# Maximum Likelihood Estimation
# &
# Maximum A Posteriori Probability Estimation

# Recap on Perceptron

Binary classifier: sign$(w^\top x)$

**The Perceptron Alg:**

Initialize $w_0 = 0$

For $t = 0 \rightarrow \infty$

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Theorem:

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Q: does the data need to be i.i.d?

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No i.i.d assumption, and indeed data $\{x_1, y_1 \ldots, x_T, y_T\}$ can be selected by an Adversary (as long as it is separable)!!!

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Yes! As long as margin exists!

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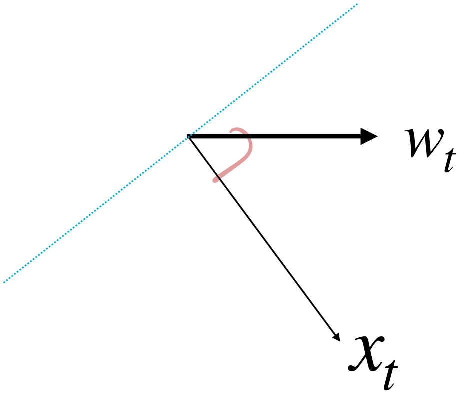
When we make a mistake, i.e., $y_t(w_t^\top x_t) < 0$ (e.g., $y_t = -1$, $w_t^\top x_t > 0$)



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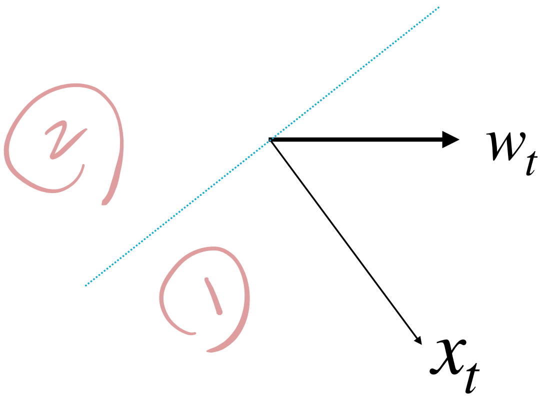


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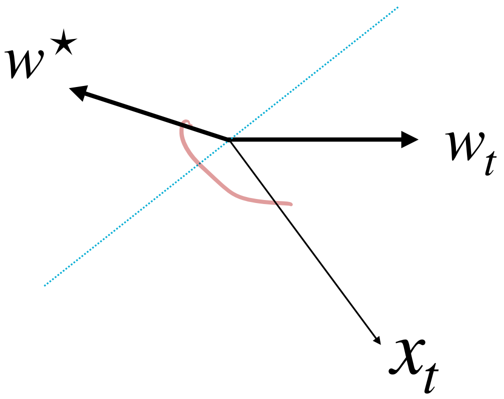


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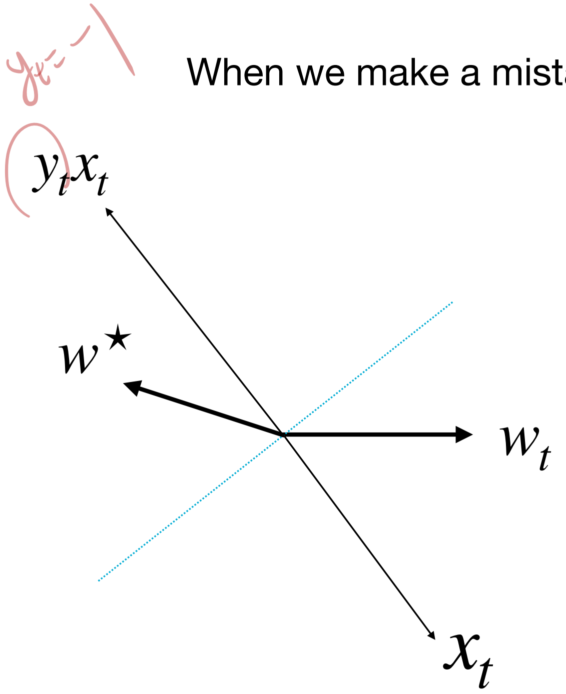


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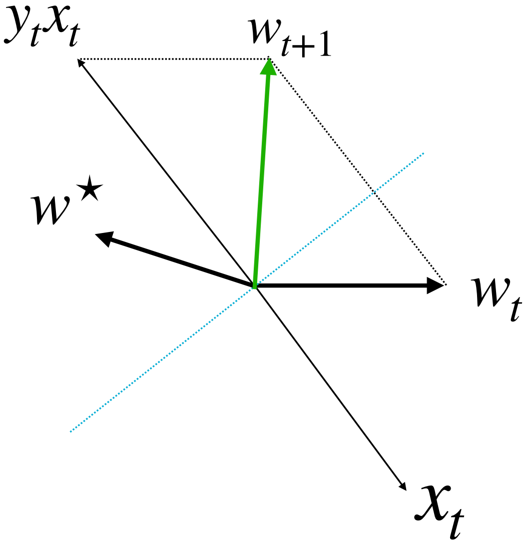


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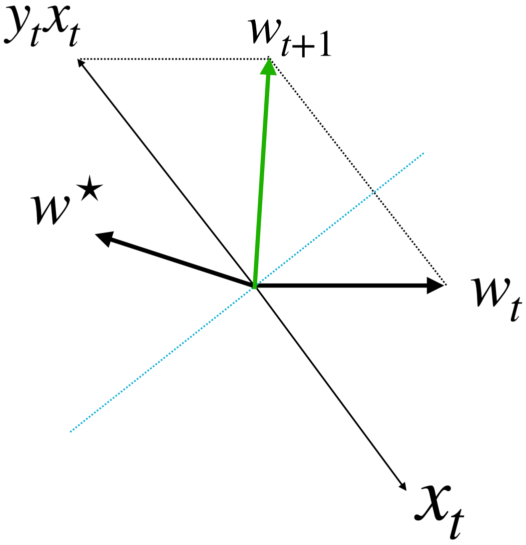


We should track how the $\cos(\theta_t)$ is changing:

$$\cos(\theta_t) = \frac{w_t^\top w^\star}{\|w_t\|_2}$$

Q: What does $w^\star$ look like?

# Outline for today:

1. Maximum Likelihood estimation (MLE)

2. Maximum a posteriori probability (MAP)

3. Example: MLE and MAP for classification

# Ex 1: Estimating the probability of a coin flip

We toss a coin n times (independently), we observe the following outcomes:

$\mathscr{D} = \{y_i\}_{i=1}^n, y_i \in \{-1,1\}$ ( $y_i = 1$ means head in $i$'s trial, -1 means tail)

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Q: assume $y_i \sim$ Bernoulli($\theta^\star$), how to estimate $\theta^\star$ given $\mathscr{D}$?

$$\hookrightarrow \begin{cases} +1 & \text{wp } \theta^\star \\ -1 & \text{wp } 1-\theta^\star \end{cases}$$

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$$\hat{\theta} \approx \frac{\sum_{i=1}^n \mathbf{1}(y_i = 1)}{n}$$

# of heads

$\hat{\theta} \to \theta^\star, n \to \infty$

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Let's make this rigorous!

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**If** the probability of getting head is $\theta \in [0,1]$, what is the probability of observing the data $\mathcal{D}$ (likelihood)?

$D = \{+1, +1, -1, +1, -1\}$

↑ ↑ ↑ ↑ ↑

$\theta \quad \theta \quad 1-\theta \quad \theta \quad 1-\theta$

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$$P(\mathscr{D} \mid \theta) = \theta^{n_1}(1-\theta)^{n-n_1}$$

$n_1 = \#(+1)$

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likelihood

# Maximum Likelihood Estimation

We have n patients, we give each the new drug, we record whether or not it's effective

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tail head

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$n_1$ = # of +1 (heads)

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$$= \arg\max_{\theta\in[0,1]} \ln(\theta^{n_1}(1-\theta)^{n-n_1})$$

$\theta = 0.01$

$n_1 = 1000$

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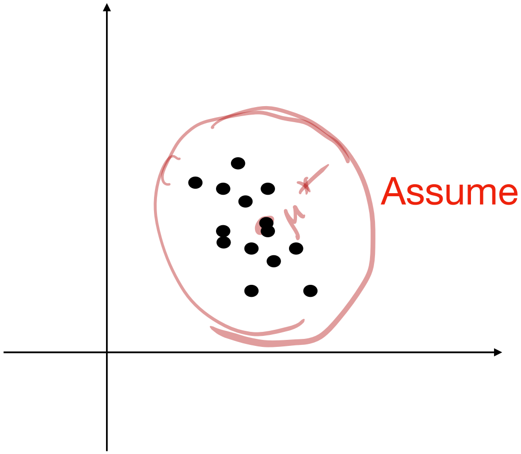
$$= \arg\max_{\theta \in [0,1]} n_1 \ln(\theta) + (n - n_1)\ln(1-\theta) = \frac{n_1}{n}$$

$$\frac{n_1}{\theta} - \frac{n-n_1}{1-\theta} = 0$$

solve for $\theta$

# Ex 2: Estimate the mean



$$\mathscr{D} = \{x_i\}_{i=1}^{n}, x_i \in \mathbb{R}^d$$

Assume data is from $\mathcal{N}(\mu^\star, I)$, want to estimate $\mu^\star$ from the data $\mathscr{D}$

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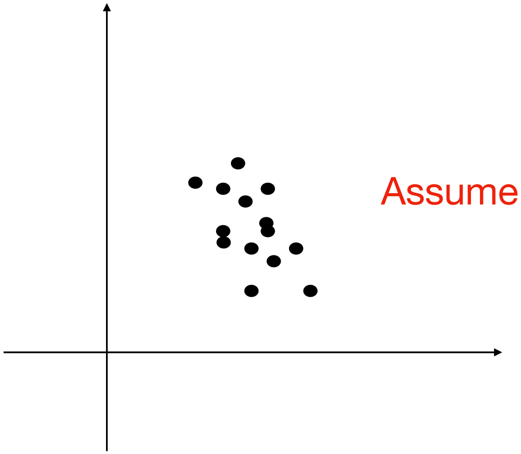

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Let's apply the MLE Principle:

Step 1: $$P(\mathscr{D} \mid \mu) = \prod_{i=1}^{n} \frac{1}{\sqrt{(2\pi)^d}} \exp\left(-\frac{1}{2}(x_i - \mu)^\top (x_i - \mu)\right)$$

likelihood of $x_i$ from $\mathcal{N}(\mu, I)$

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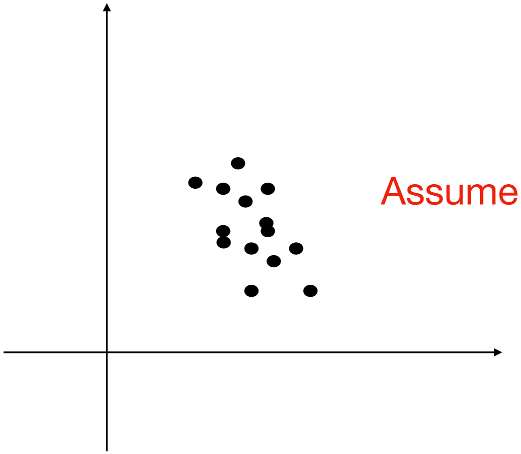

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Step 2: apply log and maximize the log-likelihood:

$$\arg\max_\mu \sum_{i=1}^n -(x_i - \mu)^\top (x_i - \mu)$$

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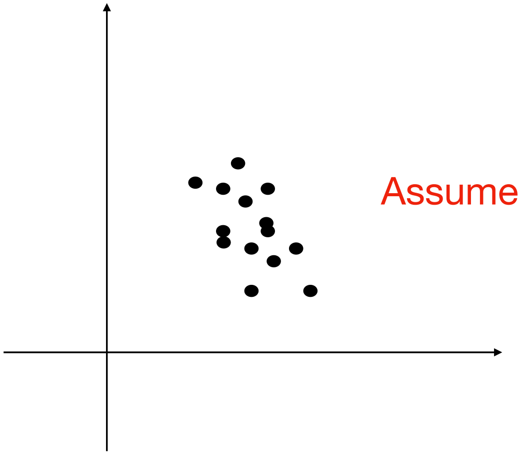

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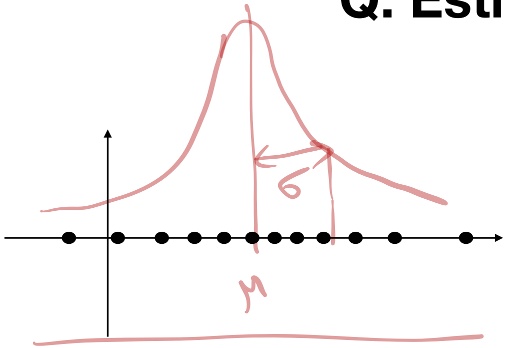
Step 2: apply log and maximize the log-likelihood:

$$\arg\max_{\mu} \sum_{i=1}^{n} -(x_i - \mu)^\top (x_i - \mu) \Rightarrow \hat{\mu}_{mle} = \sum_{i=1}^{n} x_i / n$$

# Q: Estimate the mean and variance




$$\mathscr{D} = \{x_i\}_{i=1}^{n}, x_i \in \mathbb{R}$$

Assume data is from $\mathcal{N}(\mu^\star, \sigma^2)$, want to estimate $\mu^\star, \sigma$ from the data $\mathscr{D}$

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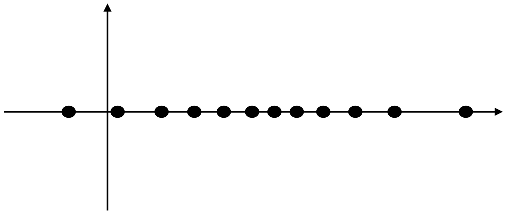

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Step 1: $P(\mathscr{D} \mid \mu, \sigma) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_i - \mu)^2/\sigma^2\right)$

likelihood of $x_i \sim N(\mu, \sigma^2)$

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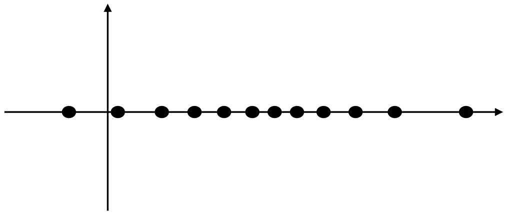

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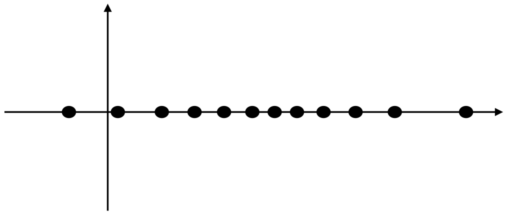
$$\ln P(D \mid \mu, \sigma) = \sum_{i=1}^{n}\left[\ln\left(\frac{1}{\sigma\sqrt{2\pi}}\right) + \ln \exp\left(-\frac{1}{2}(x_i-\mu)^2/\sigma^2\right)\right]$$

Step 2: apply log and maximize the log-likelihood:

$$\arg\max_{\mu, \sigma > 0} \sum_{i=1}^{n} \left(-(x_i - \mu)^2/\sigma^2 - \ln(\sigma)\right)$$

# Q: Estimate the mean and variance



$$\mathcal{D} = \{x_i\}_{i=1}^n, x_i \in \mathbb{R}$$

Assume data is from $\mathcal{N}(\mu^\star, \sigma^2)$, want to estimate $\mu^\star, \sigma$ from the data $\mathcal{D}$

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Step 2: apply log and maximize the log-likelihood:

$$\arg\max_{\mu,\sigma>0} \sum_{i=1}^n \left(-(x_i - \mu)^2/\sigma^2 - \ln(\sigma)\right) = ??$$

# Summary of MLE

1. MLE is consistent: if our model assumption is correct (e.g., coin flip follows some Bernoulli distribution), then $\hat{\theta}_{mle} \rightarrow \theta^{\star}$, as $n \rightarrow \infty$

$$\left| \hat{\theta}_{mle} - \theta^{*} \right| \leq \frac{1}{\sqrt{n}}$$

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2. When our model assumption is wrong (e.g., we use Gaussian to model data which is from some more complicated distribution), then MLE loses such guarantee

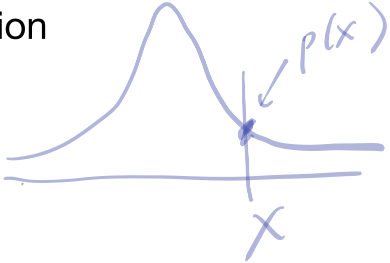
# Outline for today:

1. Maximum Likelihood estimation (MLE)

2. Maximum a Posteriori Probability (MAP)

$$\int p(x) = 1$$

3. Example: MLE and MAP for classification



# Ex: Estimating the probability of a coin flip

We toss a coin n times (independently), we observe the following outcomes:

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Example: $P(\theta)$ being a Beta distribution:

$$P(\theta) = \theta^{\alpha-1}(1-\theta)^{\beta-1}/Z,$$

where $Z = \int_{\theta\in[0,1]} \theta^{\alpha-1}(1-\theta)^{\beta-1} d_\theta$ → $\int_{\theta\in[0,1]} P(\theta)\, d\theta = 1$

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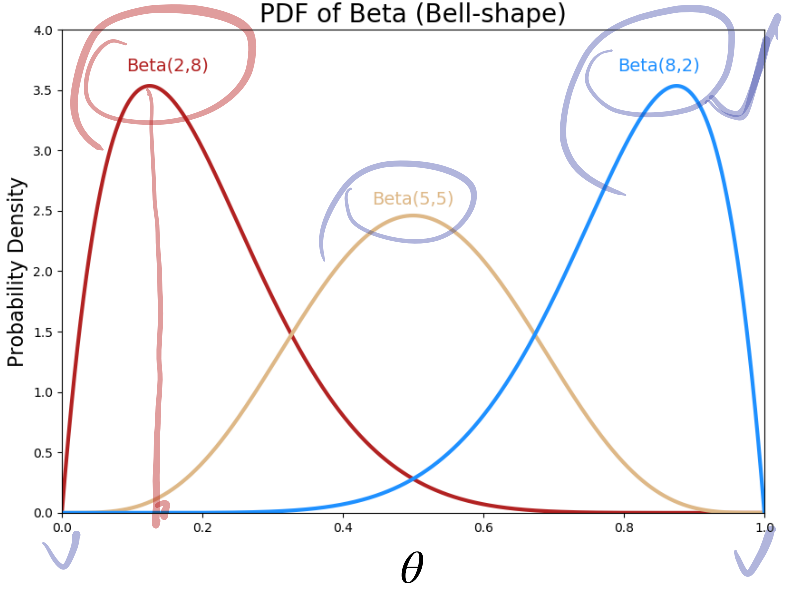

# The Posterior distribution over $\theta$

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$P(\mathscr{D})$ is independent of $\theta$

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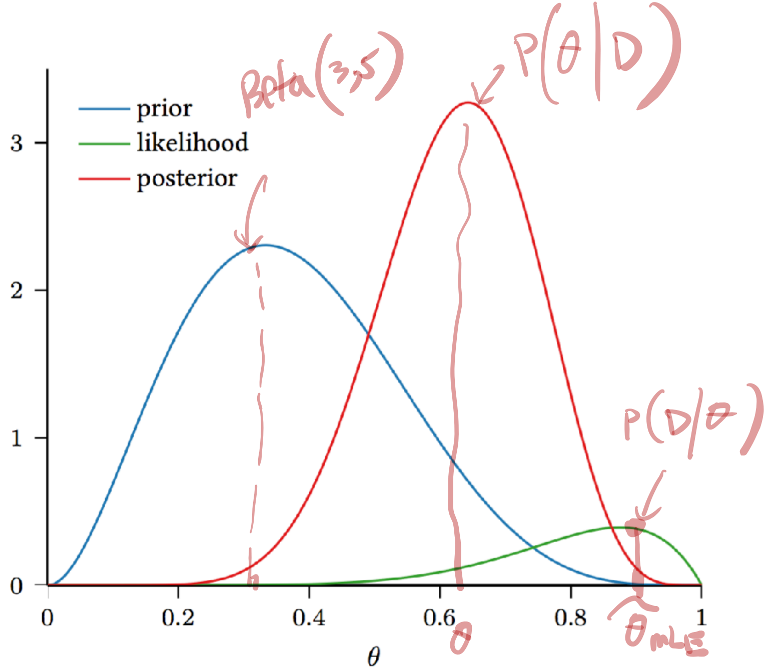

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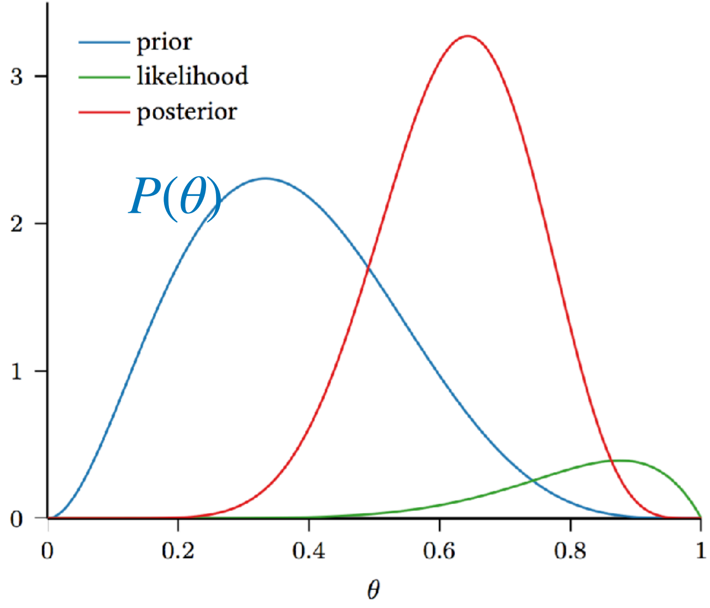

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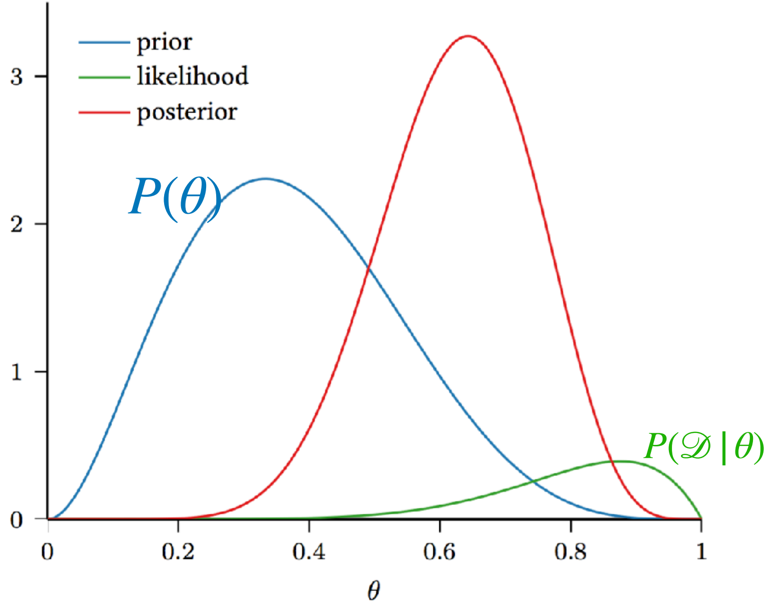

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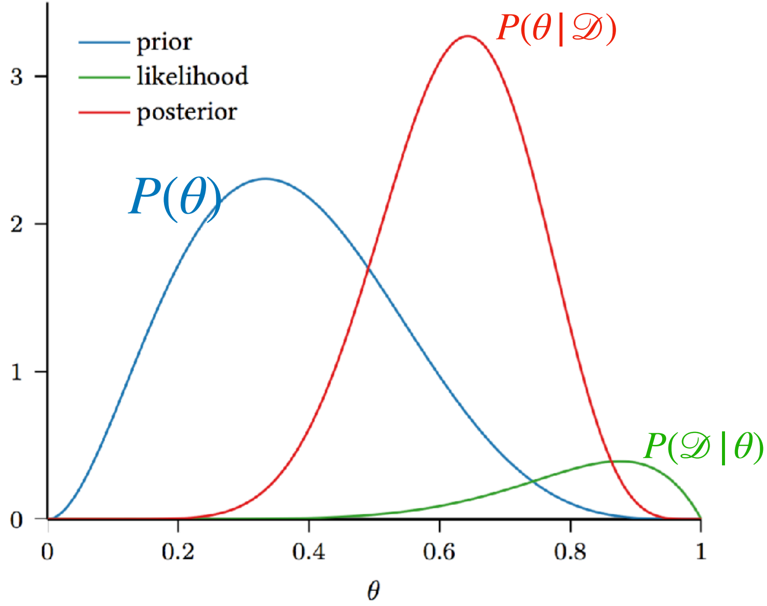

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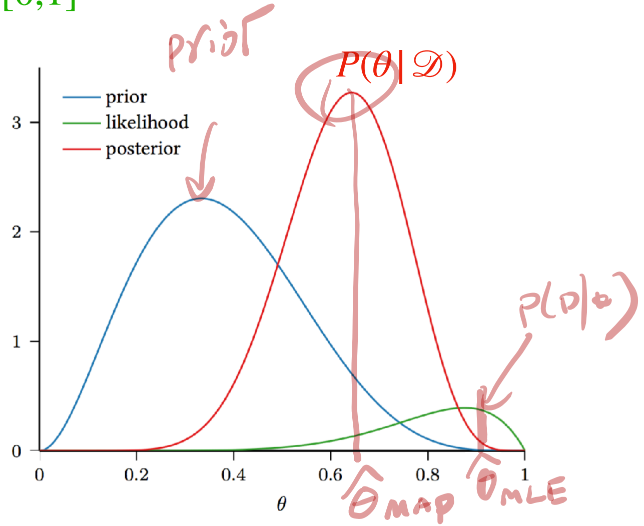

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$\arg\max_{\theta \in [0,1]} P(\theta \mid D)$

Step 4: Compute MAP $\hat{\theta}_{map} = \dfrac{n_1+\alpha-1}{n+\alpha+\beta-2}$

$\left(MLE = n_1/n\right)$

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\# of heads: $n_1 + \alpha - 1$

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\# of Exp: $n + \alpha - 1 + \beta - 1$

$(\alpha - 1, \beta - 1)$ can be understood as some fictions flips: we had $\alpha - 1$ hallucinated heads, and $\beta - 1$ hallucinated tails

# Some considerations on prior distributions

1. In coin flip example, when $n \to \infty, \hat{\theta}_{map} = \frac{n_1 + \alpha - 1}{n + \alpha + \beta - 2} \to \frac{n_1}{n}$(i.e.,$\hat{\theta}_{mle}$)

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3. In general, not so easy to set up a good prior….

# Outline for today:

1. Maximum Likelihood estimation (MLE)

2. Maximum a posteriori probability (MAP)

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# Binary Classification

$x \sim P(x) \quad y \sim P(y|x)$

Given labeled dataset $\{x_i, y_i\}_{i=1}^n, x_i \in \mathbb{R}^d, y_i \in \{-1,1\}$, we want to estimate $P(y\,|\,x)$

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Let us assume the ground truth has the form $P(y = 1 \mid x; \theta^\star) = \dfrac{\exp((\theta^\star)^\top x)}{1 + \exp((\theta^\star)^\top x)}$

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Goal: estimate $\theta^\star$ using $\mathscr{D}$

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Using MLE:

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prior

Data likelihood

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Independent of the data

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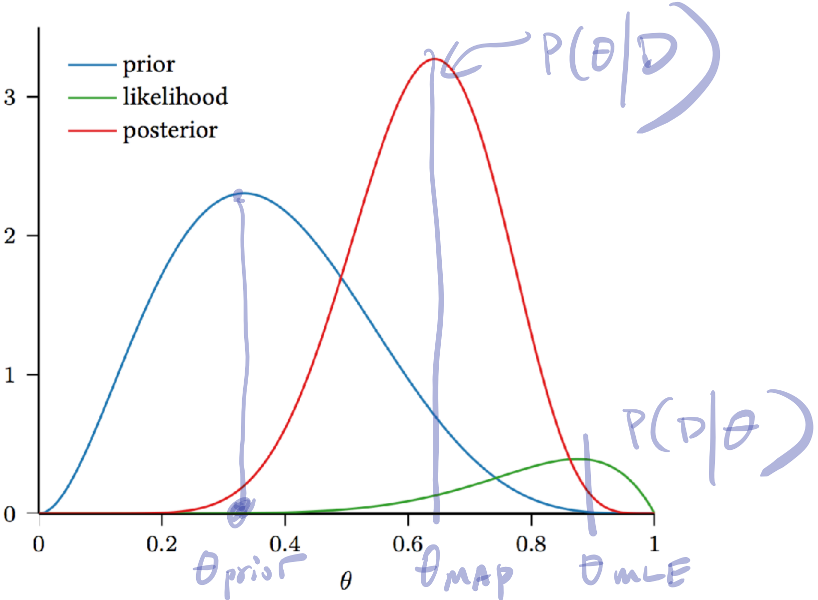

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