# Neural Network

# Announcements

# Recap on Boosting

Boosting iteratively learns a new classifier, and add it to the ensemble

Initialize $H_1 = h_1 \in \mathscr{H}$

For t = 1 …

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$H_t = \sum_{i=1}^{t} \alpha_i h_i$

For t = 1 …

Denote $\hat{\mathbf{y}} = \left[H_t(x_1), H_t(x_2), \ldots, H_t(x_n)\right]^\top \in \mathbb{R}^n$

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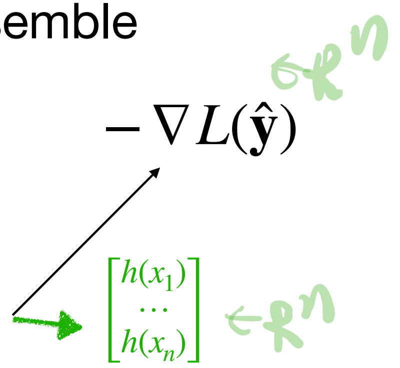

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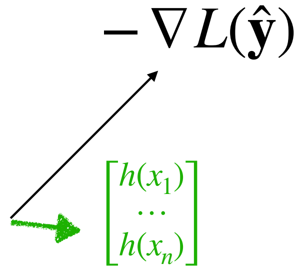


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Solve the optimization problem: $h_{t+1} = \arg\max_{h \in \mathscr{H}} \left\langle \begin{bmatrix} h(x_1) \\ \cdots \\ h(x_n) \end{bmatrix}, -\nabla L(\hat{\mathbf{y}}) \right\rangle$

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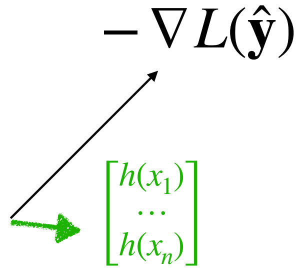


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$$H_{t+1} = H_t + \alpha h_{t+1}$$

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$\sum_i p_i = 1$

$p_i \geq 0$

**2. Add new learner to the ensemble:**

$$H_{t+1} = H_t + \frac{1}{2} \ln \frac{1 - \epsilon}{\epsilon} \cdot h_{t+1}$$

$\epsilon = \sum_{i: h_{t+1}(x_i) \neq y_i} p_i$

# Outline of Today

1. Analysis of Boosting

2. Multilayer feedforward Neural Network

3. Training a neural network

# The definition of Weak learning

Each weaker learning optimizes its own data:

$$\widetilde{\mathscr{D}} = \{p_i, x_i, y_i\}, \text{ where } \sum_i p_i = 1, p_i \geq 0, \forall i$$

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Assume that weaker learner's loss $\epsilon := \sum_{i=1}^{n} p_i \mathbf{1}\{h_{t+1}(x_i) \neq y_i\} \leq \frac{1}{2} - \gamma, \;\; \gamma > 0$

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Q: assume $\mathscr{H}$ is symmetric, i.e., $h \in \mathscr{H}$ iff $-h \in \mathscr{H}$, why does the above always hold?

# Weaker learnability implies approximating gradient well

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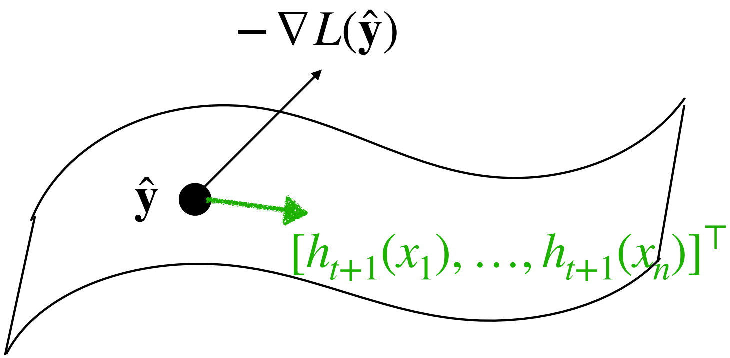

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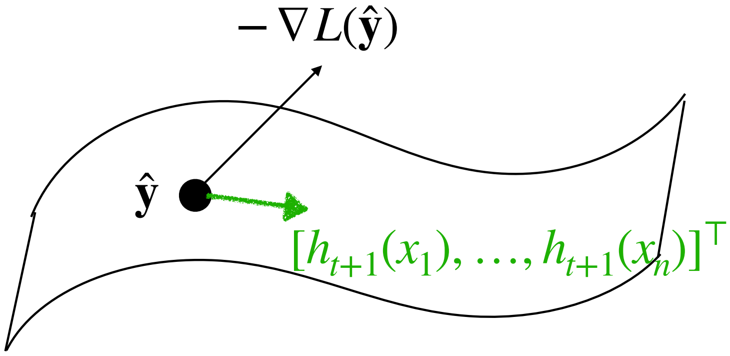

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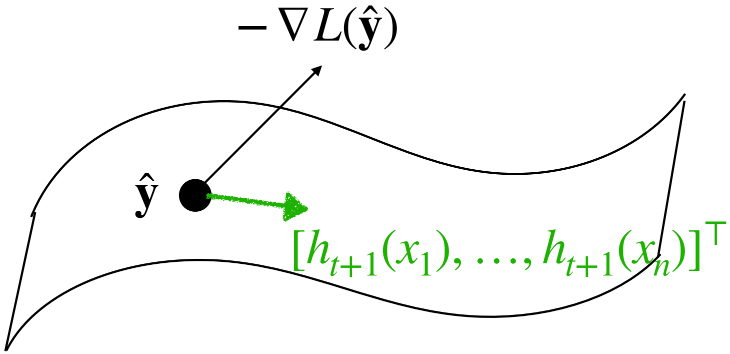

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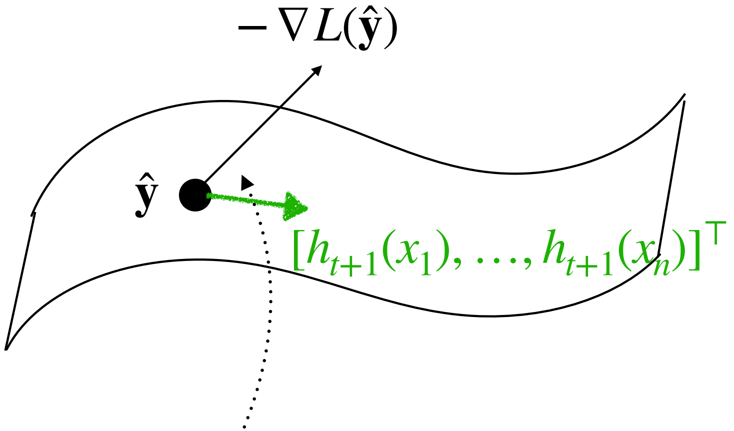


Within 90 degree, so improve the objective!

# Formal Convergence of AdaBoost

Then after T iterations, for the original exp loss, we have

$$\frac{1}{n}\sum_{i=1}^{n}\exp(-H_T(x_i)\cdot y_i) \leq n(1-4\gamma^2)^{T/2}$$

(Proof in lecture note, optional)

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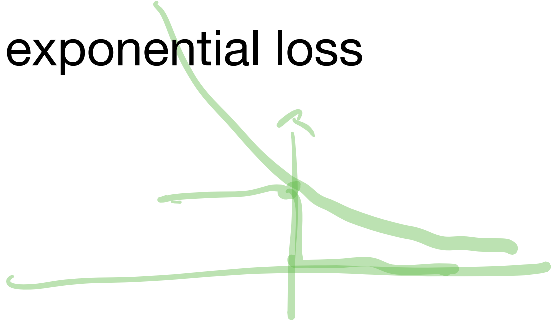

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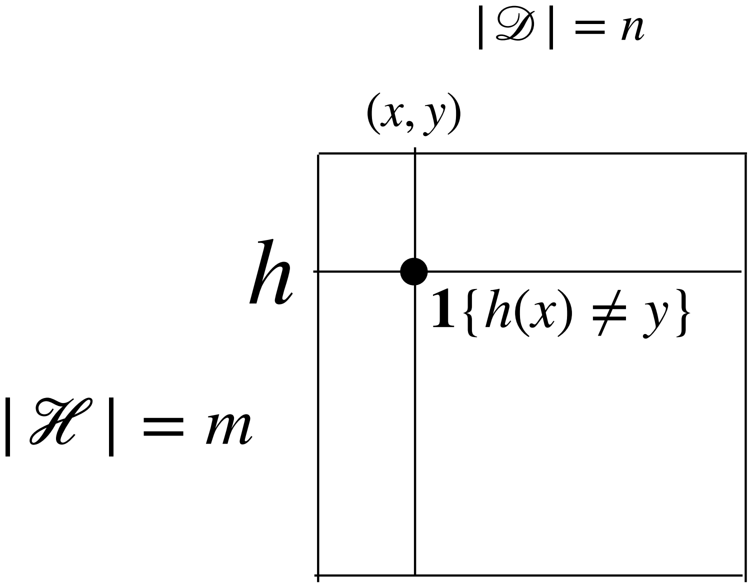
$$\frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{\text{sign}(H_T(x_i)) \neq y_i\} \le \frac{1}{n}\sum_{i=1}^{n} \exp(-H_T(x_i)\cdot y_i) \le n(1-4\gamma^2)^{T/2}$$

Avg number of mistakes

(Proof in lecture note, optional)

# Thinking about Boosting via two player zero sum game



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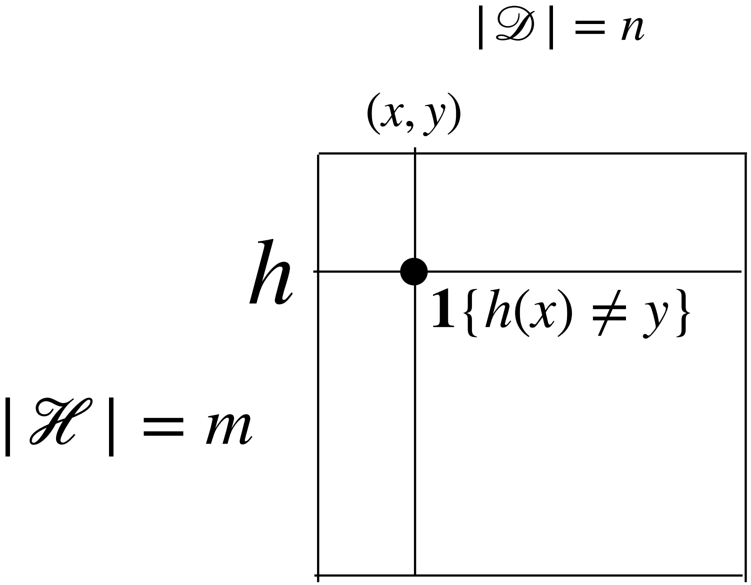


Row player plays hypothesis $h \in \mathcal{H}$

Column player plays example $(x, y)$

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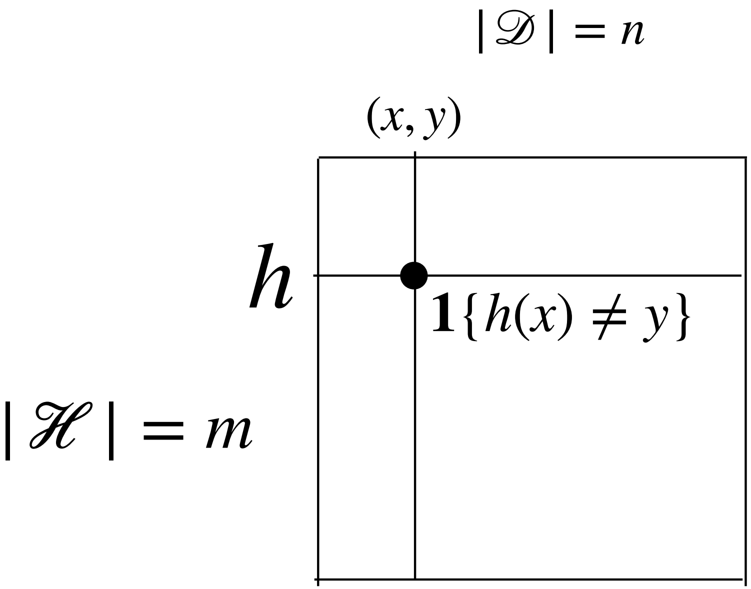


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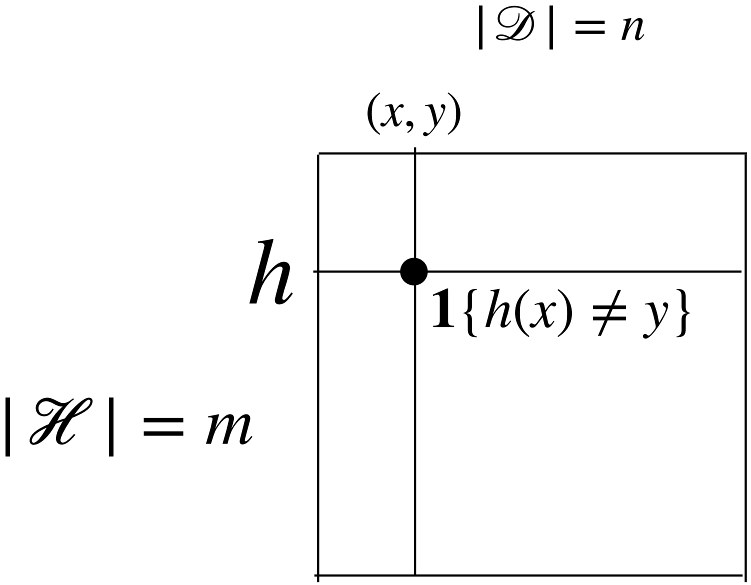


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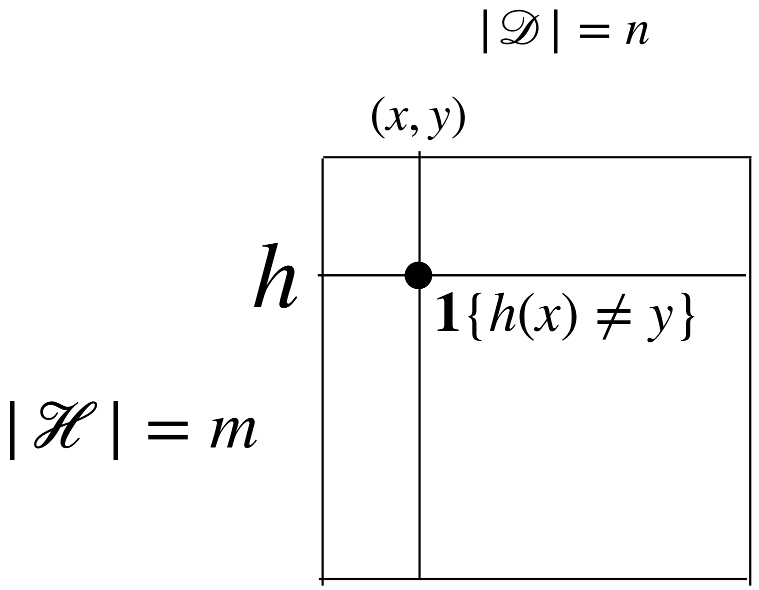


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Boosting can be understood as running some specific algorithm to find the Nash equilibrium of the game

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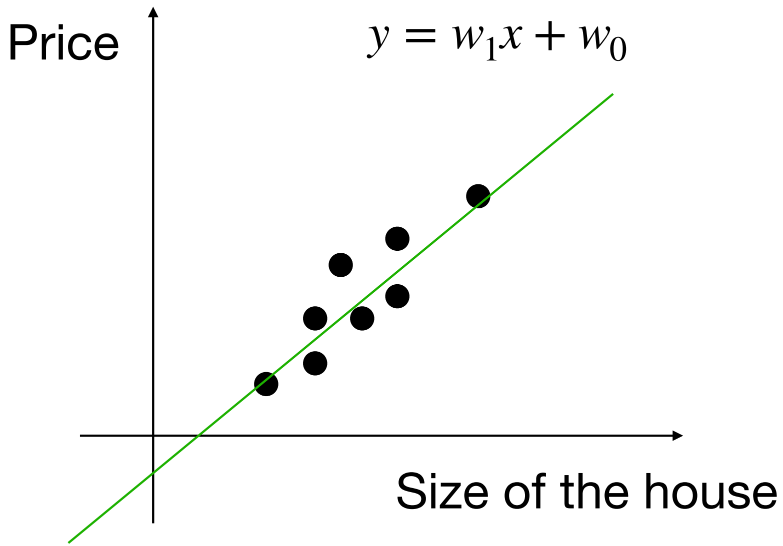

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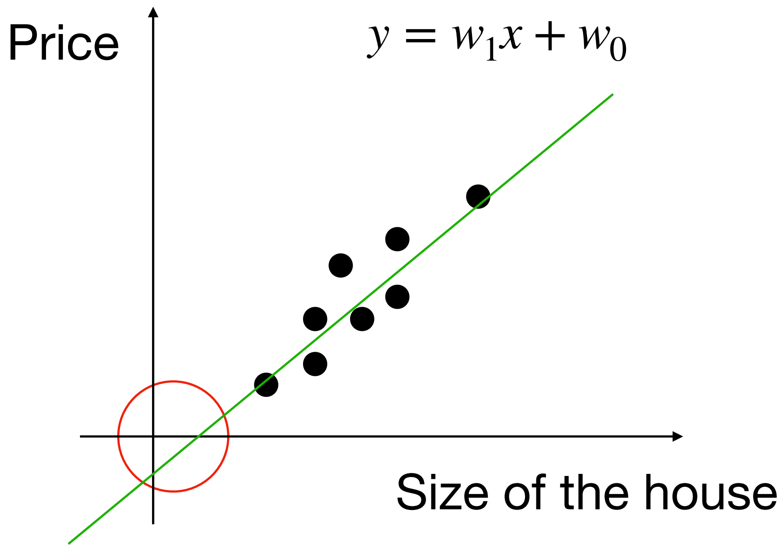


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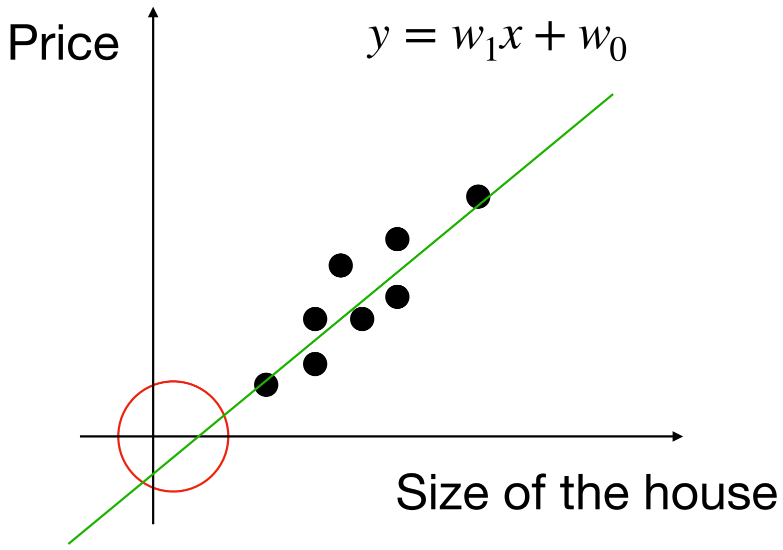


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We can fix this with a simple nonlinear function

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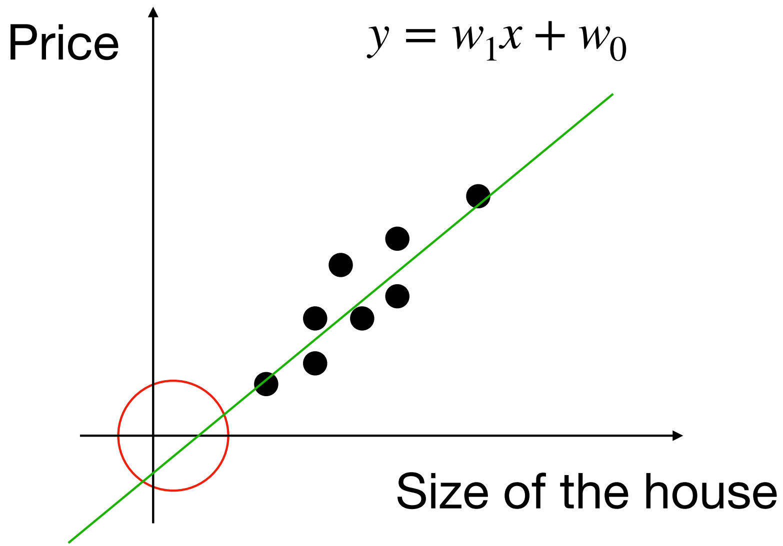


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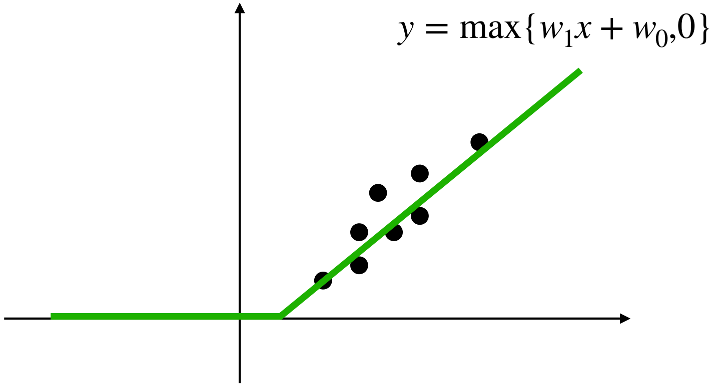

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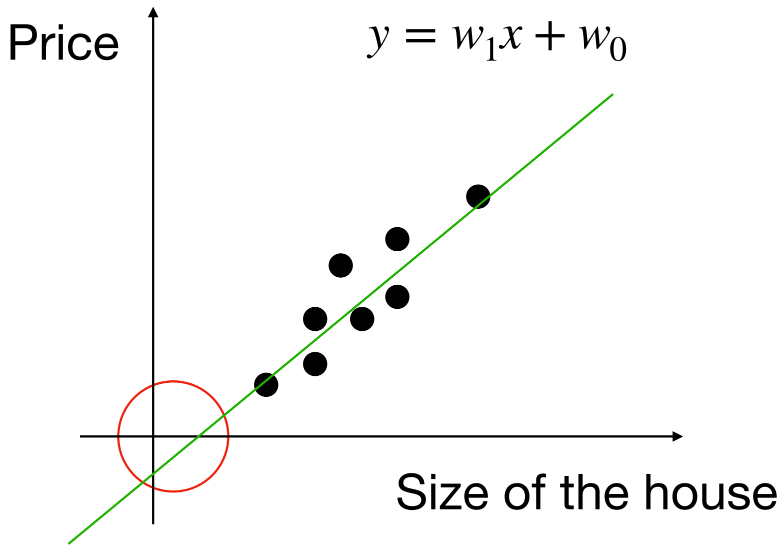


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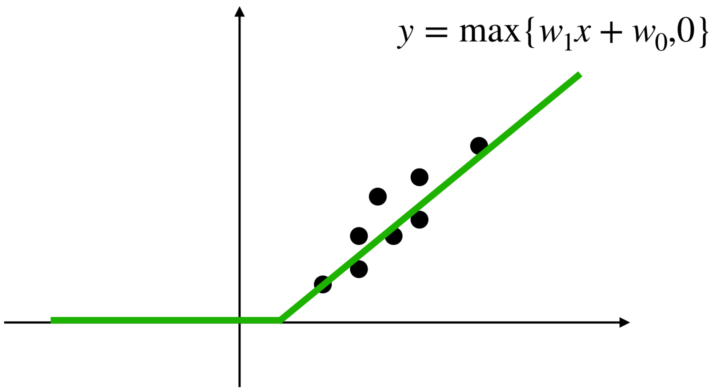
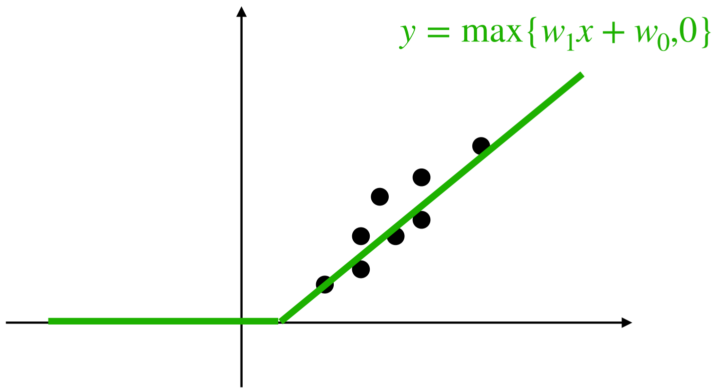



rectified linear unit (ReLU)

# A single neuron network



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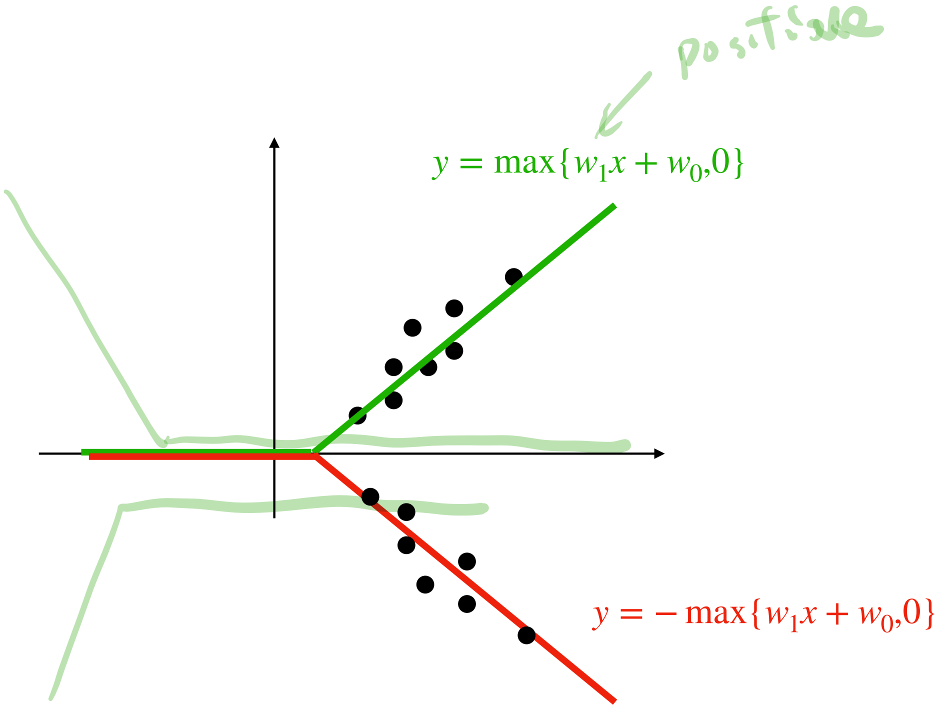

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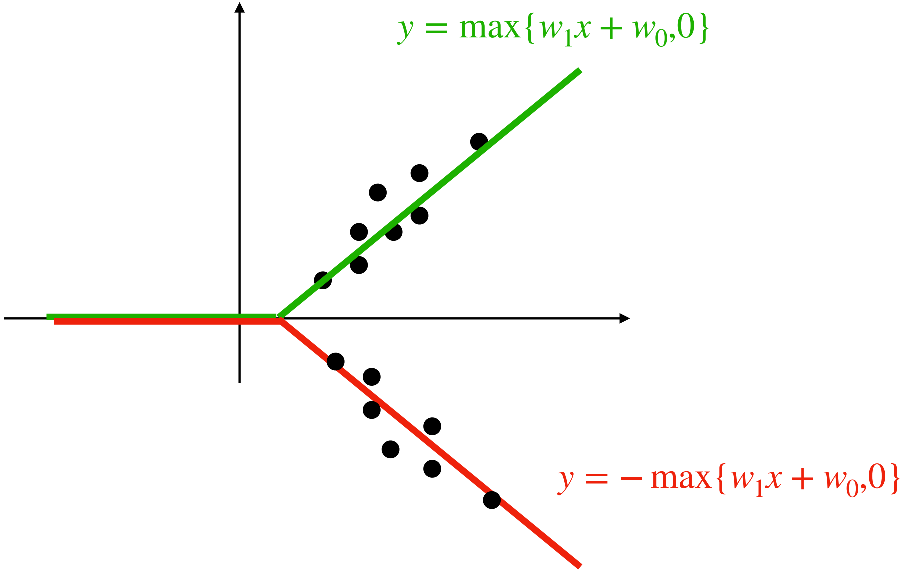


$$y = a \max\{w_1 x + w_0, 0\} + b$$

$(a, w_1, w_0, b)$

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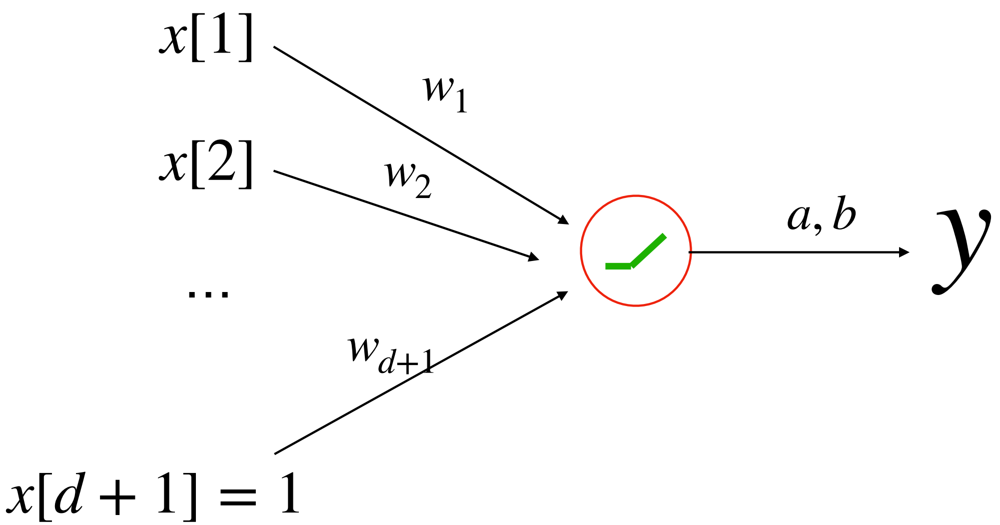

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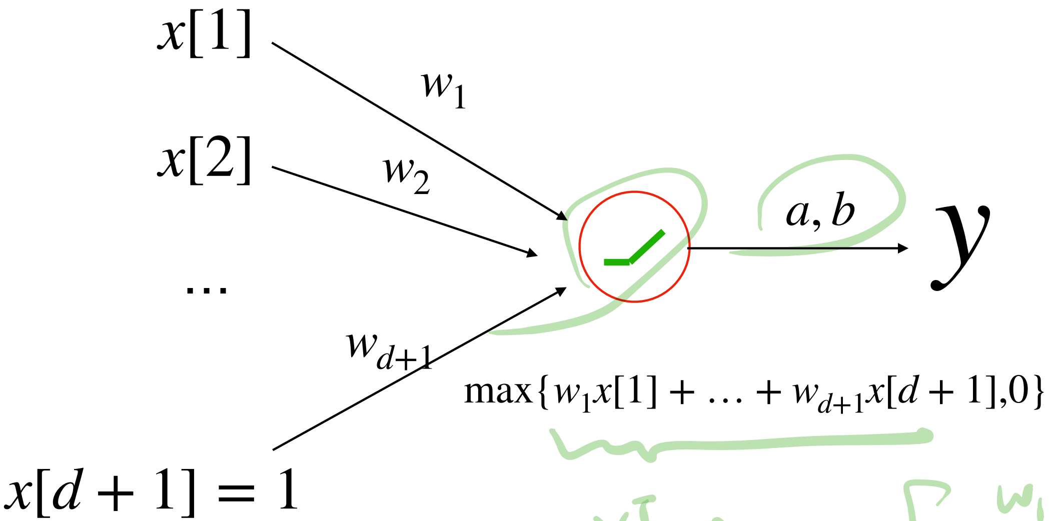


$X^T w$, $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_{d+1} \end{bmatrix} \in \mathbb{R}^{d+1}$

$X \in \mathbb{R}^{d+1}$

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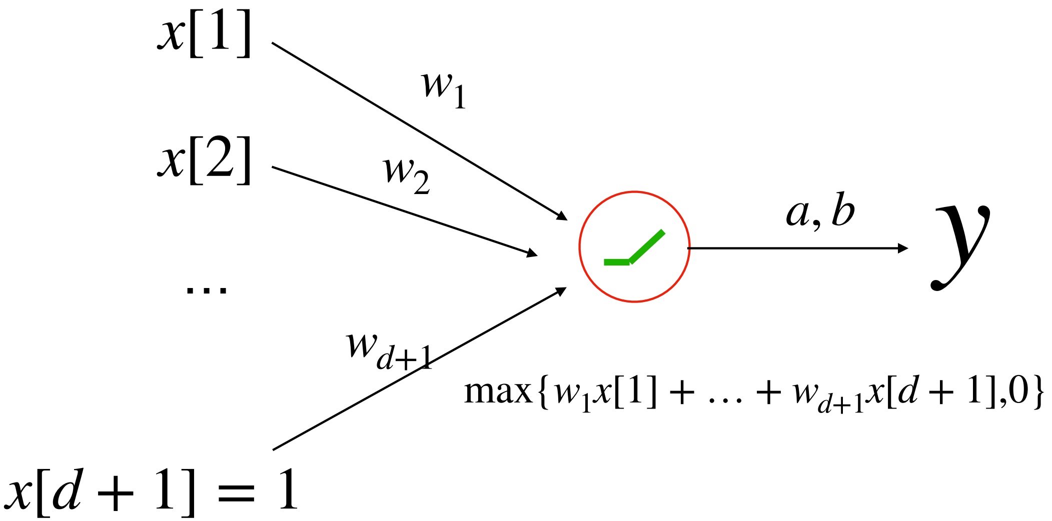
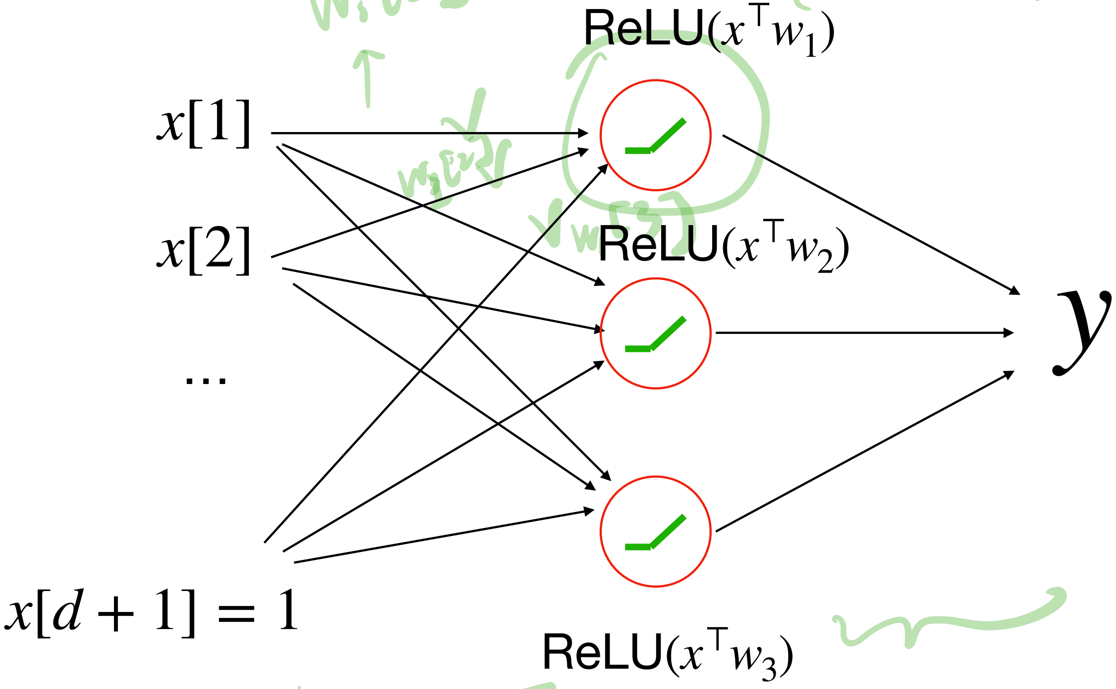



$$y = a\text{ReLU}(w^\top x) + b$$

# Let us stack multiple neurons together




$$y = \sum_{i=1}^{3} a_i \text{ReLU}(x^\top w_i) + b$$

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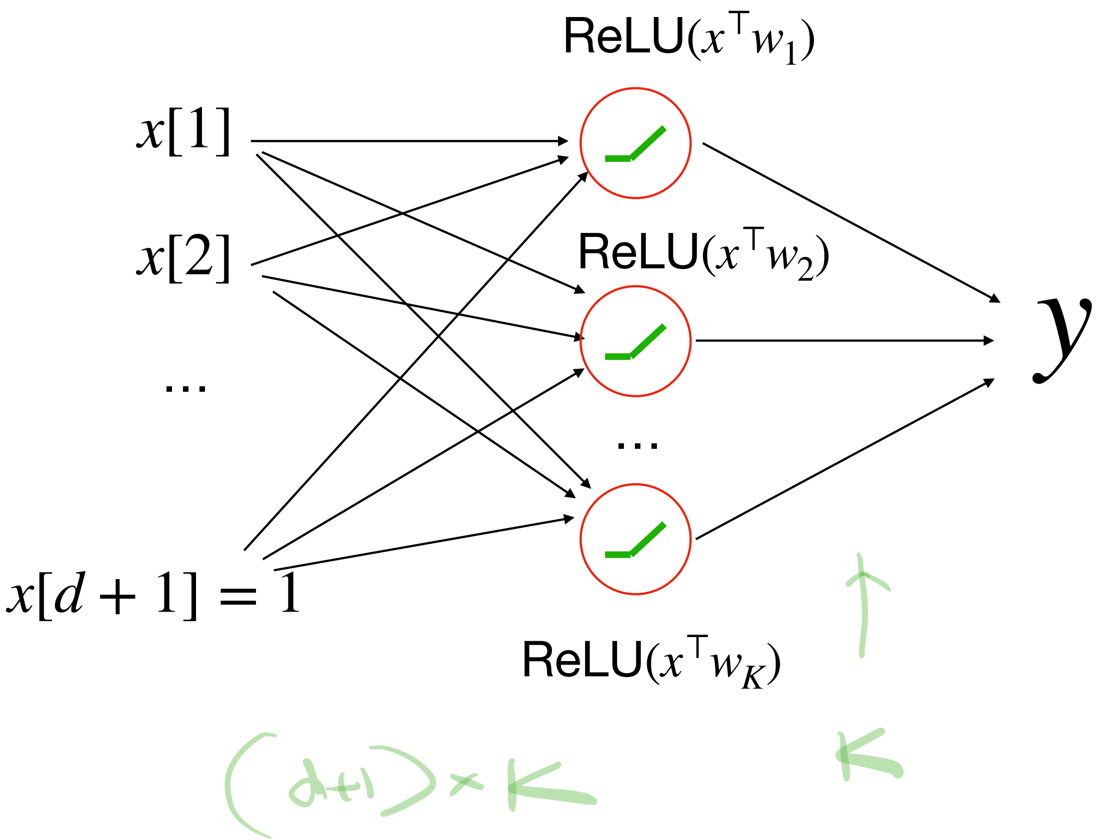

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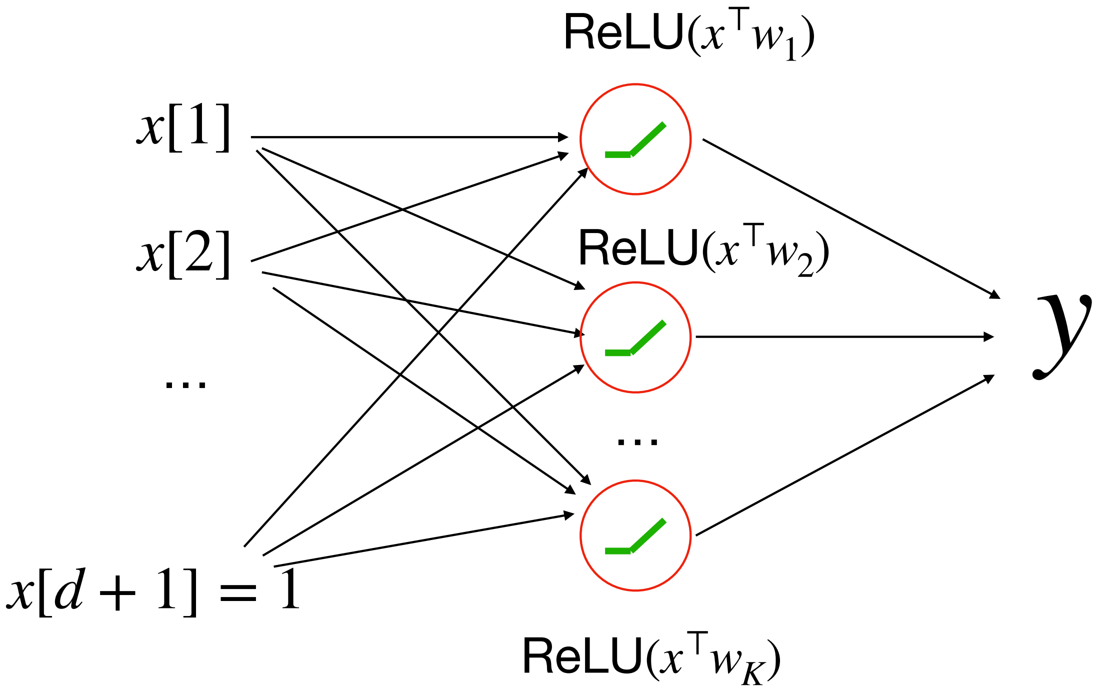


Vectorized form:

Define $W = \begin{bmatrix} (w_1)^\top \\ \cdots \\ (w_K)^\top \end{bmatrix} \in \mathbb{R}^{K \times d}$ (+1)

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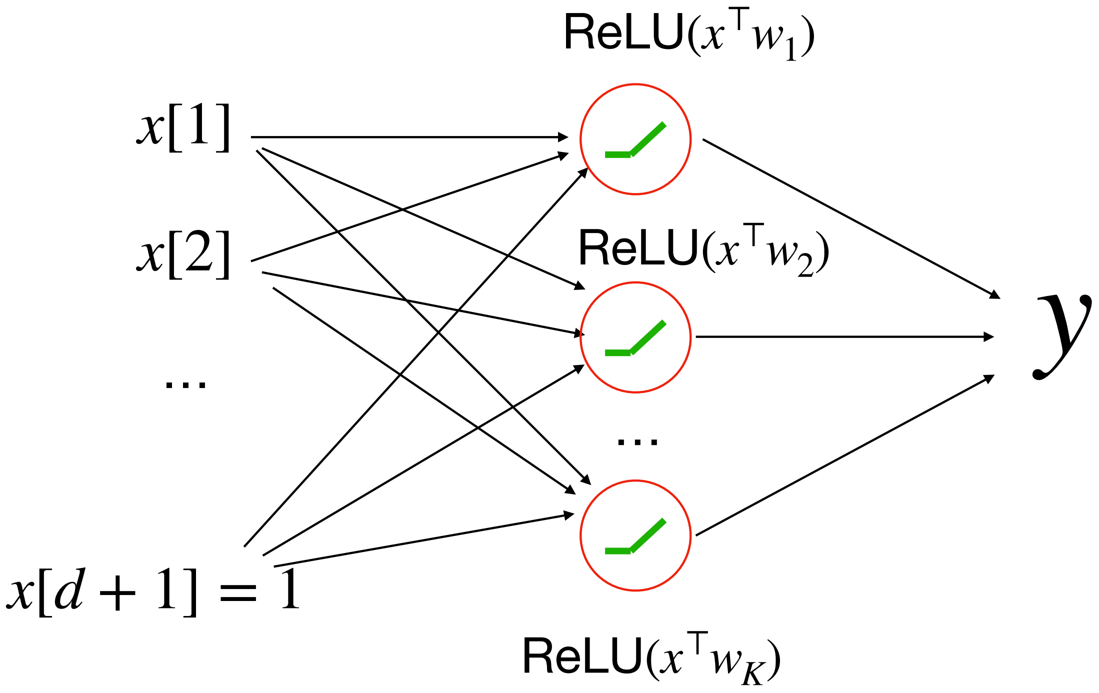


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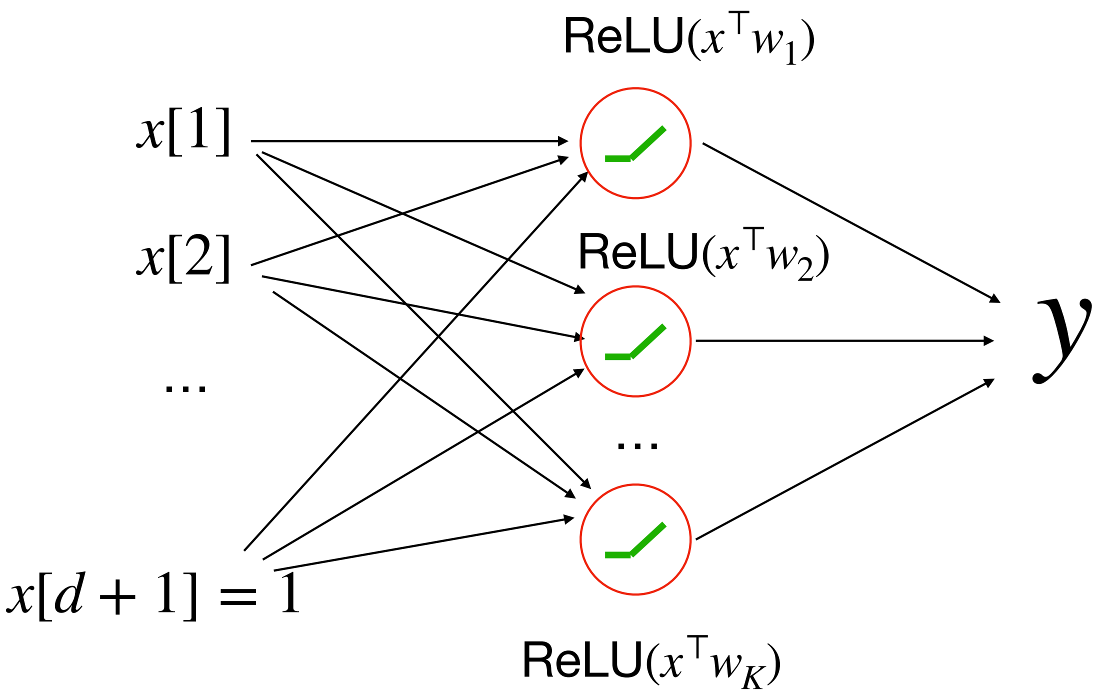


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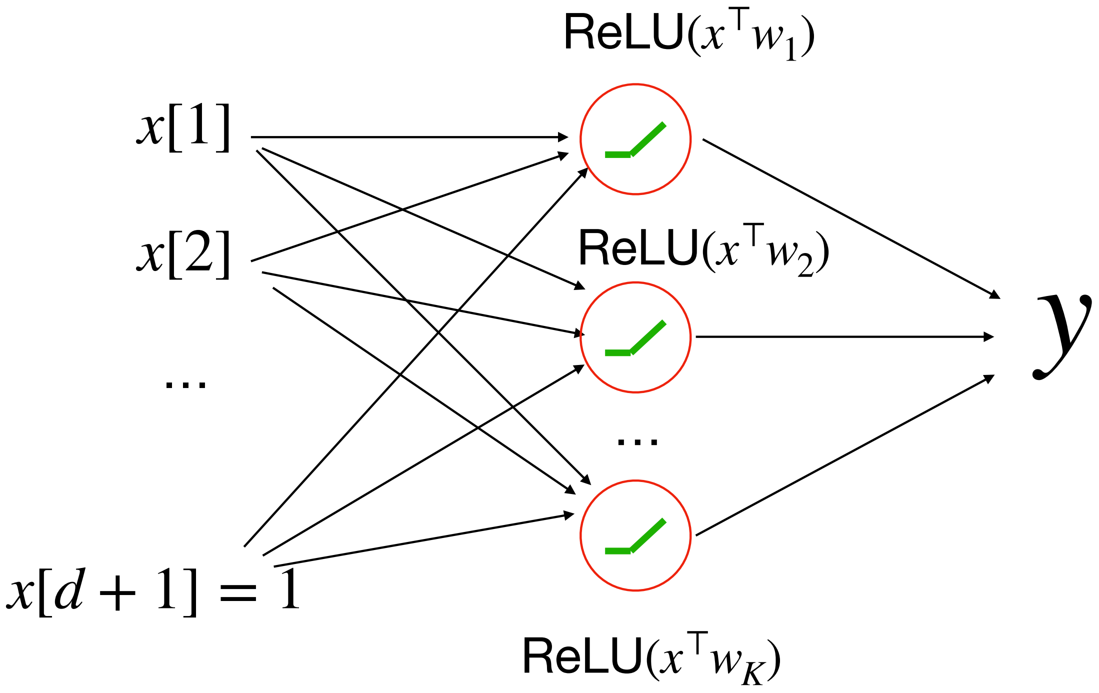


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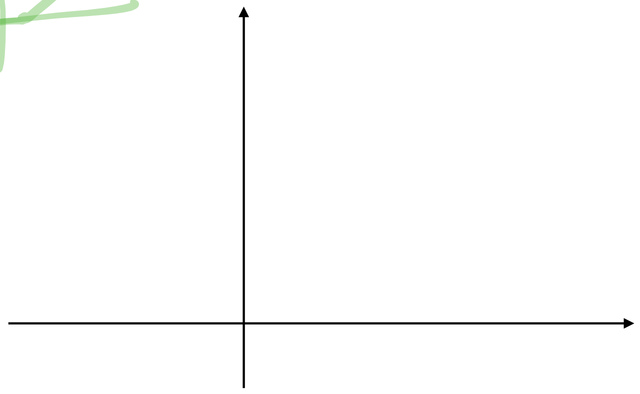
Learnable feature $\phi(x)$

# What does a neural network approximate

$$y = \alpha^\top \left(\text{ReLU}(Wx)\right) + b$$

$$= \sum_{i=1}^{K} a_i \, \text{Relu}\left(w_i^T x\right) + b$$



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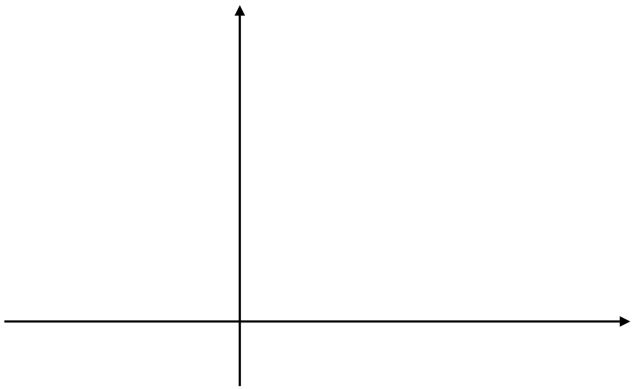
It's a pieces wise linear functions



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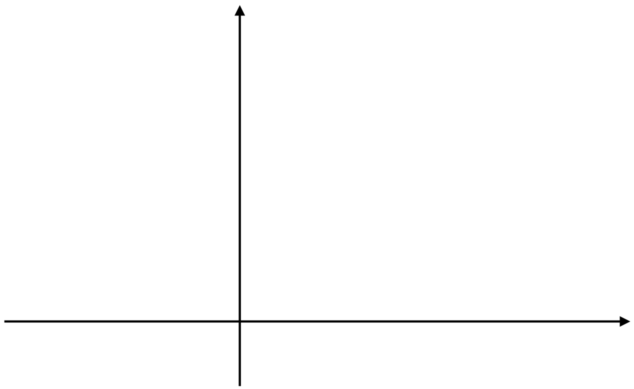

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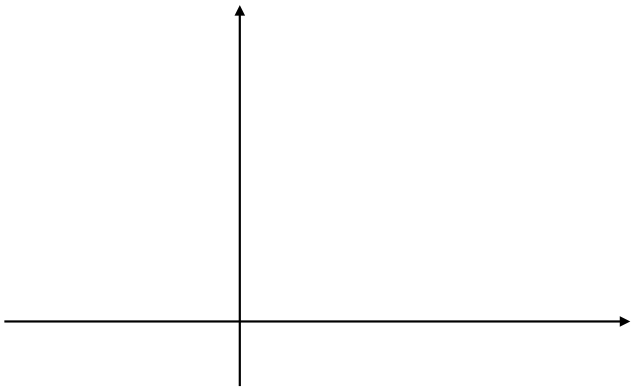

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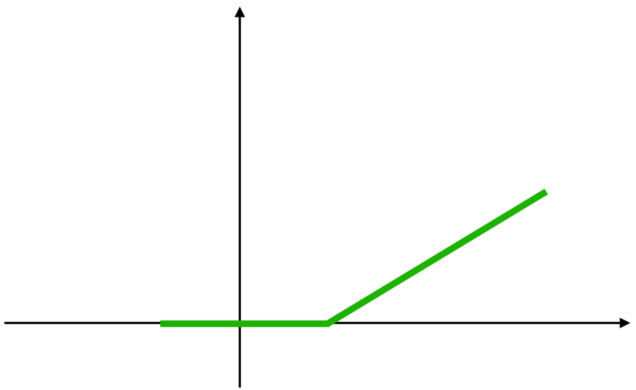

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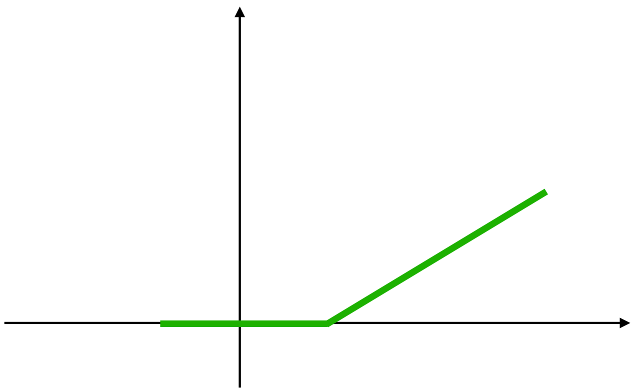

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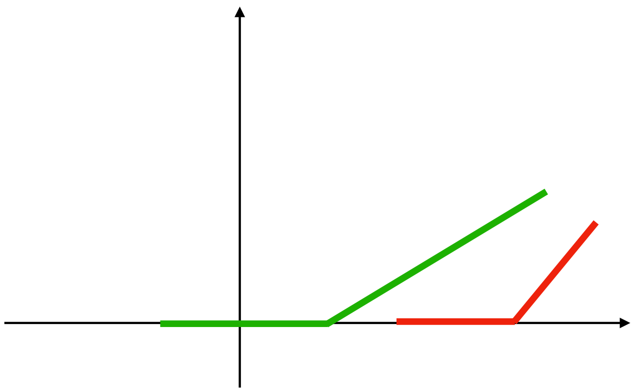

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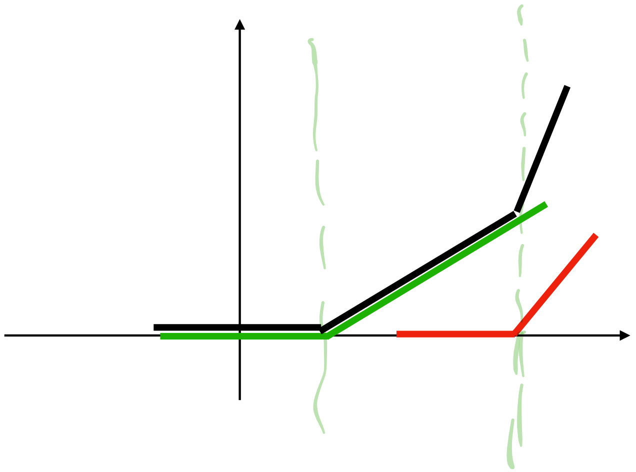

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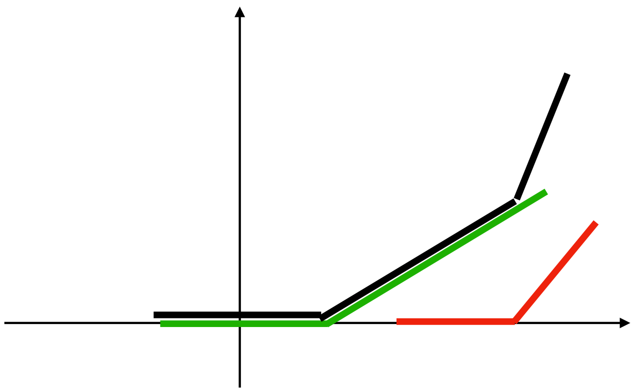

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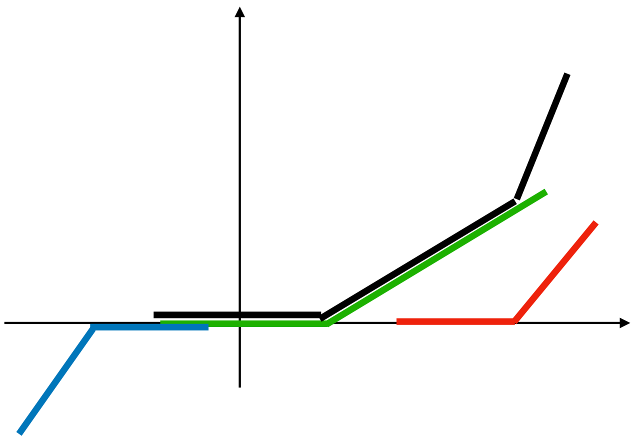

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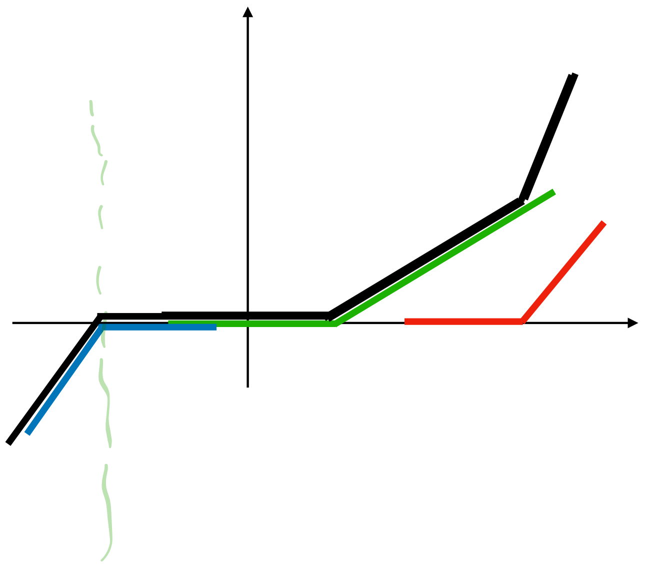

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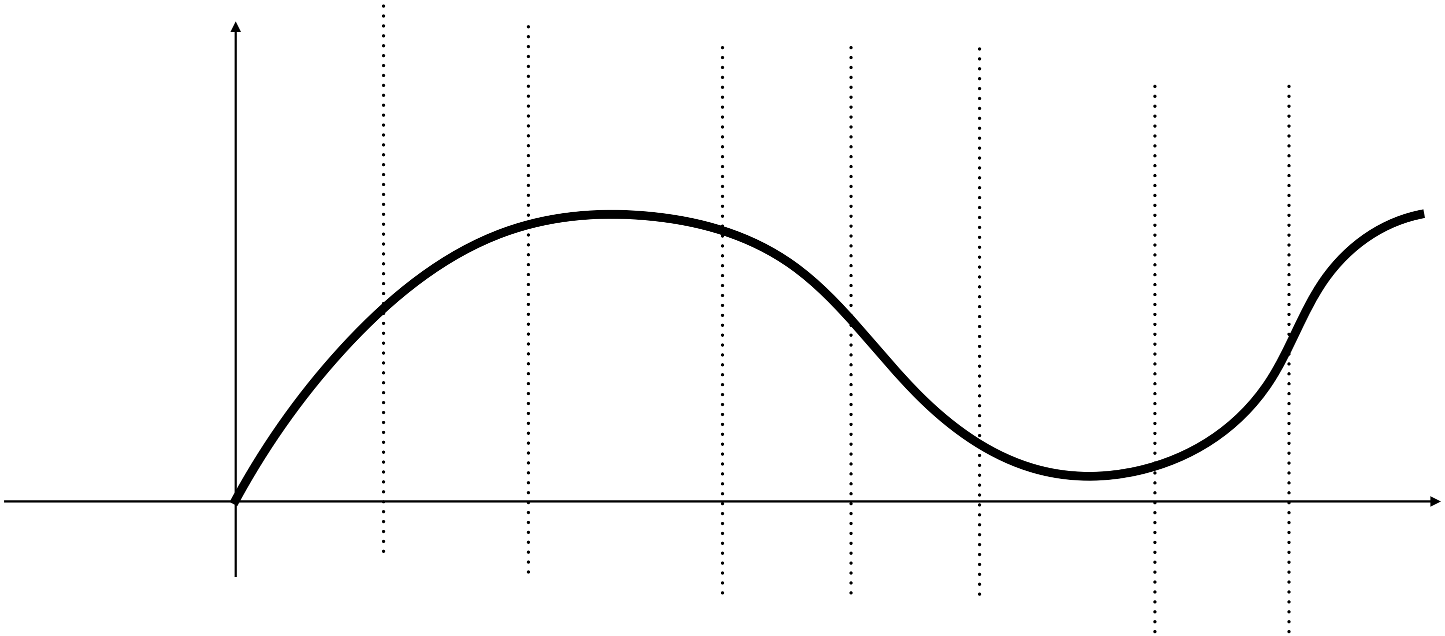

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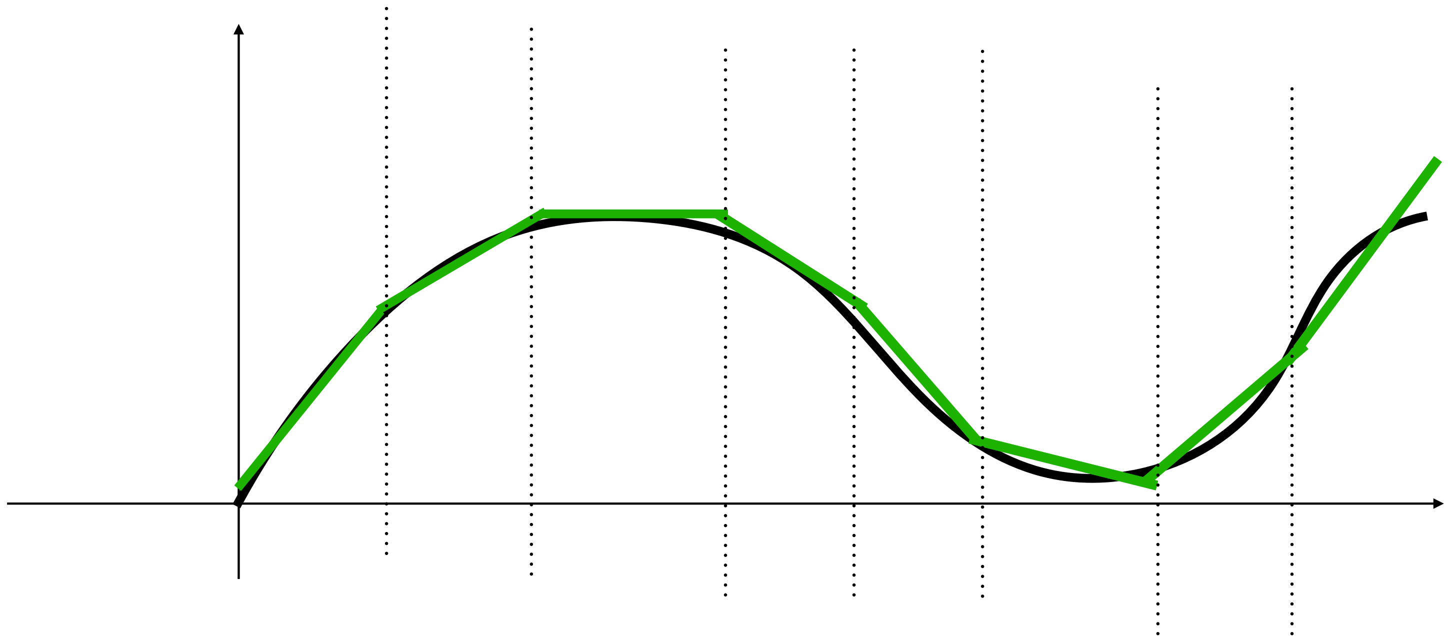

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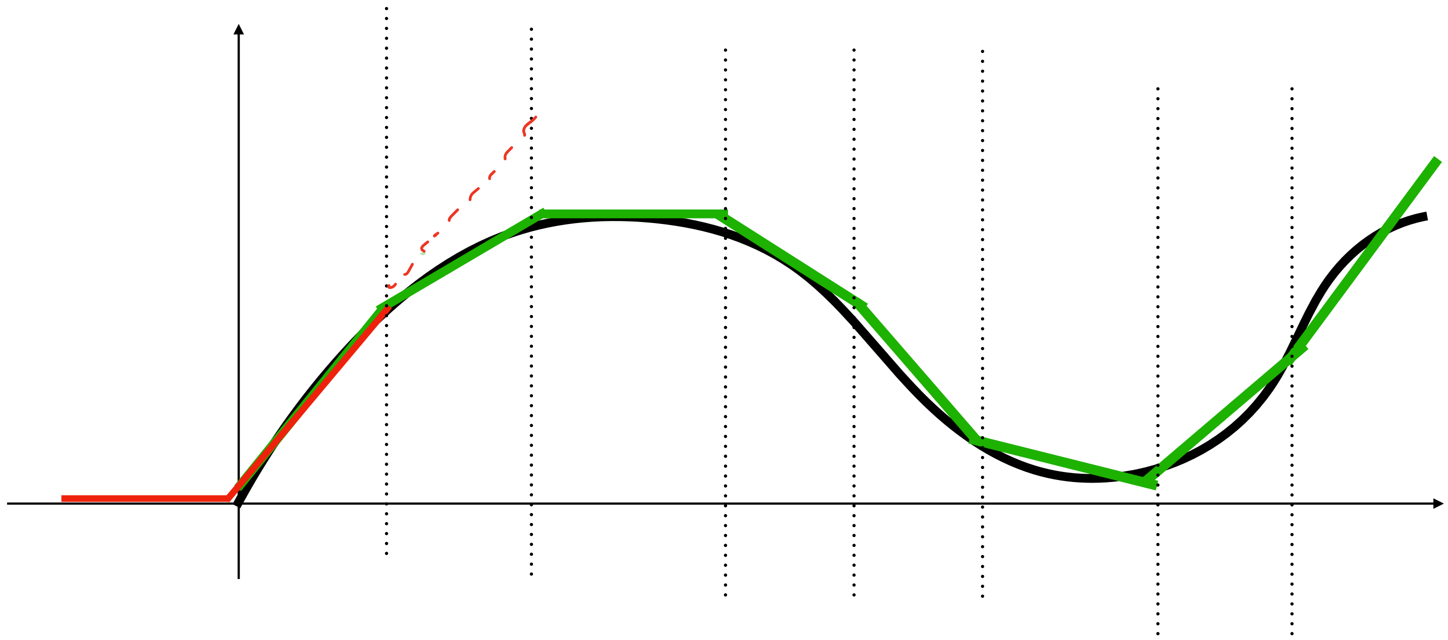

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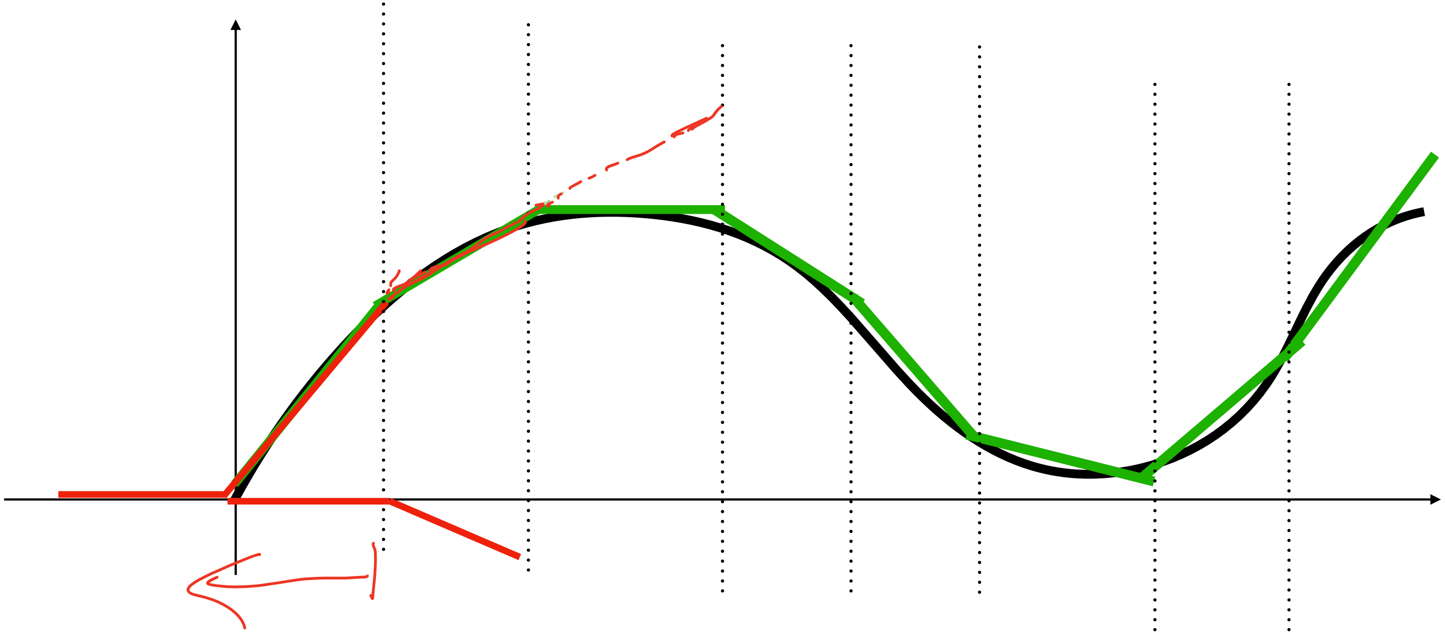

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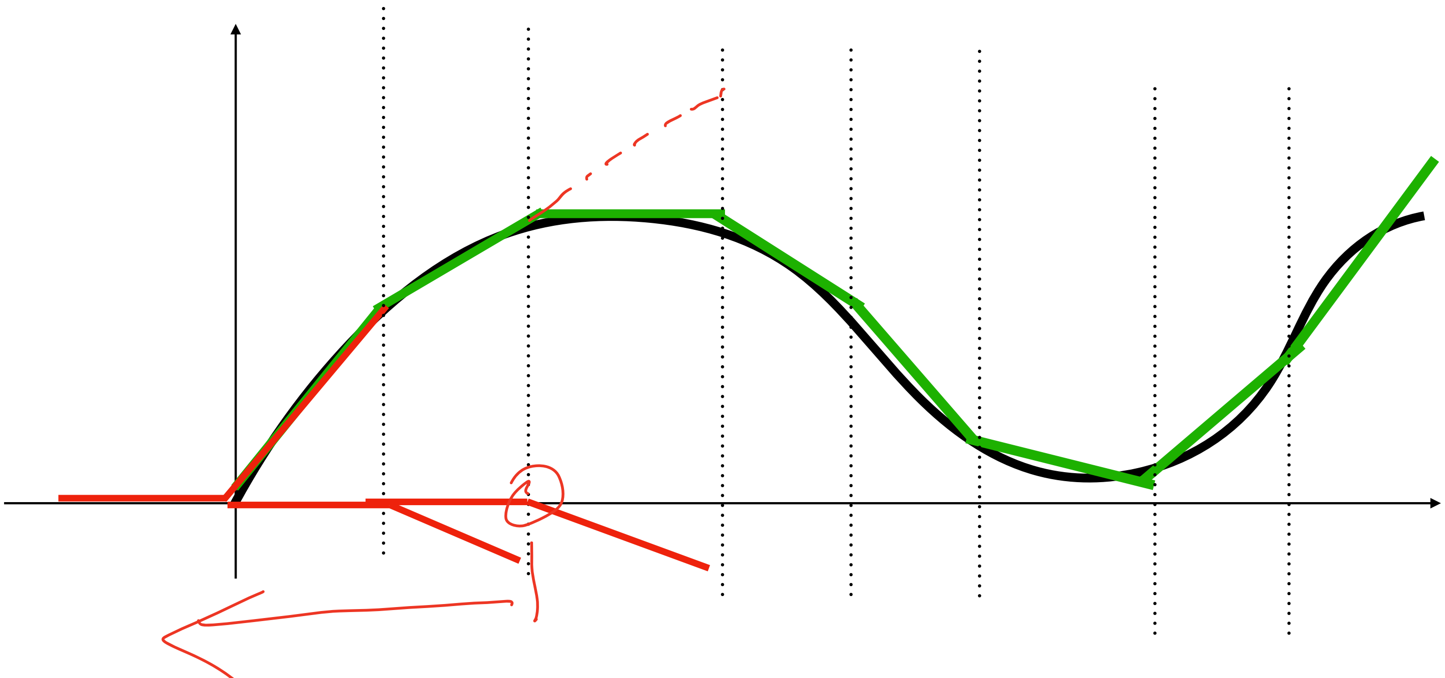

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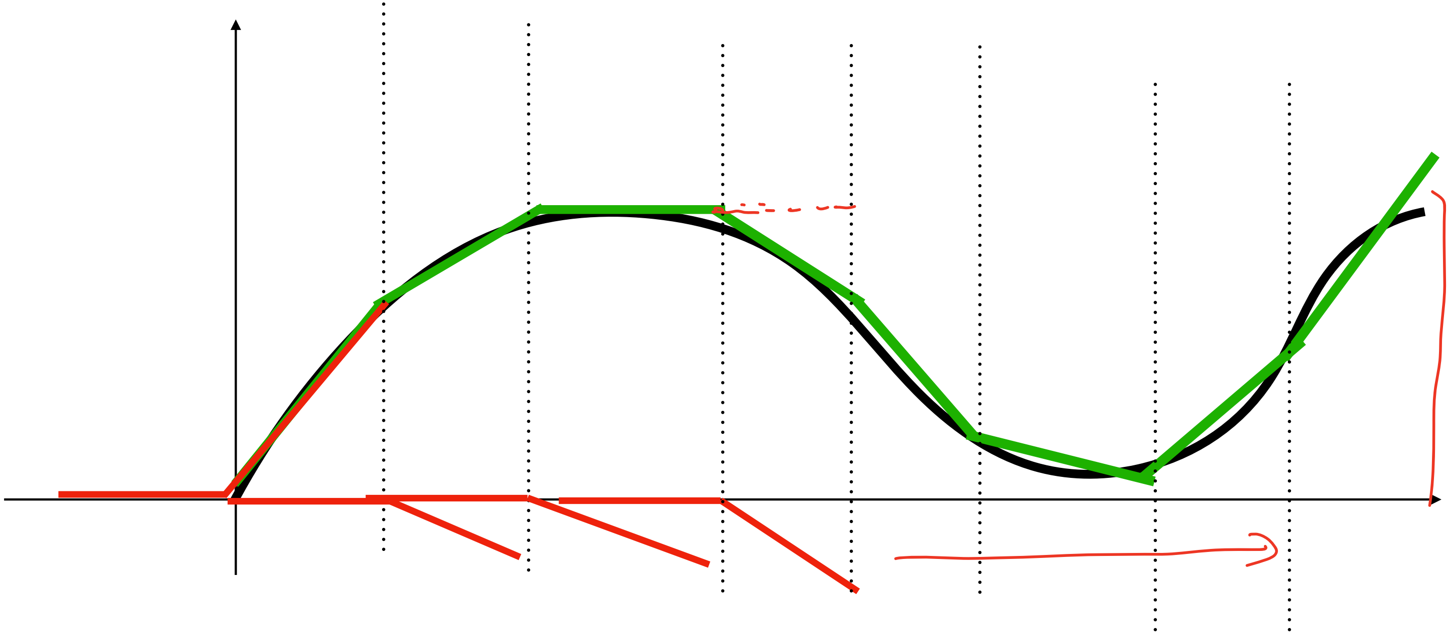
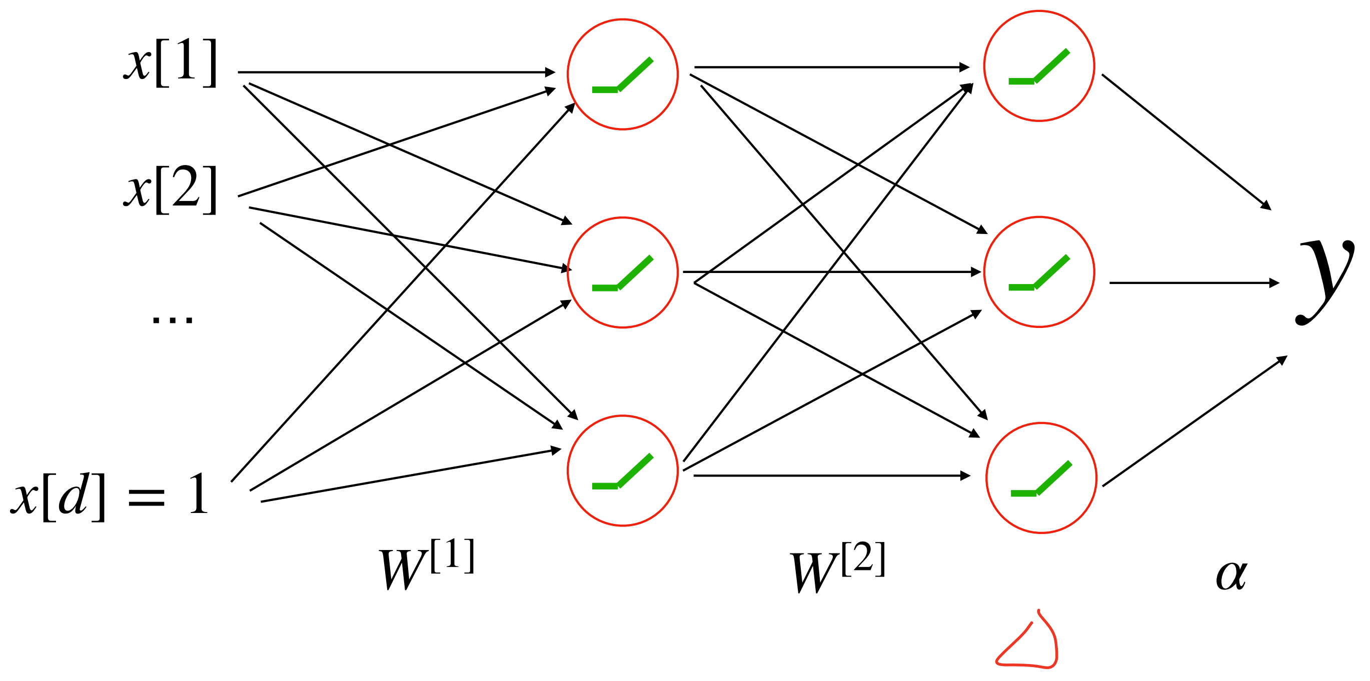


# A multi-layer fully connected neural network



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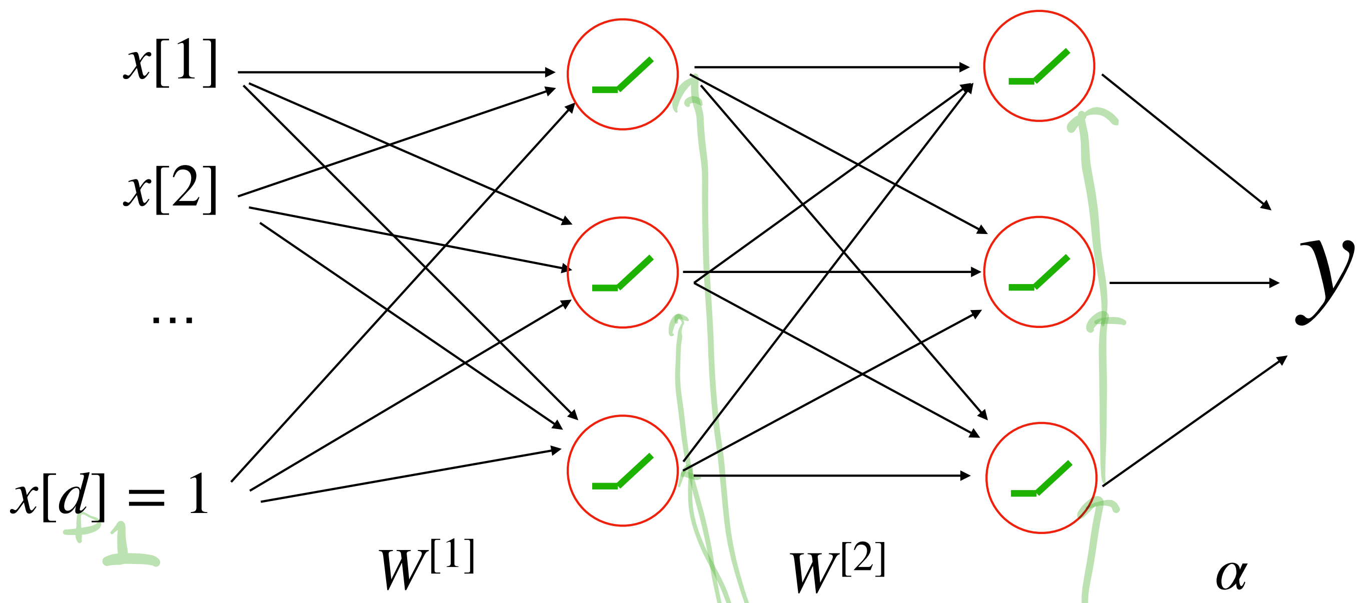


$$y = \alpha^{\top}\mathrm{ReLU}\left(W^{[2]}\mathrm{ReLU}\left(W^{[1]}x\right)\right) + b$$

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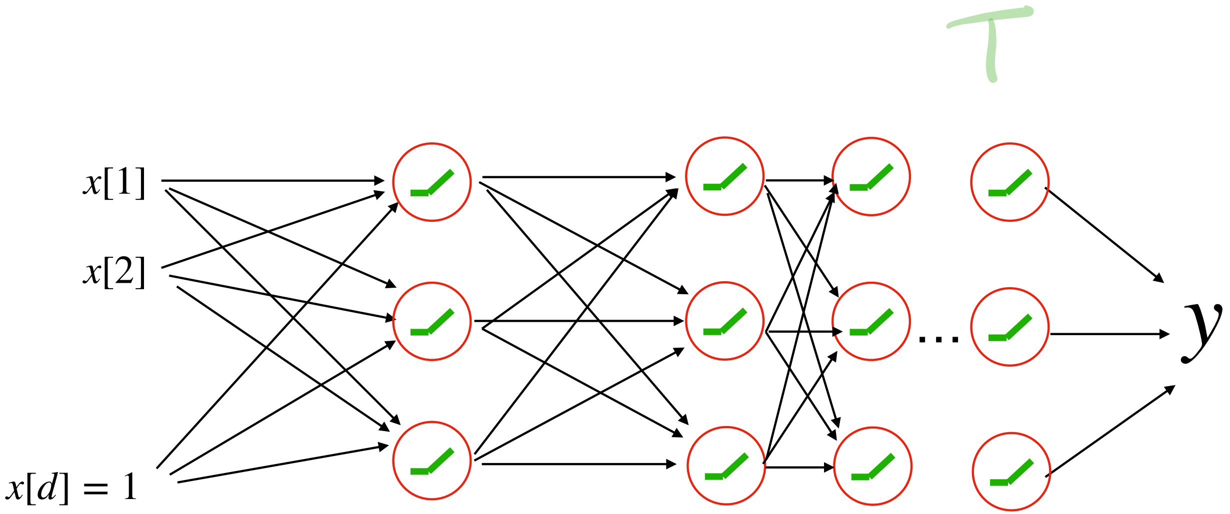

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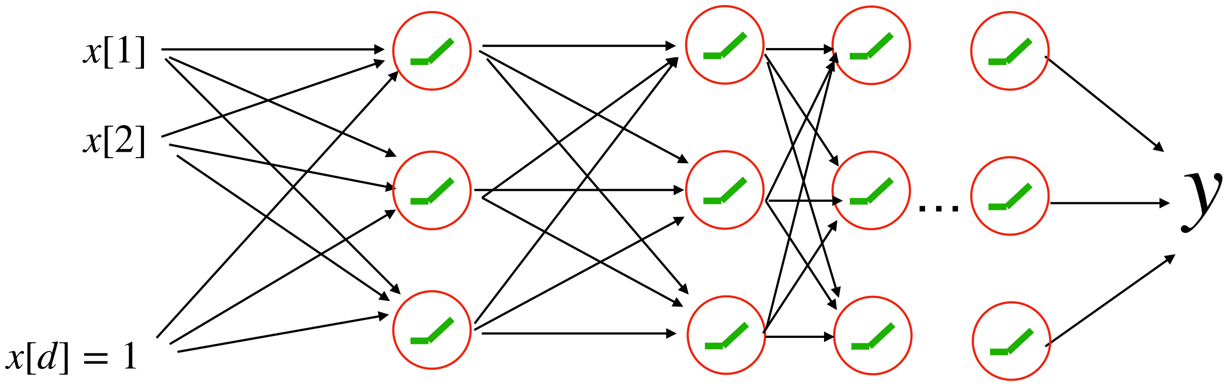


Define it by a forward pass:

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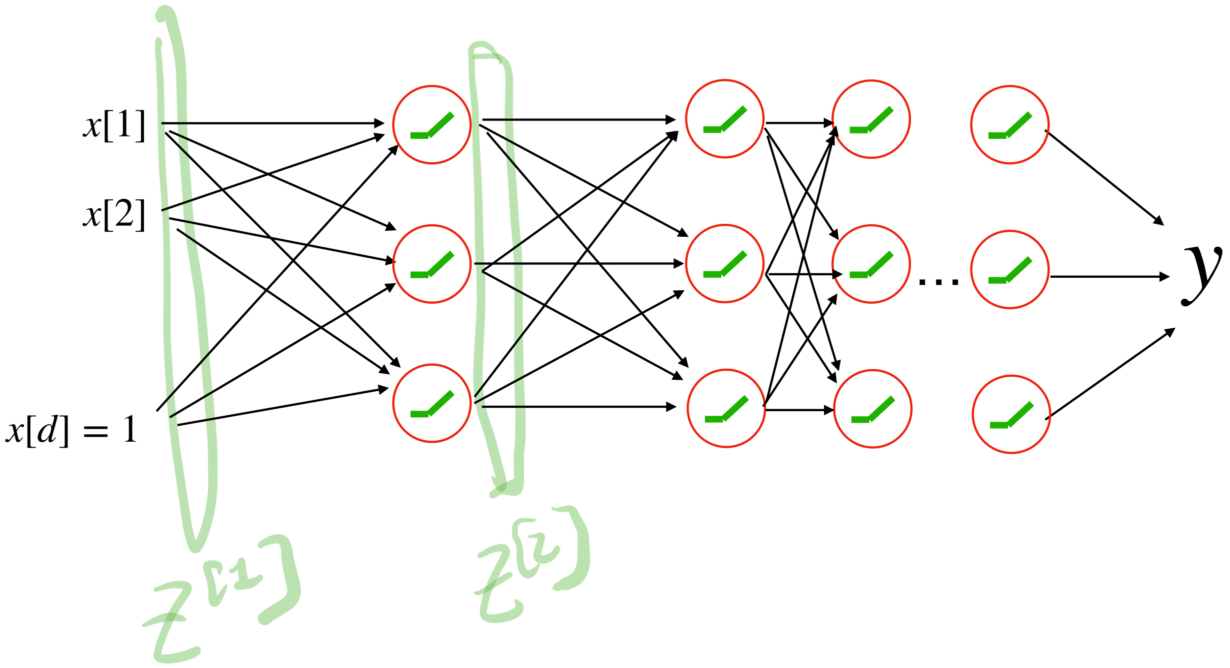


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$$z^{[1]} = x$$

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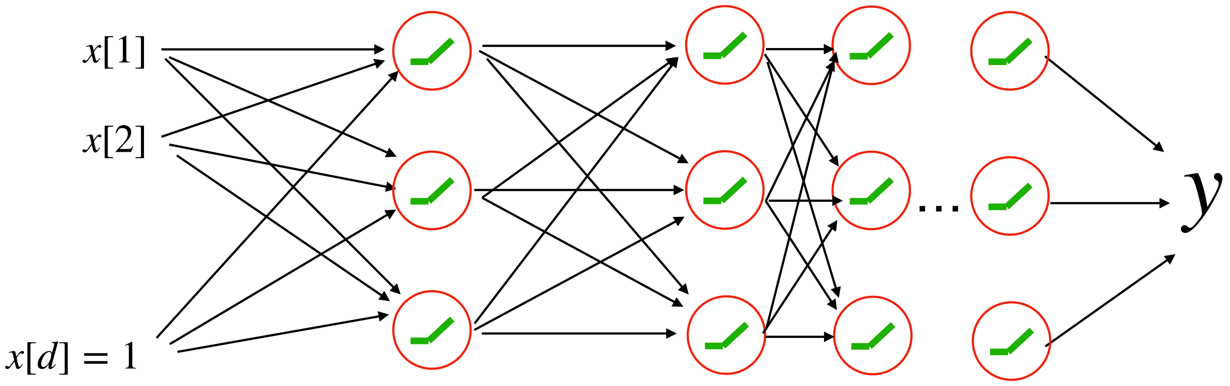


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For t = 1 to T-1:

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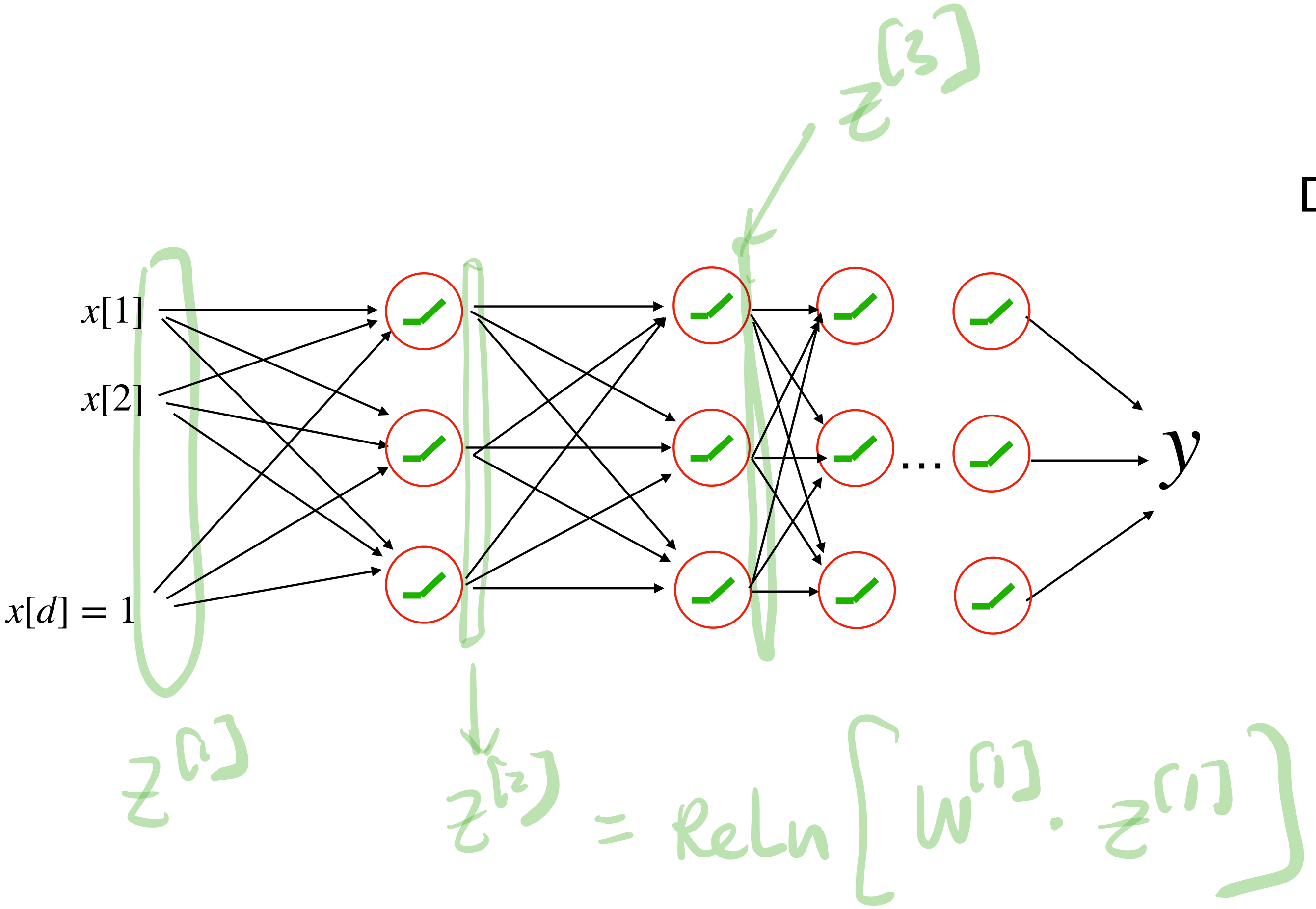


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$$z^{[t+1]} = \text{ReLU}\left(W^{[t]} z^{t}\right)$$

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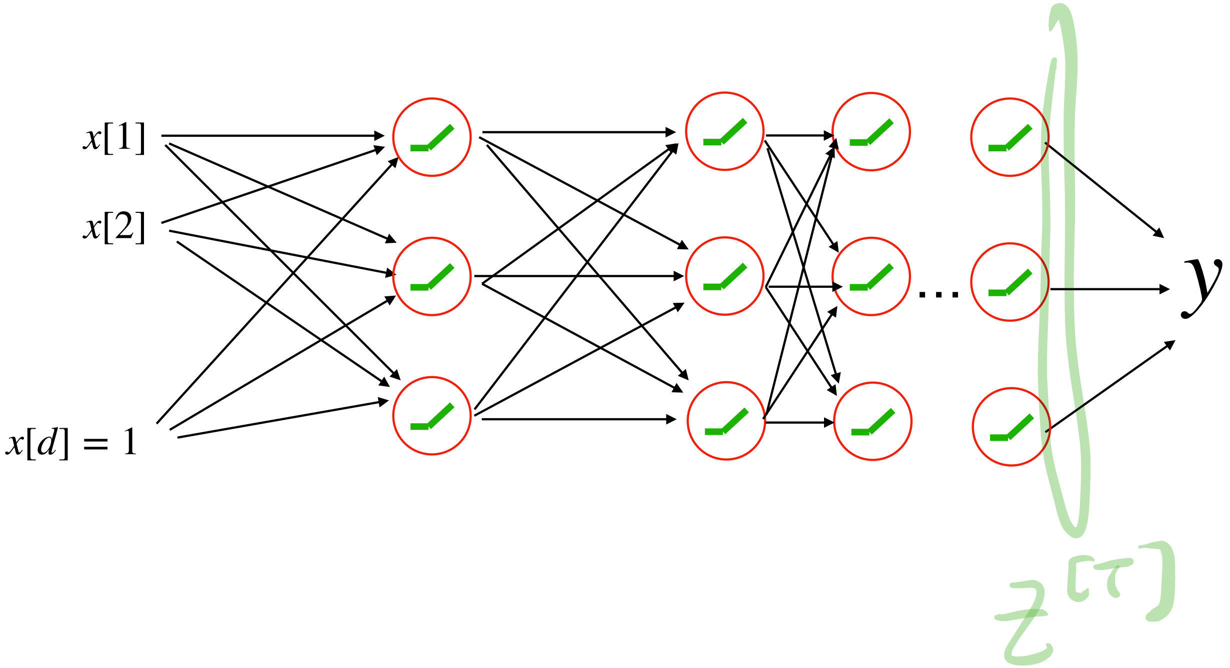


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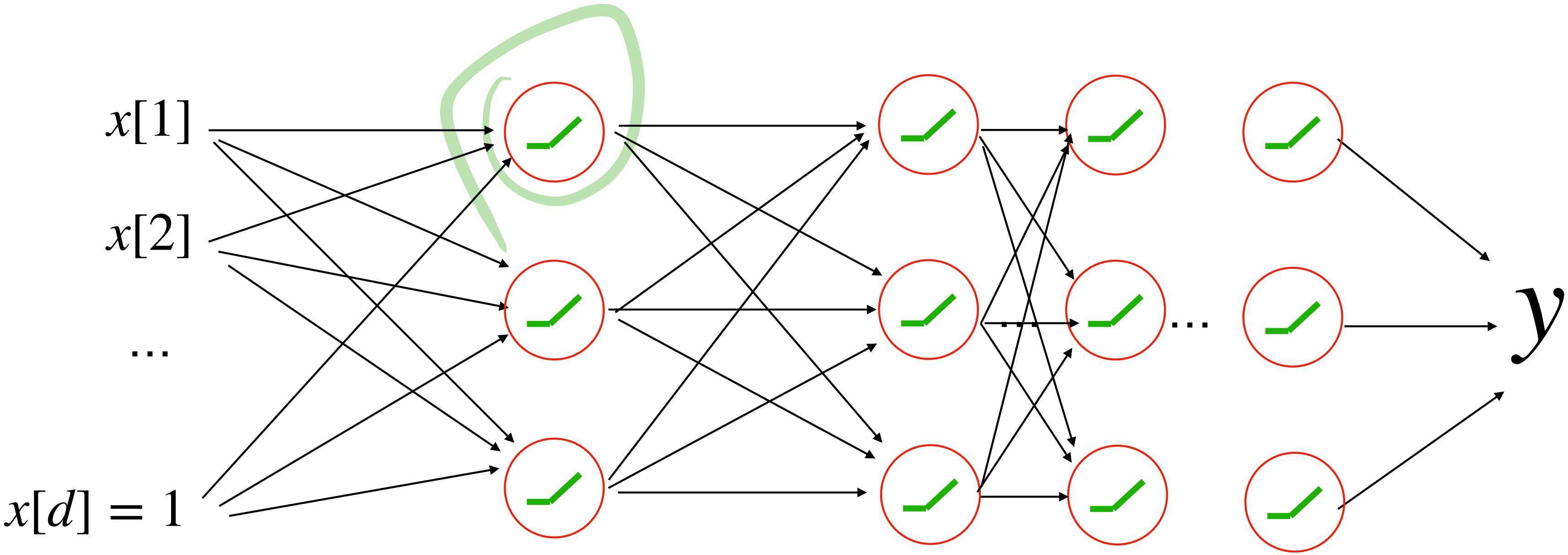
# The benefits of going deep



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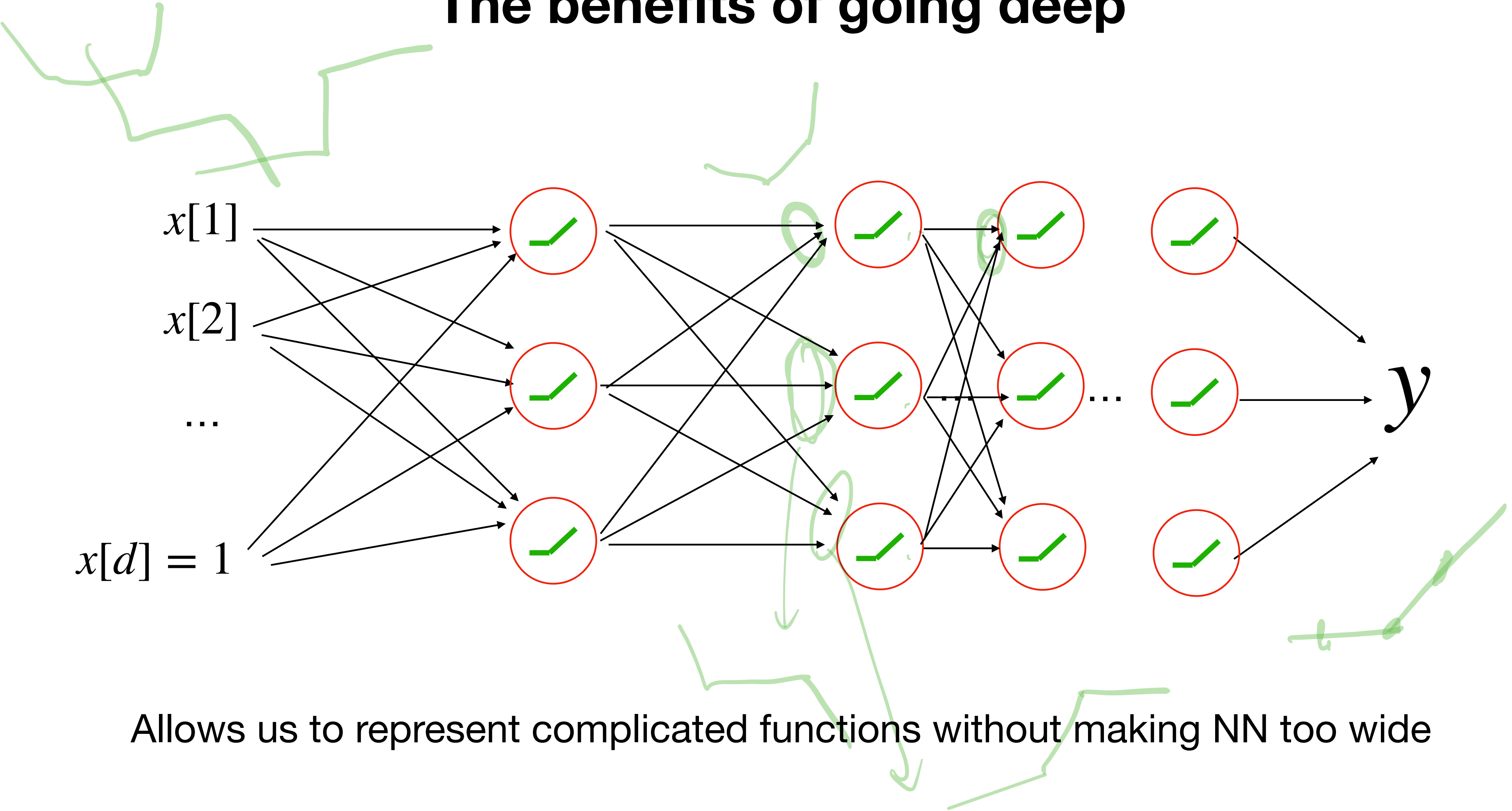


Allows us to represent complicated functions without making NN too wide