

Statistical Learning Theory: Weighted Experts and Bandits

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Reading: Mitchell Chapter 7.5

Expert Learning Model

- Setting
 - N experts named $H = \{h_1, \dots, h_N\}$
 - Each expert h_i takes an action $y = h_i(x_t)$ in each round t and incurs loss $\Delta_{t,i}$
 - Algorithm can select which expert's action to follow in each round
- Interaction Model
 - FOR t from 1 to T
 - Algorithm selects expert h_{i_t} according to strategy $A(w_t)$ and follows its action y
 - Experts incur losses $\Delta_{t,1} \dots \Delta_{t,N}$
 - Algorithm incurs loss Δ_{t,i_t}
 - Algorithm updates w_t to w_{t+1} based on $\Delta_{t,1} \dots \Delta_{t,N}$

Regret

- Idea
 - N experts named $H = \{h_1, \dots, h_N\}$
 - Compare performance of A to best expert i^* in hindsight.
- Regret
 - Overall loss of best expert i^* in hindsight is

$$\Delta_T^* = \min_{i^* \in [1..N]} \sum_{t=1}^T \Delta_{t,i^*}$$

- Loss of algorithm A at time t is

$$\Delta_{t,i}$$

for algorithm that picks recommendation of expert $i = A(w_t)$ at time t .

- Regret is difference between loss of algorithm and best fixed expert in hindsight

$$\text{Regret}(T) = \sum_{t=1}^T \Delta_{t,A(w_t)} - \min_{i^* \in [1..N]} \sum_{t=1}^T \Delta_{t,i^*}$$

Weighted Majority Algorithm

- Setting
 - N experts named $H = \{h_1, \dots, h_N\}$
 - Binary actions $y = \{+1, -1\}$ given input x , zero/one loss
 - There may be no expert in H that acts perfectly
- Algorithm
 - Initialize $w_1 = (1, 1, \dots, 1)$
 - FOR $t = 1$ TO T
 - Predict the same y as majority of $h_i \in H$, each weighted by $w_{t,i}$
 - FOREACH $h_i \in H$
 - IF h_i incorrect THEN $w_{t+1,i} = w_{t,i} * \beta$
 - ELSE $w_{t+1,i} = w_{t,i}$
- Mistake Bound
 - How close is the number of mistakes the Weighted Majority Algorithm makes to the number of mistakes of the best expert in hindsight?

Exponentiated Gradient Algorithm for Expert Setting (EG)

- Setting
 - N experts named $H = \{h_1, \dots, h_N\}$
 - Any actions, any loss function
 - There may be no expert in H that acts perfectly
- Algorithm
 - Initialize $w_1 = \left(\frac{1}{N}, \dots, \frac{1}{N}\right)$
 - FOR t from 1 to T
 - Algorithm randomly picks i_t from $P(I_t = i_t) = w_{t,i}$
 - Experts incur losses $\Delta_{t,1} \dots \Delta_{t,N}$
 - Algorithm incurs loss Δ_{t,i_t}
 - Algorithm updates w for all experts i as
$$\forall i, w_{t+1,i} = w_{t,i} \exp(-\eta \Delta_{t,i})$$
Then normalize w_{t+1} so that $\sum_j w_{t+1,j} = 1$.

Expected Regret

- Idea
 - Compare performance to best expert in hindsight
- Regret
 - Overall loss of best expert i^* in hindsight is

$$\Delta_T^* = \min_{i^* \in [1..N]} \sum_{t=1}^T \Delta_{t,i^*}$$

- Expected loss of algorithm $A(w_t)$ at time t is

$$E_{A(w_t)}[\Delta_{t,i}] = w_t \Delta_t$$

for randomized algorithm that picks recommendation of expert i at time t with probability $w_{t,i}$.

- Regret is difference between expected loss of algorithm and best fixed expert in hindsight

$$\text{ExpectedRegret}(T) = \sum_{t=1}^T w_t \Delta_t - \min_{i^* \in [1..N]} \sum_{t=1}^T \Delta_{t,i^*}$$

Regret Bound for Exponentiated Gradient Algorithm

- Theorem

The expected regret of the exponentiated gradient algorithm in the expert setting is bounded by

$$\text{ExpectedRegret}(T) \leq \Delta^{\max} \sqrt{2 T \log(|H|)}$$

where $\Delta^{\max} = \max\{\Delta_{t,i}\}$ and $\eta = \frac{\sqrt{\log(N)}}{\Delta\sqrt{2T}}$.

Bandit Learning Model

- Setting
 - N bandits named $H = \{h_1, \dots, h_N\}$
 - Each bandit h_i takes an action in each round t and incurs loss $\Delta_{t,i}$
 - Algorithm can select which bandit's action to follow in each round
- Interaction Model
 - FOR t from 1 to T
 - Algorithm selects expert h_{i_t} according to strategy A_{w_t} and follows its action y
 - Bandits incur losses $\Delta_{t,1} \dots \Delta_{t,N}$
 - Algorithm incurs loss Δ_{t,i_t}
 - Algorithm updates w_t to w_{t+1} based on Δ_{t,i_t}

Key difference compared to Expert Model

Bar

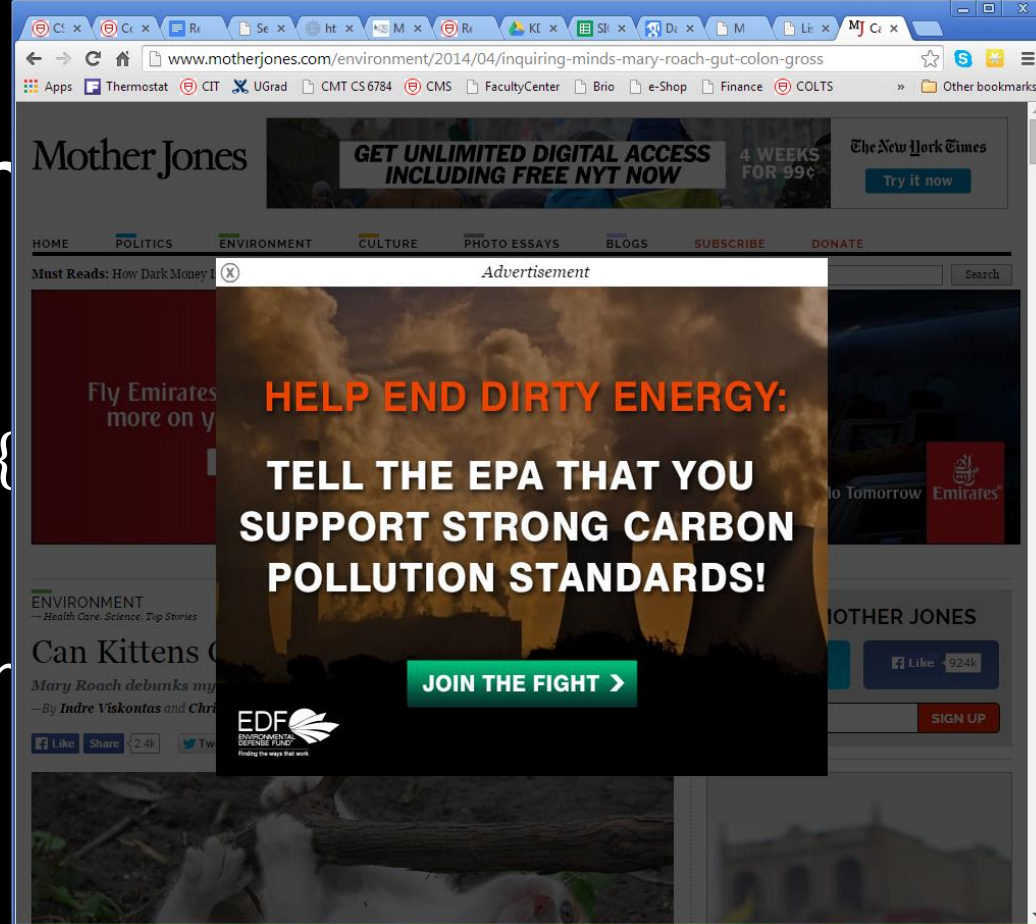
- Setting

- N bandits named $H = \{$
- Each bandit h_i takes an loss $\Delta_{t,i}$
- Algorithm can select wh each round

- Interaction Model

- FOR t from 1 to T

- Algorithm selects expert h_{i_t} according to strategy A_{w_t} and follows its action y
- Bandits incur losses $\Delta_{t,1} \dots \Delta_{t,N}$
- Algorithm incurs loss Δ_{t,i_t}
- Algorithm updates w_t to w_{t+1} based on Δ_{t,i_t}



Key difference compared to Expert Model

Exponentiated Gradient Algorithm for Bandit Setting (EXP3)

- Initialize $w_1 = \left(\frac{1}{N}, \dots, \frac{1}{N}\right)$, $\gamma = \min \left\{ 1, \sqrt{\frac{N \log N}{(e-1)\Delta T}} \right\}$
- FOR t from 1 to T
 - Algorithm randomly picks i_t with probability $P(i_t) = (1 - \gamma)w_{t,i} + \gamma/N$
 - Experts (aka Bandits) incur losses $\Delta_{t,1} \dots \Delta_{t,N}$
 - Algorithm incurs loss Δ_{t,i_t}
 - Algorithm updates w for bandit i_t as
$$w_{t+1,i_t} = w_{t,i_t} \exp\left(-\eta\Delta_{t,i_t}/P(i_t)\right)$$
Then normalize w_{t+1} so that $\sum_j w_{t+1,j} = 1$.

Other Online Learning Problems

- Stochastic Experts
- Stochastic Bandits
- Online Convex Optimization
- Partial Monitoring