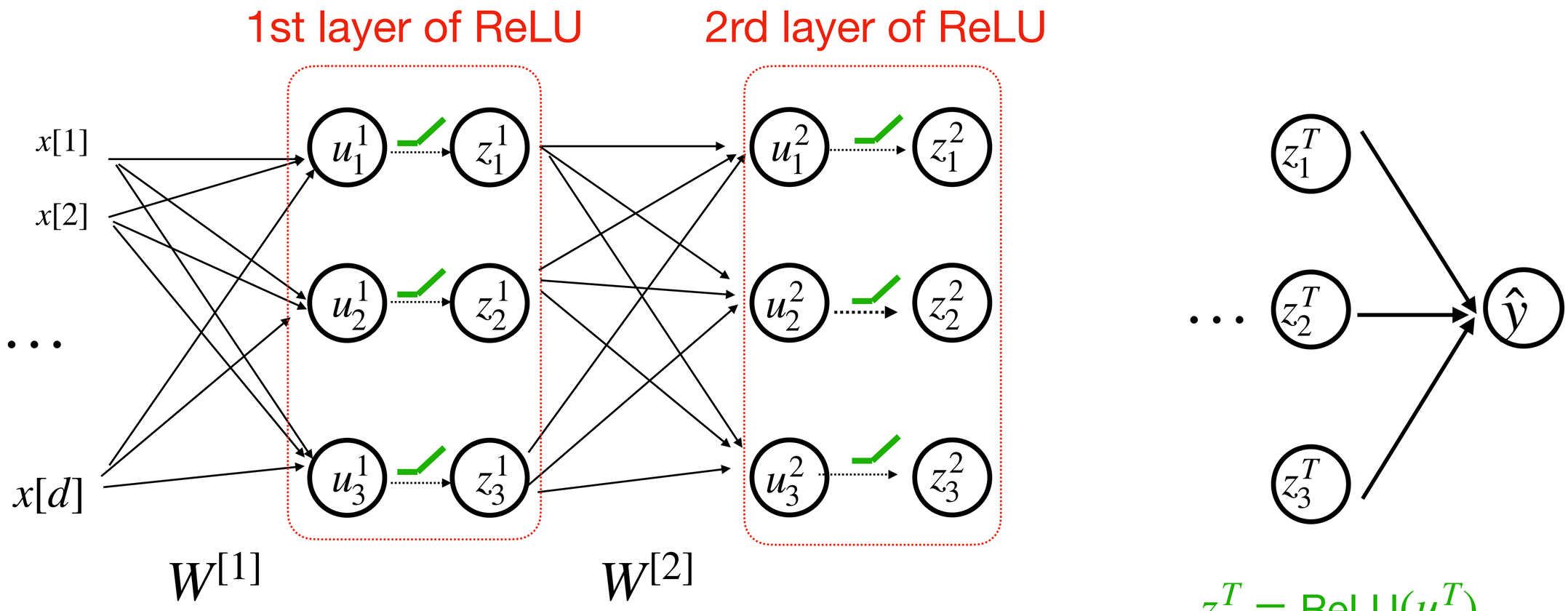

$z^T = \text{ReLU}(u^T)$

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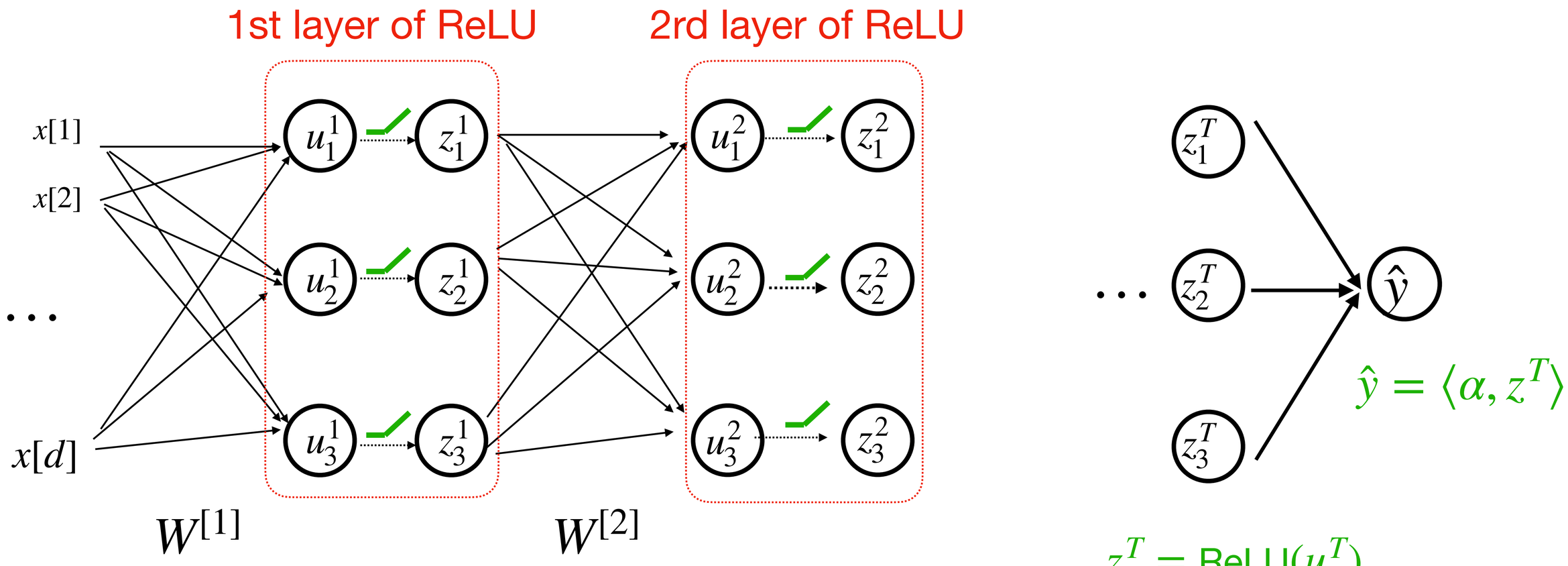
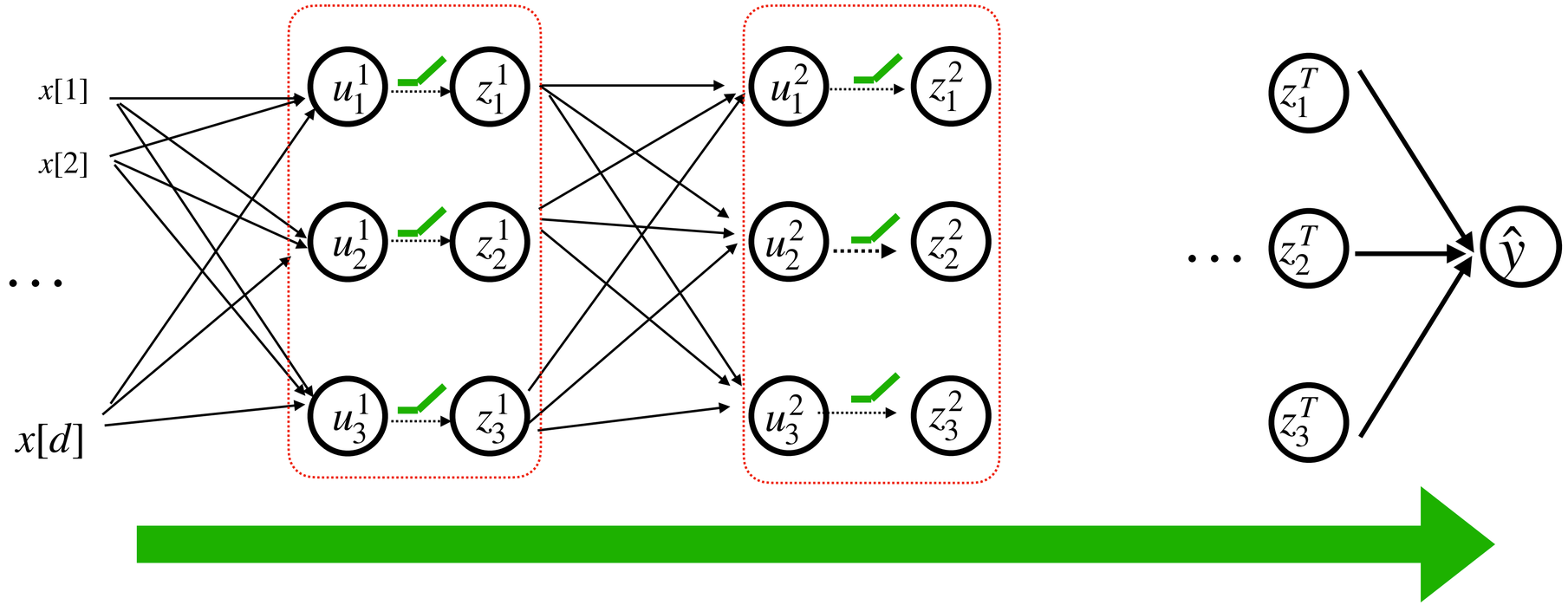


Summary of the forward pass



All nodes' values (i.e., z, u, $\hat{y}$) are computed and stored

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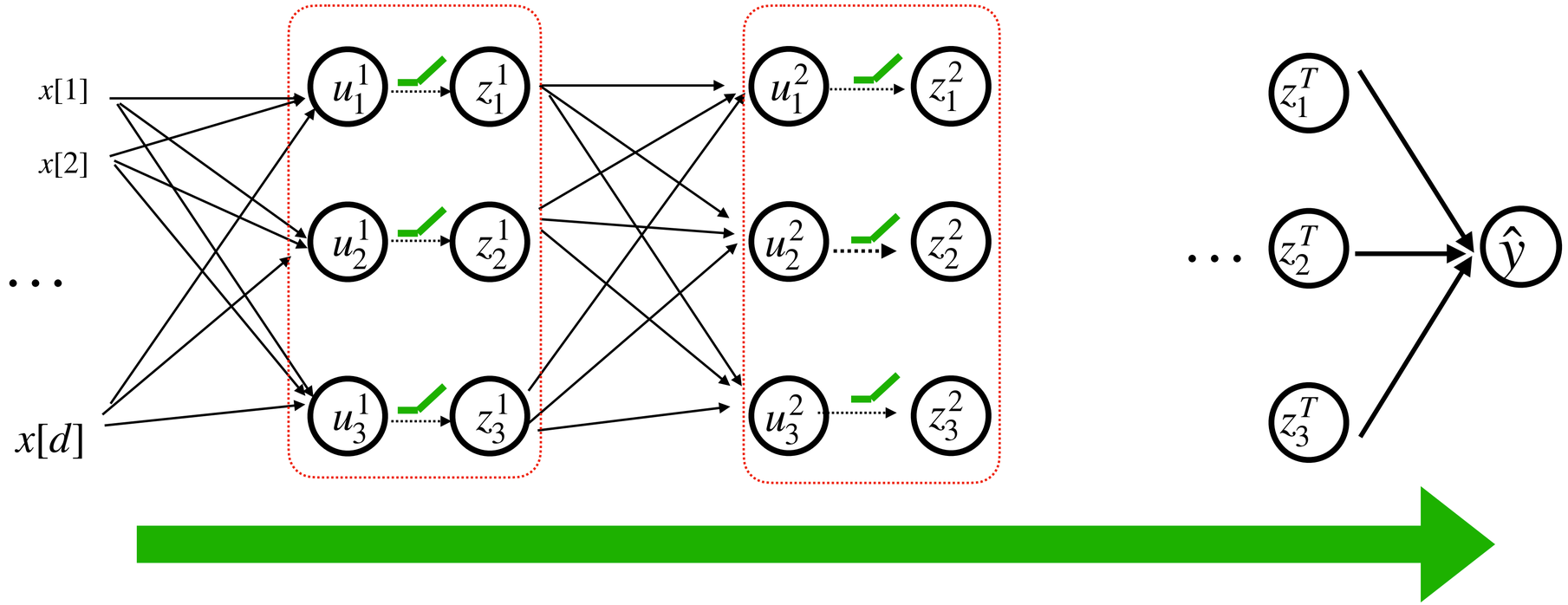

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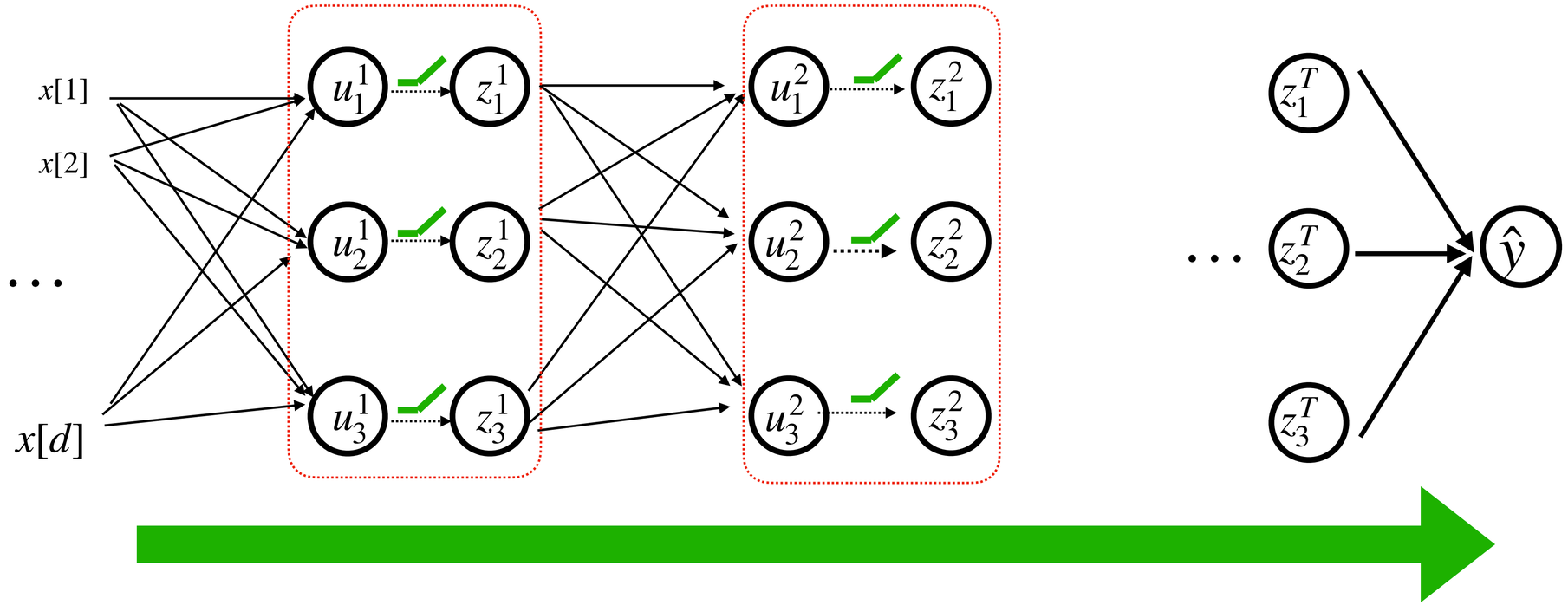

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The backward Pass

Claim: to compute $\partial \hat{y} / \partial w, \forall$ edge w, it suffices to compute $\partial \hat{y} / \partial z, \forall$ node z.

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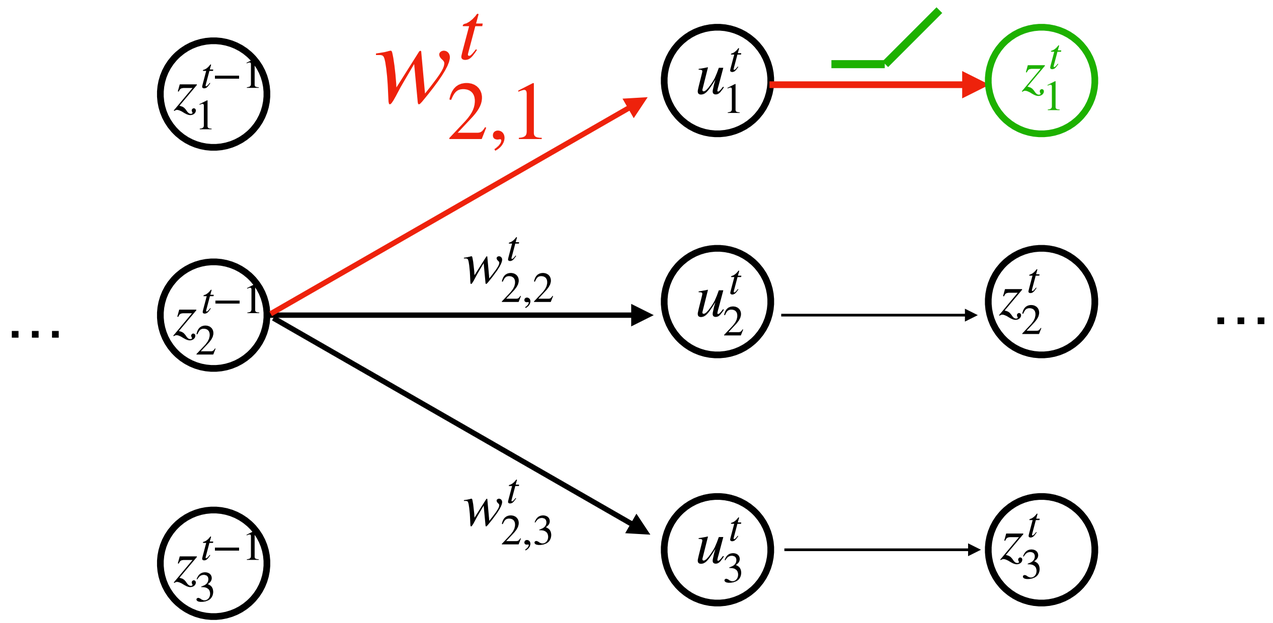

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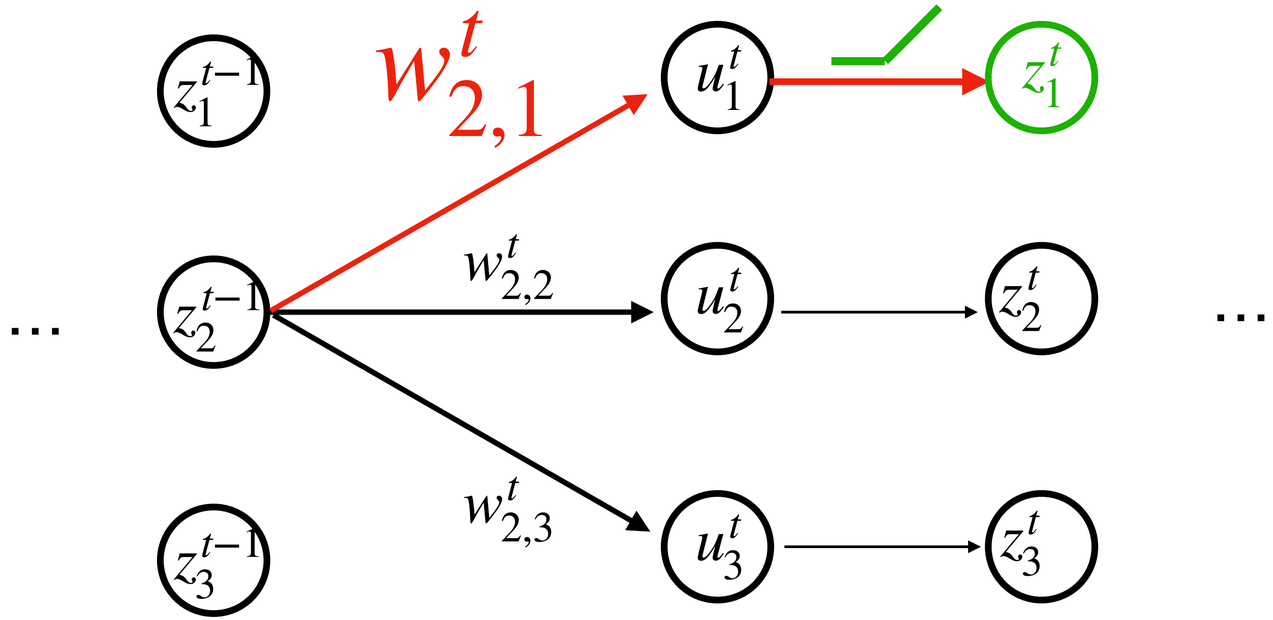

Proof:

WLOG consider $\partial\hat{y}/\partial w_{2,1}^t$

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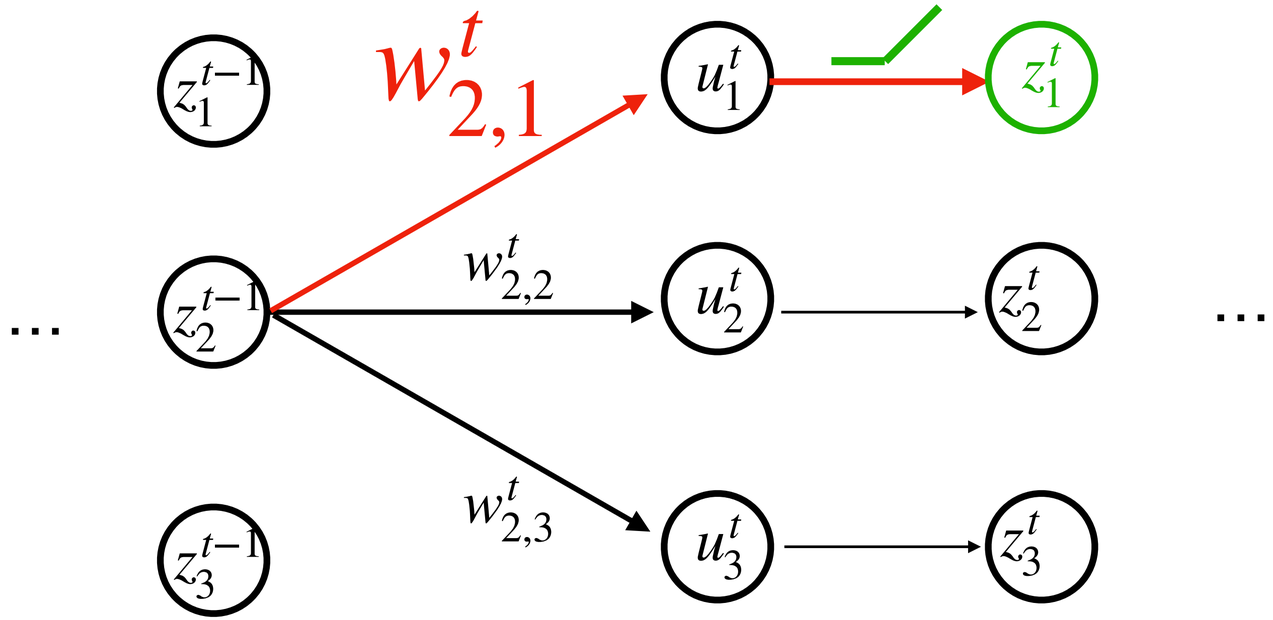

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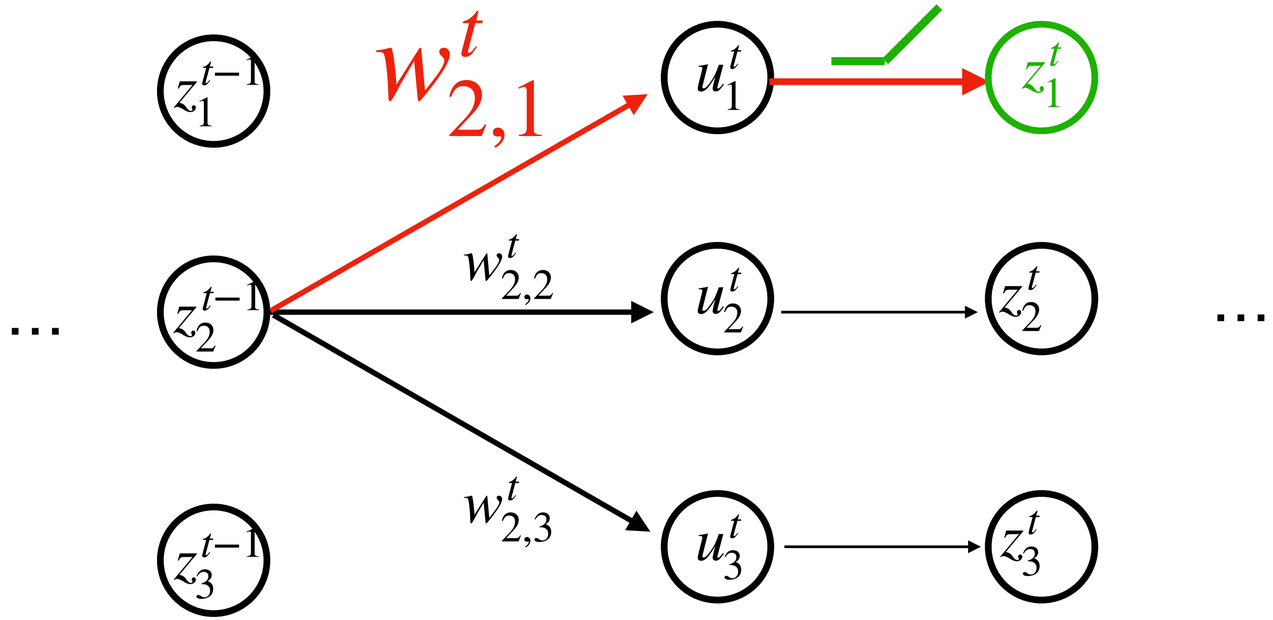

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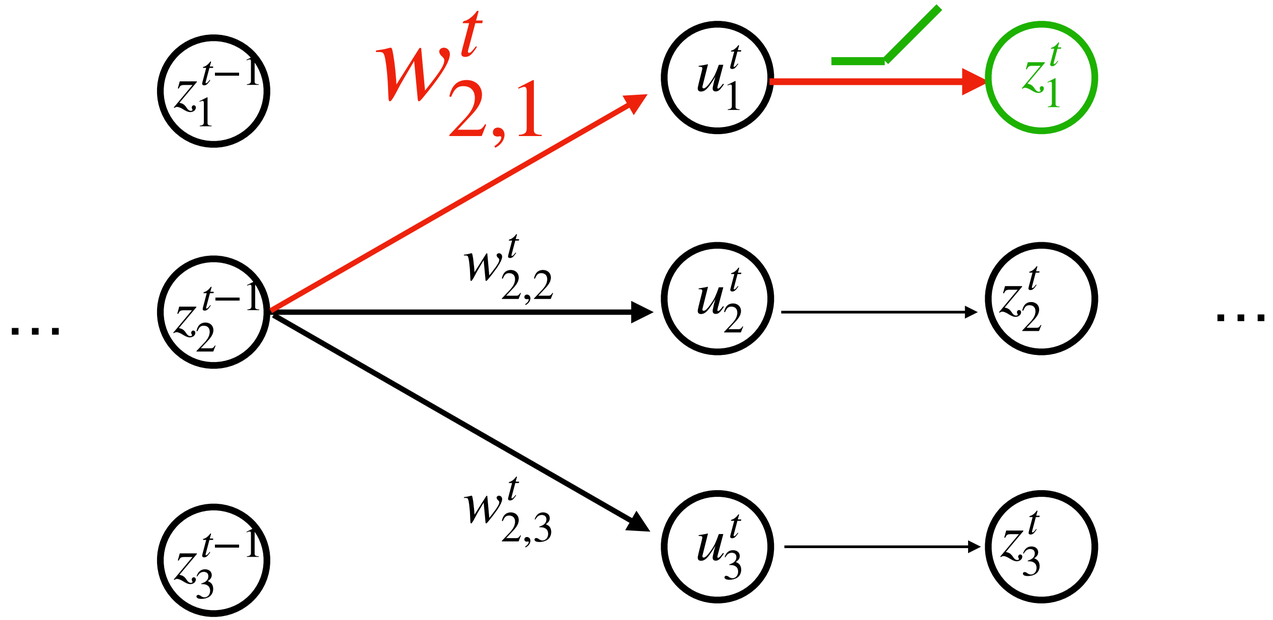

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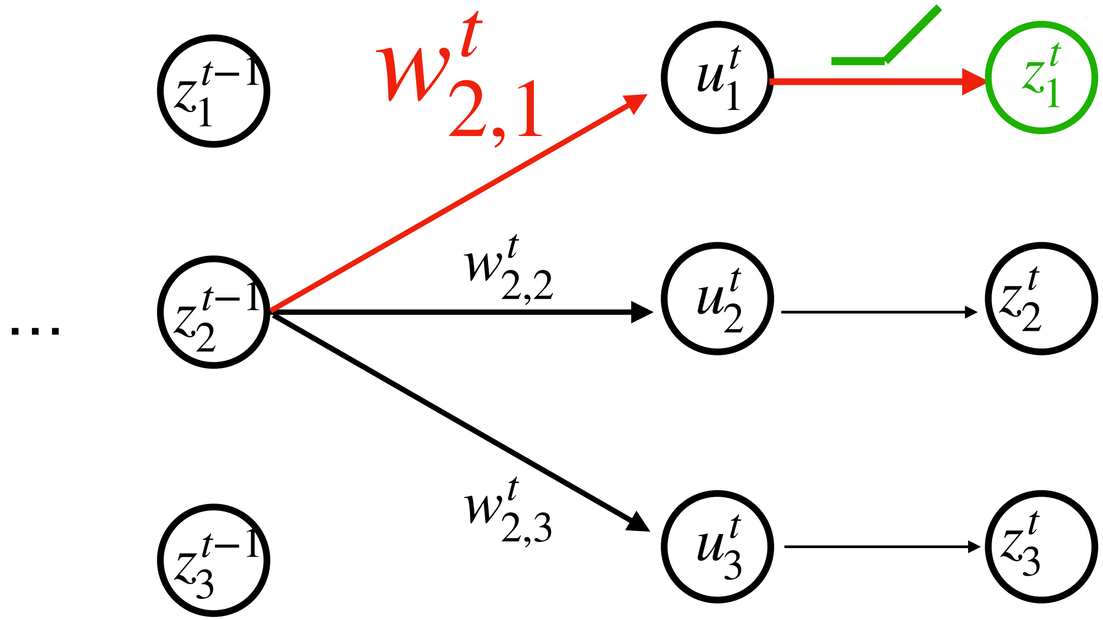

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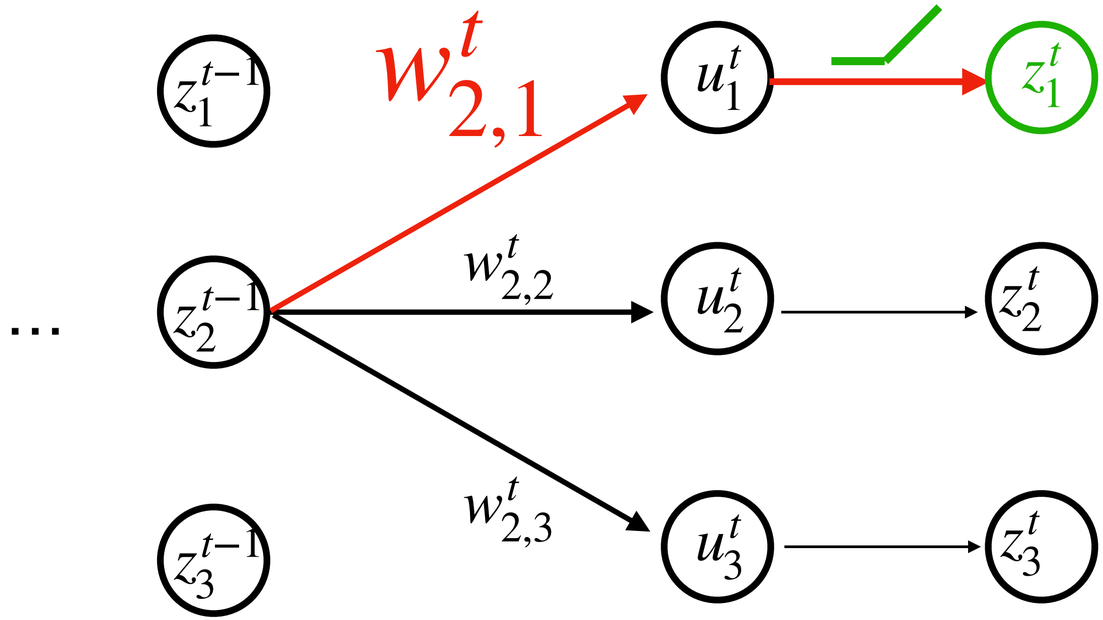

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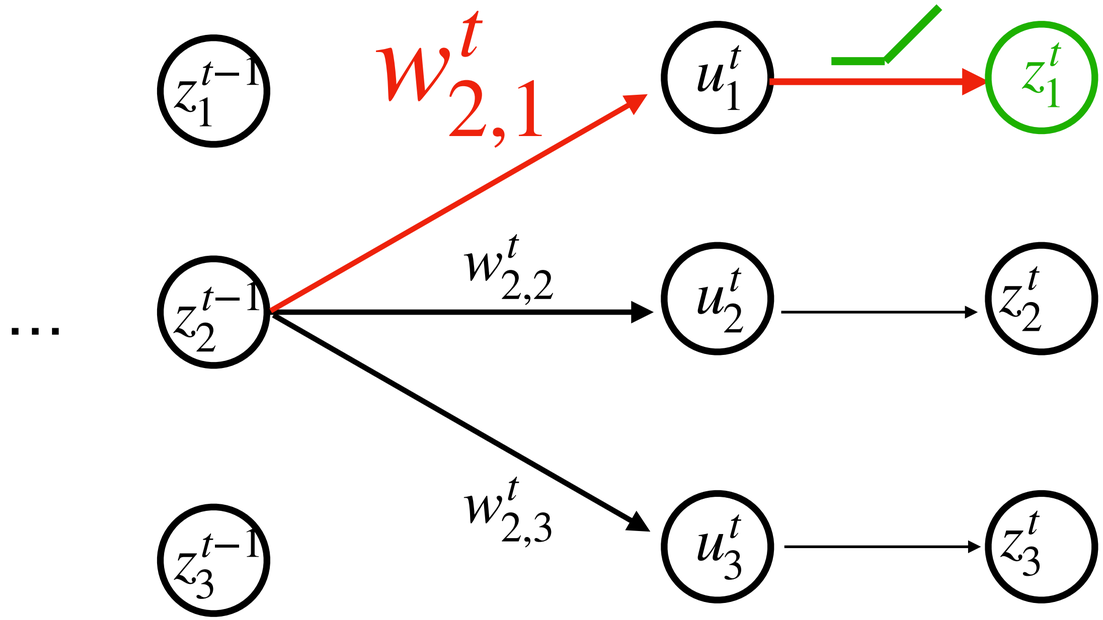

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Known from forward pass

Given by assumption

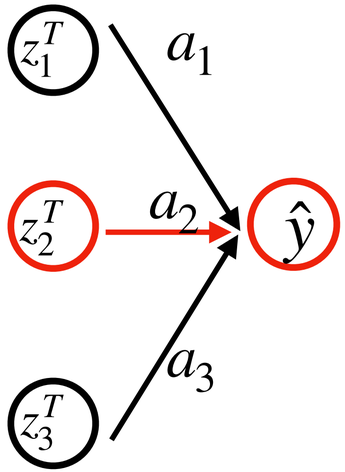
Derivative of ReLU

The backward Pass

We compute $\partial \hat{y} / \partial z^t$ backwards in time from $t = T$ to $t = 1$:

The backward Pass: base case

Base case: compute $\partial \hat{y} / \partial z^T$, for all node z at T-th Layer



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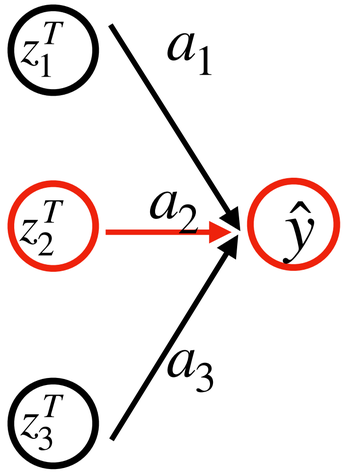
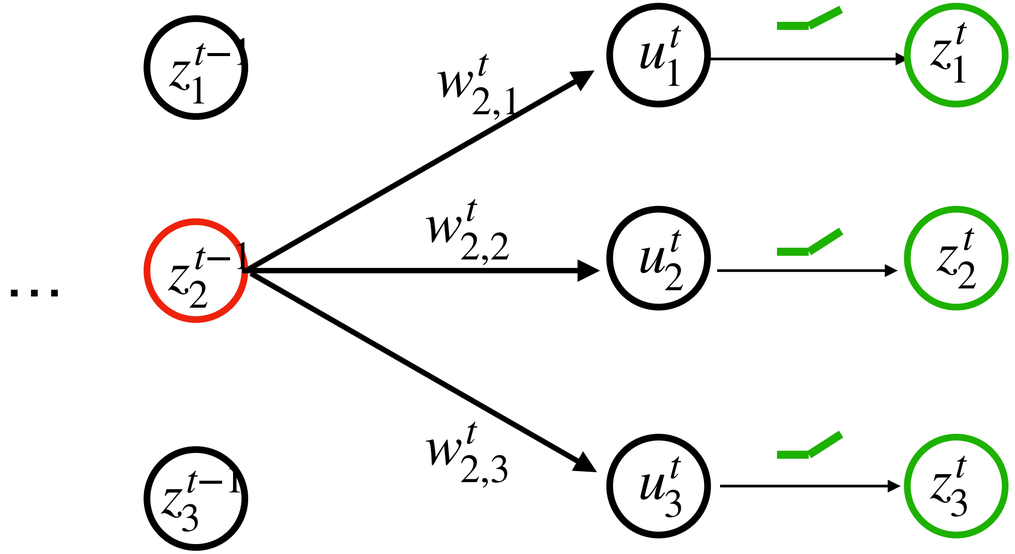


$$\partial \hat{y} / \partial z_i^T = a_i$$

The backward Pass: induction step

Assume that we have computed $\partial \hat{y} / \partial z_i^t, \forall i$



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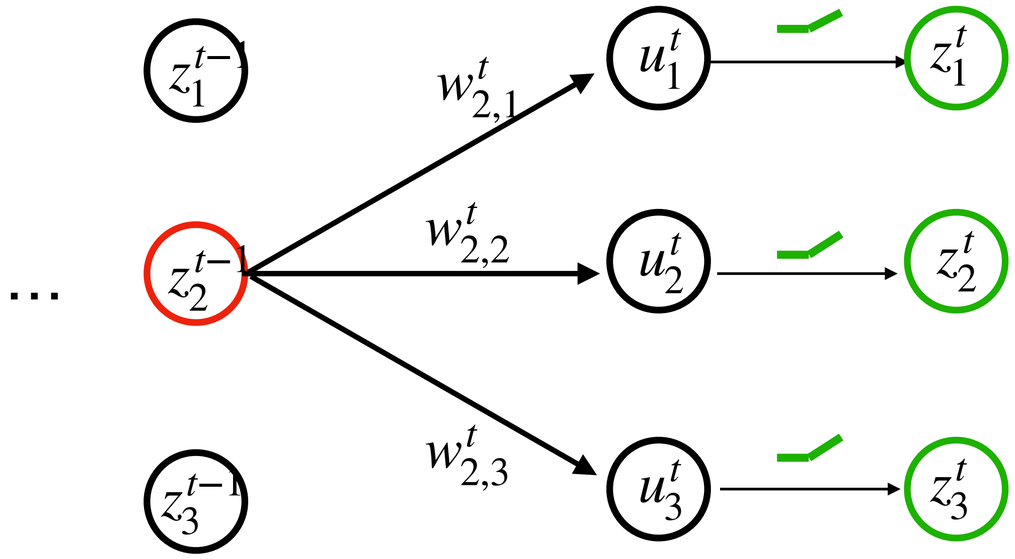

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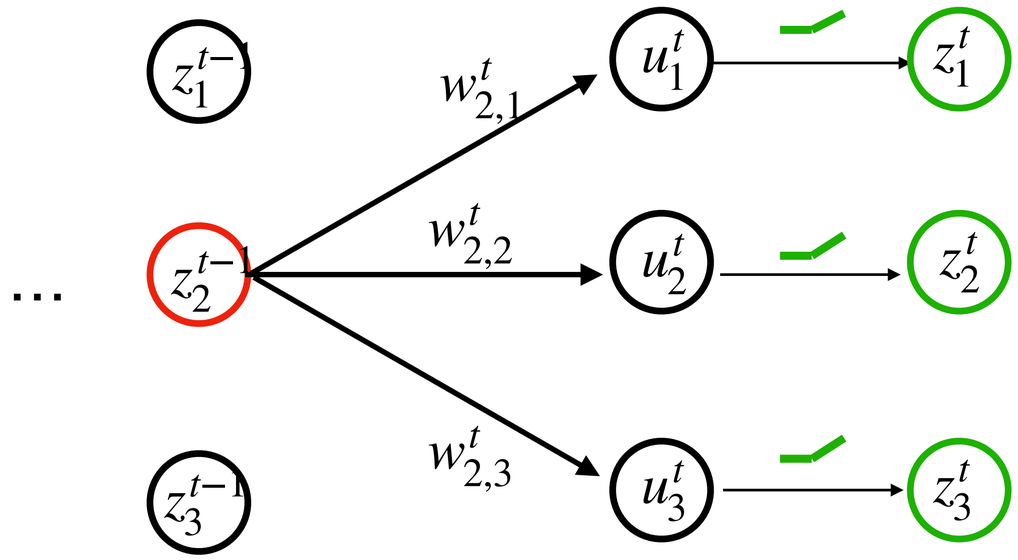

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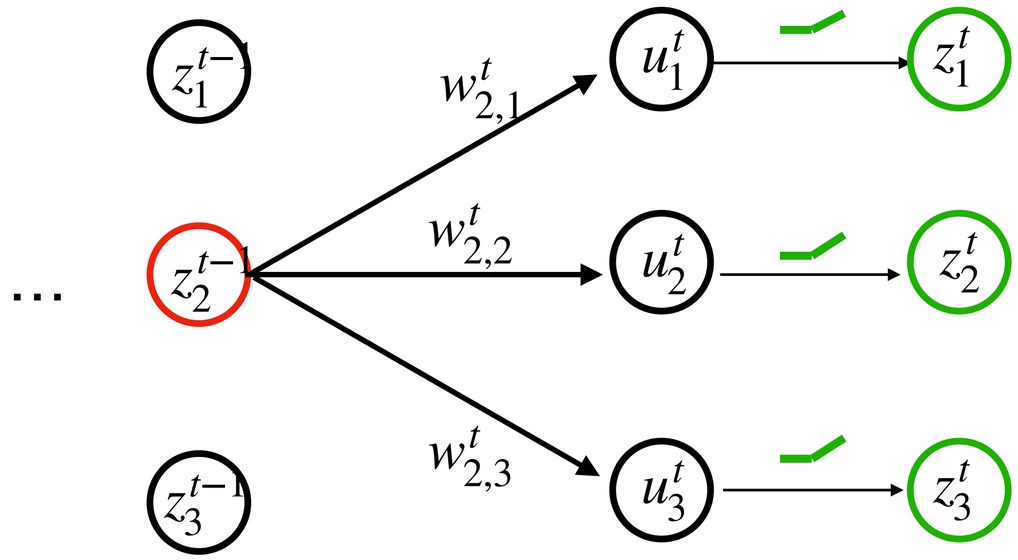

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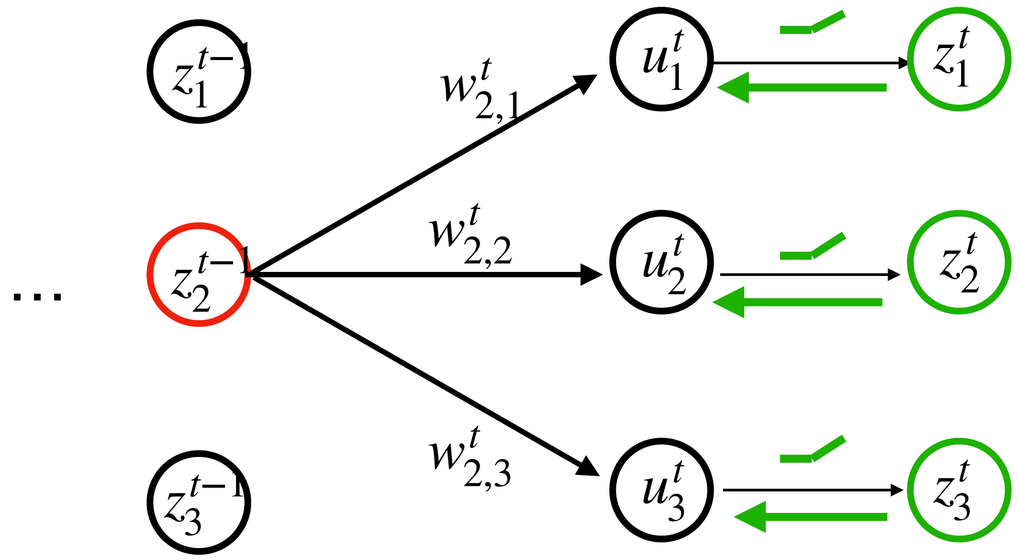

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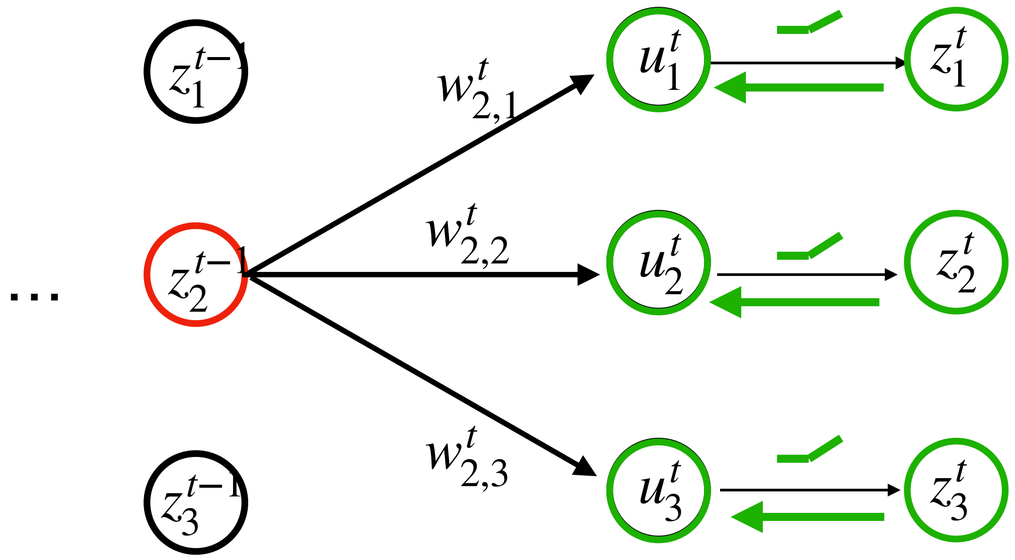

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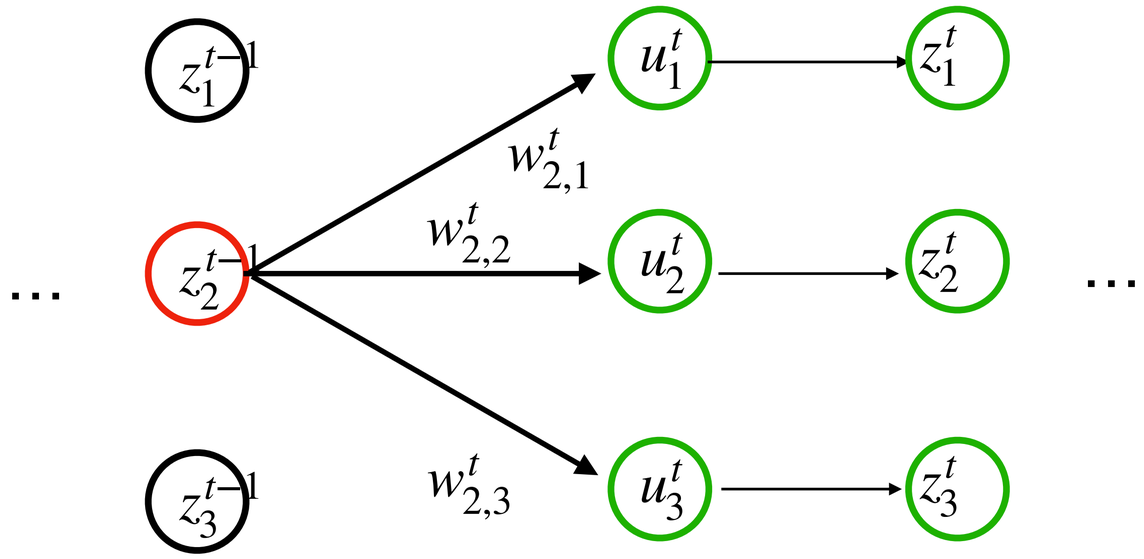

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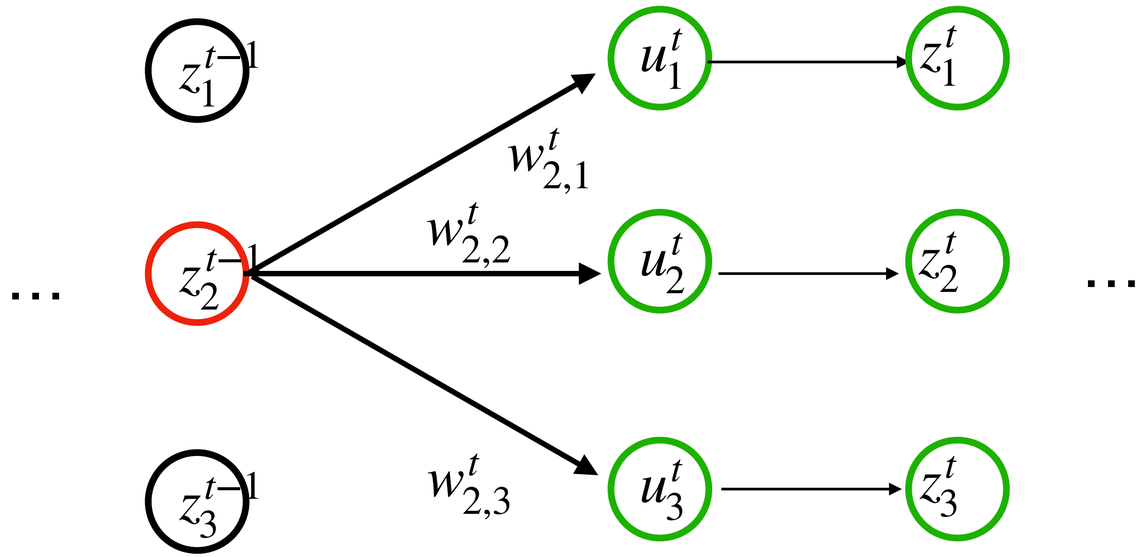

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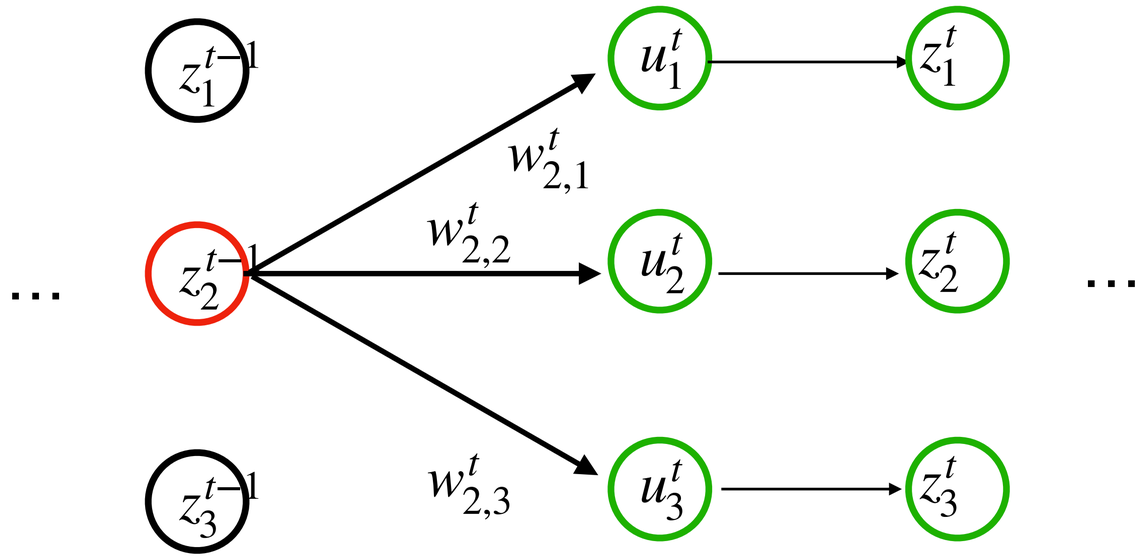

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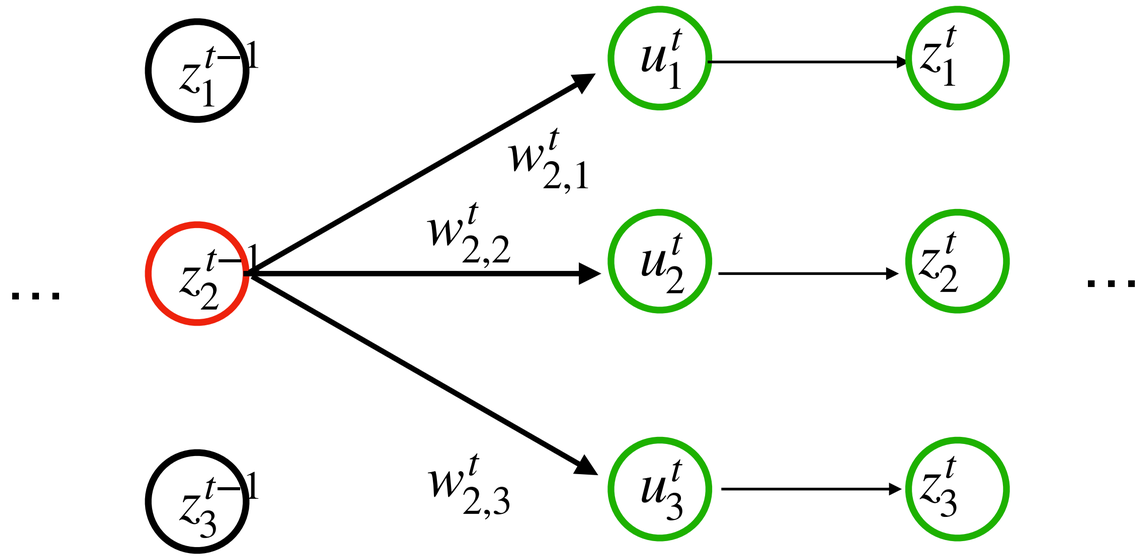

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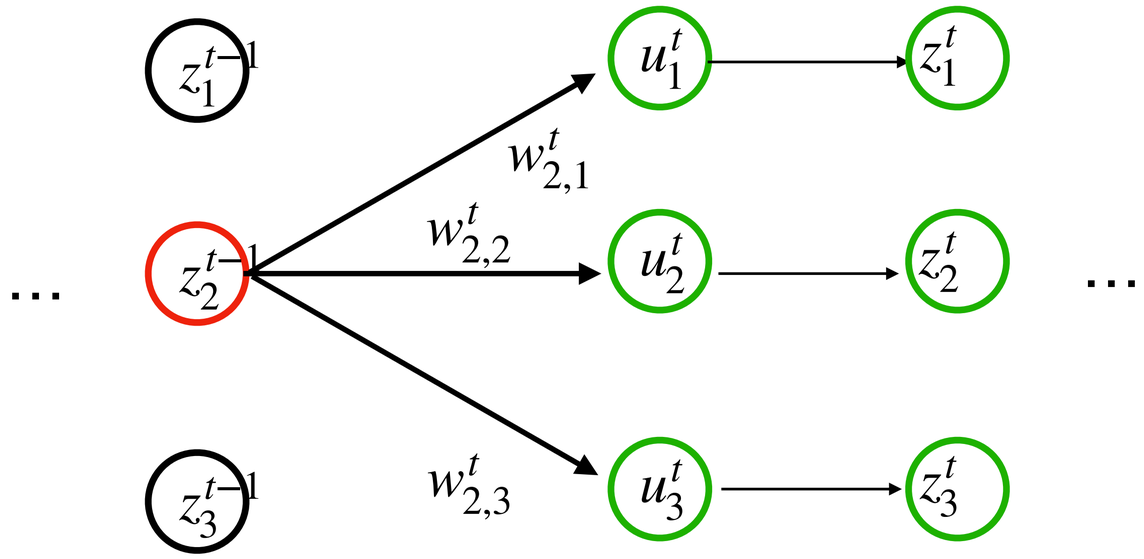

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We are done at node z_2^{t-1}!

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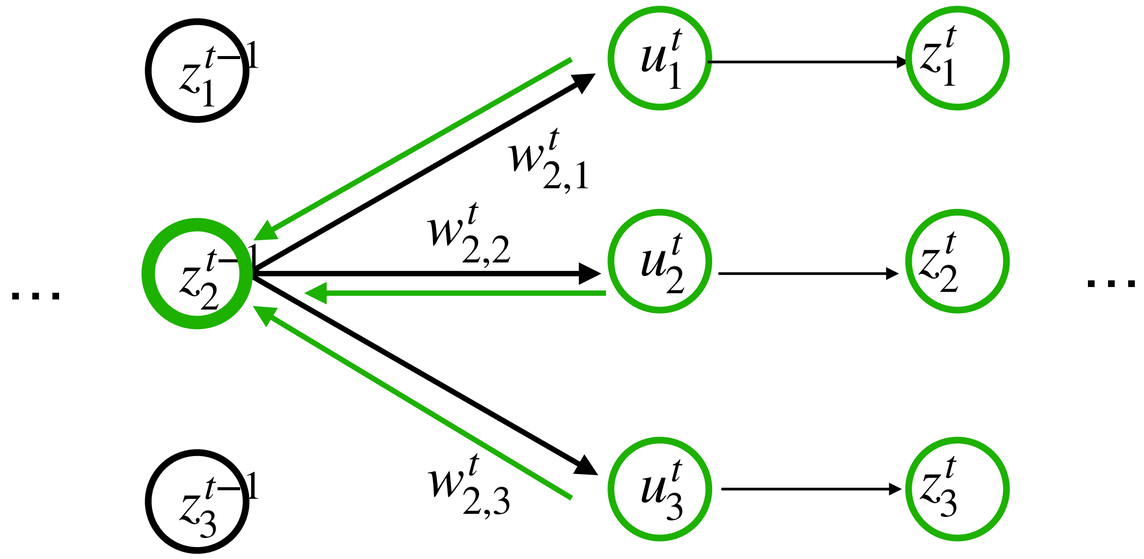

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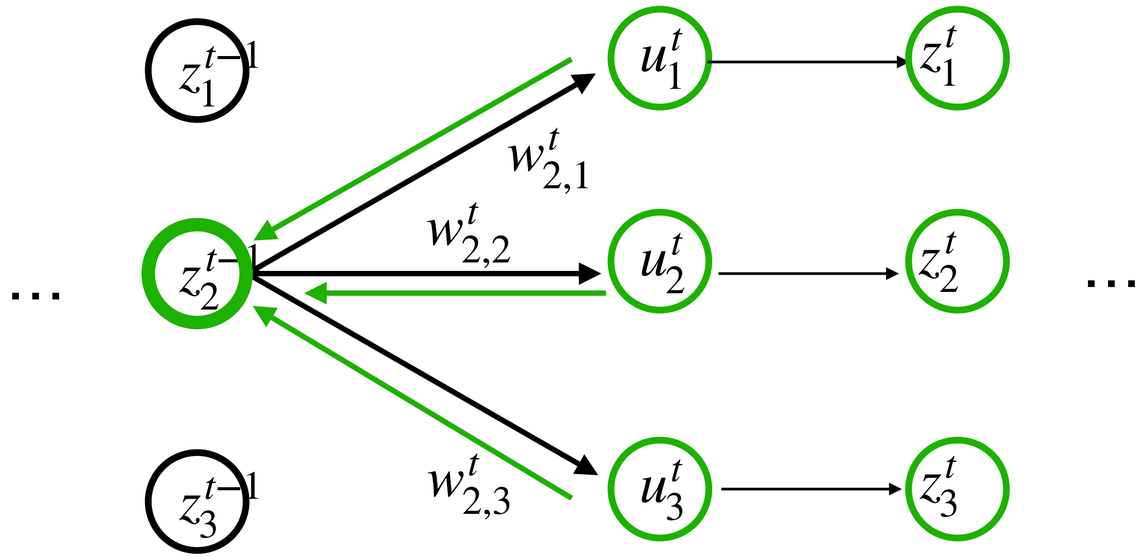

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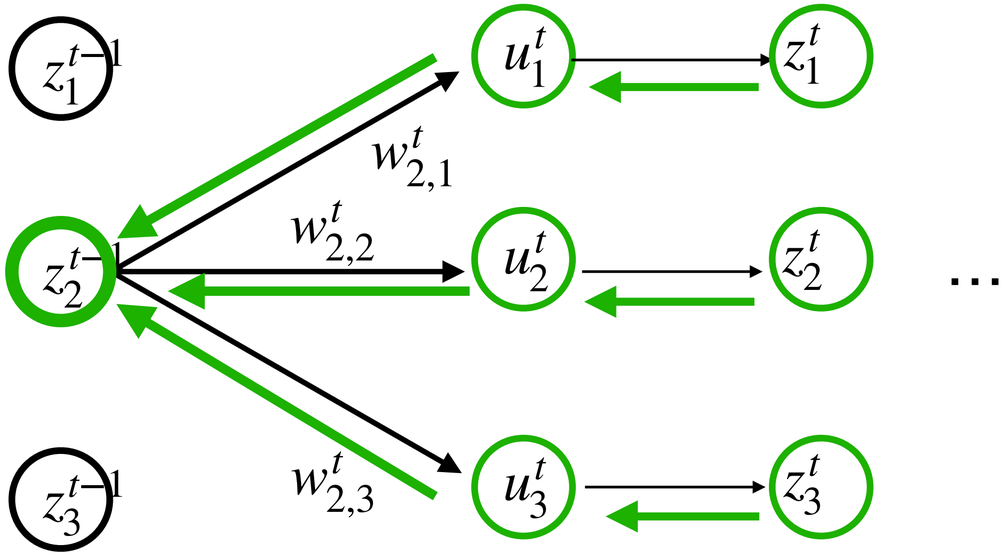
Repeat this for all $z_i^{t-1}, \forall i$

Summary of backward pass



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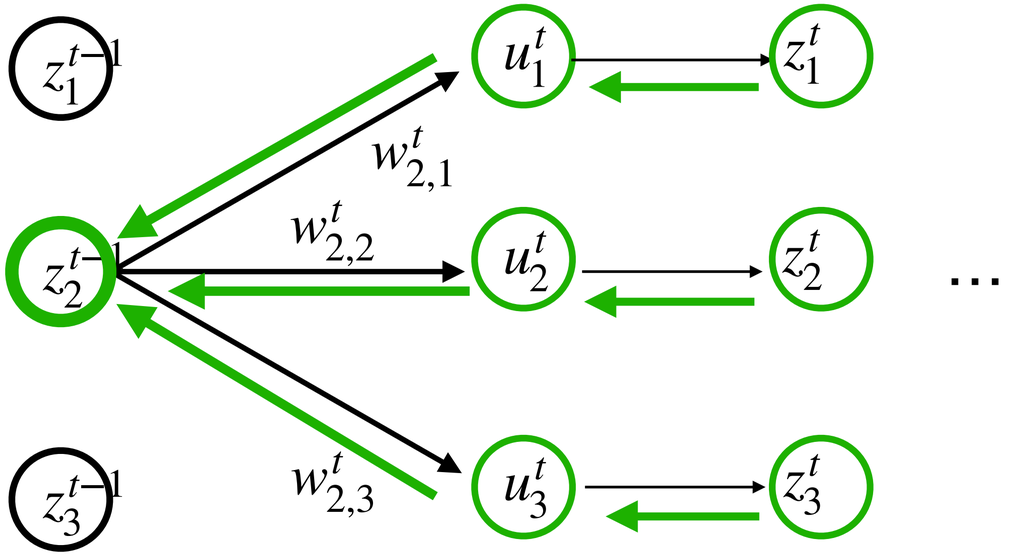

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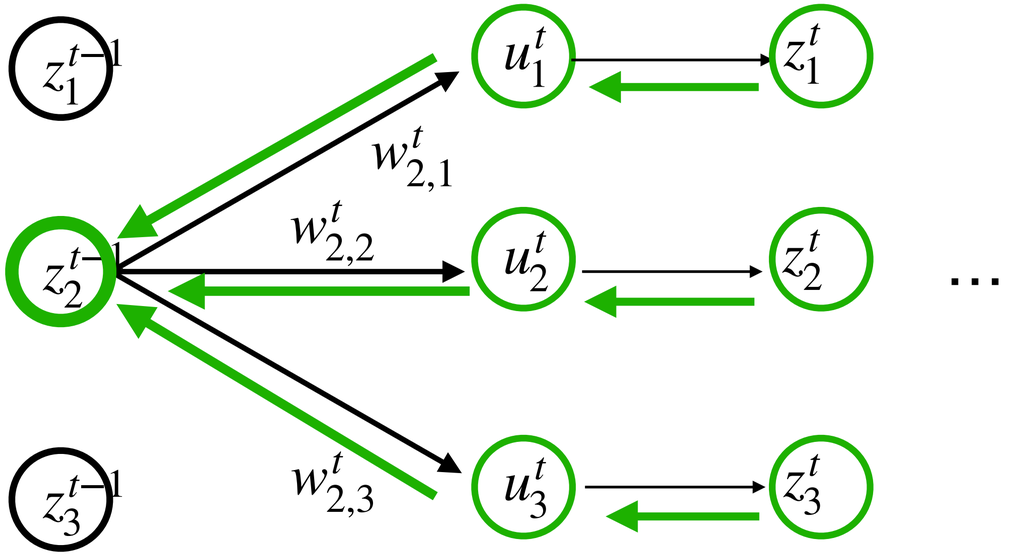

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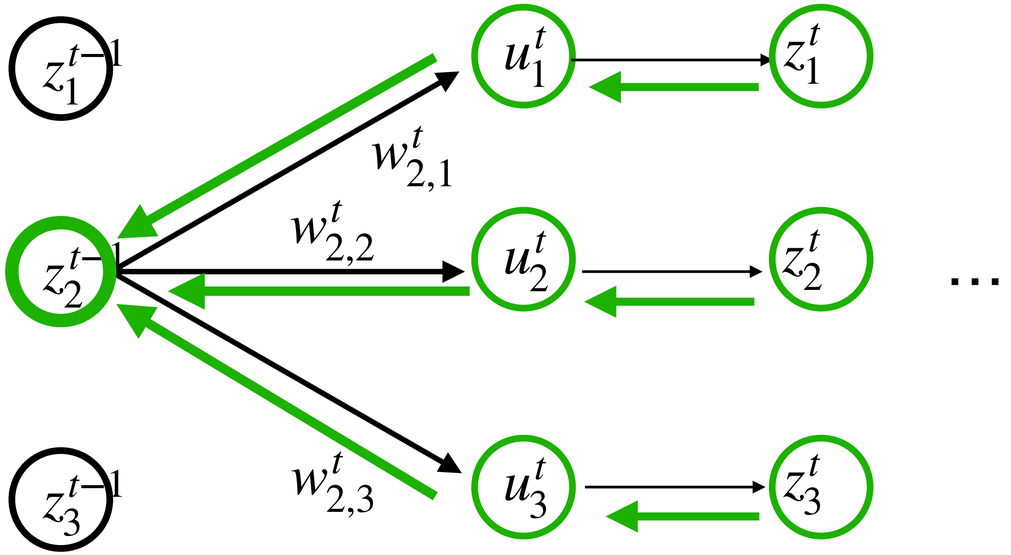

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Exercise: can you express backward pass in matrix-vector format?

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Forward pass: $x = z^0 \rightarrow u^1 \rightarrow z^1 \rightarrow \ldots \rightarrow z^t \rightarrow u^{t+1} \rightarrow z^{t+1} \ldots \rightarrow z^T \rightarrow \hat{y}$

Summary for today

1. Naively compute all derivatives wrt edges using chain rule takes $(E+V)^2$ time

2. Backpropagation: forward pass & backward pass takes $O(E+V)$ time

Forward pass: $x = z^0 \rightarrow u^1 \rightarrow z^1 \rightarrow \ldots \rightarrow z^t \rightarrow u^{t+1} \rightarrow z^{t+1} \ldots \rightarrow z^T \rightarrow \hat{y}$

Backward pass: $\frac{\partial \hat{y}}{\partial z^T} \rightarrow \frac{\partial \hat{y}}{\partial z^{T-1}} \rightarrow \ldots \frac{\partial \hat{y}}{\partial z^1}$