PID control, linear systems

CS 4758

- Review of Potential Fields and RRTs.

- Discrete time Linear systems:
  - reachability, controllability.
- Bang-bang controller.
- PID controller.
  - Stability of system with P controller.

The Bang-Bang Controller

- Push back, against the *direction* of the error
  - with constant action *u*
- Error is $e = x - x_{set}$

$$e < 0 \Rightarrow u := on \Rightarrow \dot{x} = F(x, on) > 0$$

$$e > 0 \Rightarrow u := off \Rightarrow \dot{x} = F(x, off) < 0$$

- To prevent chatter around e = 0,

$$e < -\varepsilon \Rightarrow u := on$$

$$e > +\varepsilon \Rightarrow u := off$$

- Household thermostat. Not very subtle.

Reachability for discrete-time LDS

DT system $x(t+1) = Ax(t) + Bu(t)$, $x(t) \in \mathbf{R}^n$

$$x(t) = \mathcal{C}_t \begin{bmatrix} u(t-1) \\ \vdots \\ u(0) \end{bmatrix}$$

where $\mathcal{C}_t = \begin{bmatrix} B & AB & \cdots & A^{t-1}B \end{bmatrix}$

so reachable set at t is $\mathcal{R}_t = \mathrm{range}(\mathcal{C}_t)$

by C-H theorem, we can express each A^k for $k \geq n$ as linear combination of $A^0, \ldots, A^{n-1}$

hence for $t \geq n$, $\mathrm{range}(\mathcal{C}_t) = \mathrm{range}(\mathcal{C}_n)$

thus we have

$$\mathcal{R}_t = \begin{cases} \mathrm{range}(\mathcal{C}_t) & t < n \\ \mathrm{range}(\mathcal{C}) & t \geq n \end{cases}$$

where $\mathcal{C} = \mathcal{C}_n$ is called the *controllability matrix*

- any state that can be reached can be reached by $t = n$
- the reachable set is $\mathcal{R} = \mathrm{range}(\mathcal{C})$

Controllable system

system is called *reachable* or *controllable* if all states are reachable ($i.e.$, $\mathcal{R} = \mathbf{R}^n$)

system is reachable if and only if $\mathbf{Rank}(\mathcal{C}) = n$

example: $x(t+1) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$

controllability matrix is $\mathcal{C} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$

hence system is not controllable; reachable set is

$$\mathcal{R} = \mathrm{range}(\mathcal{C}) = \{ \, x \mid x_1 = x_2 \, \}$$

The PID Controller

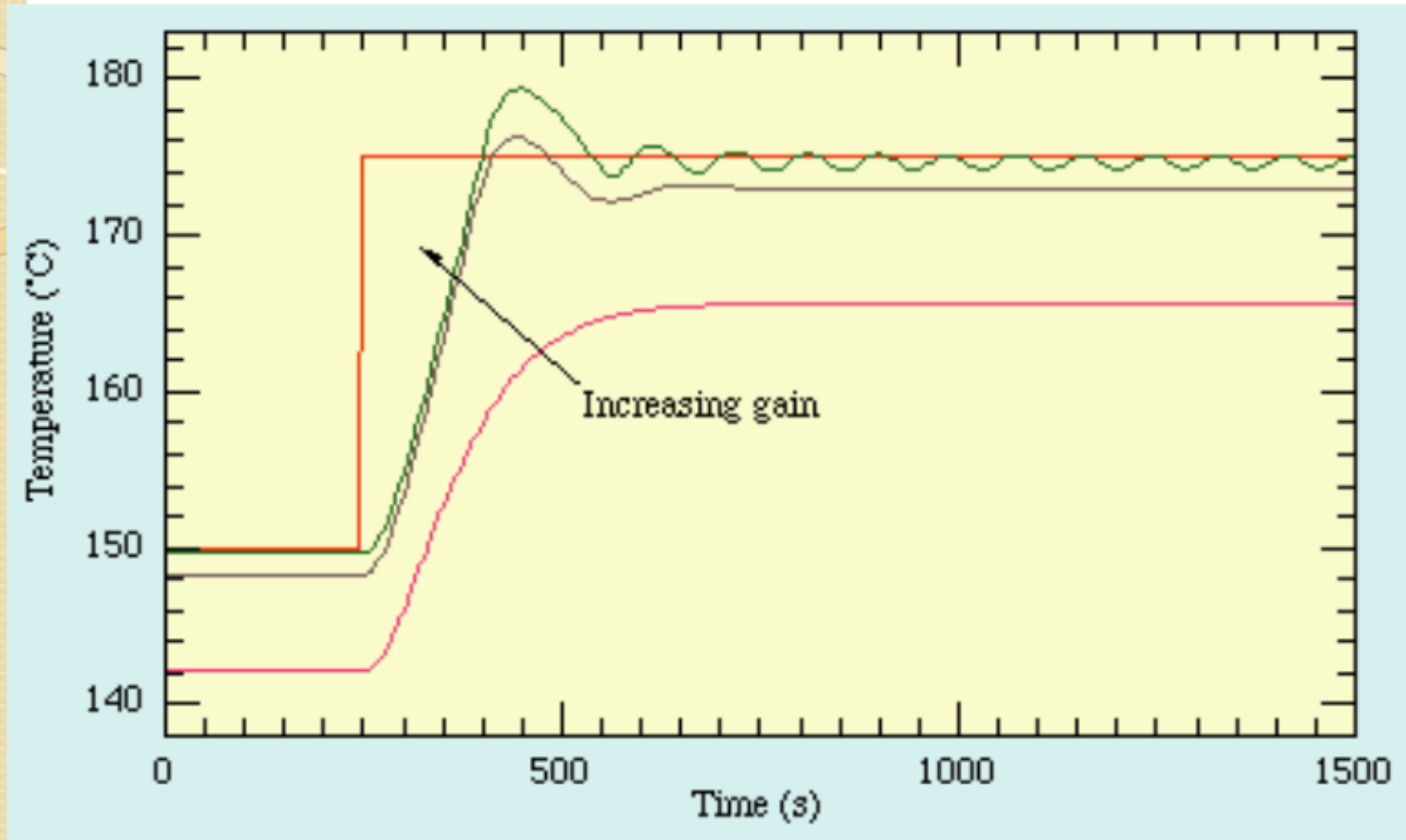
- A weighted combination of Proportional, Integral, and Derivative terms.

$$u(t) = -k_P e(t) - k_I \int_0^t e dt - k_D \dot{e}(t)$$

- The PID controller is the workhorse of the control industry. Tuning is non-trivial.

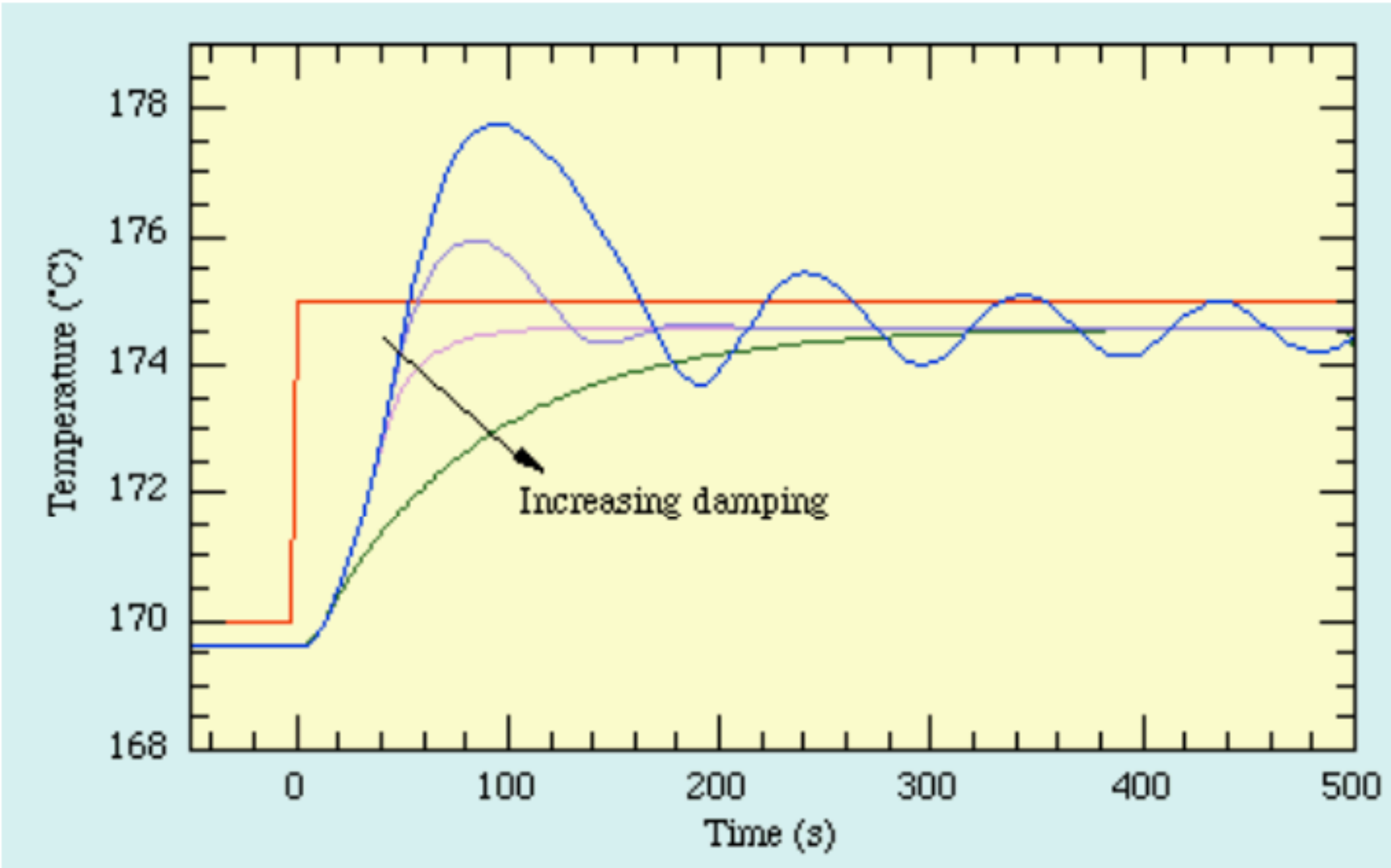
where, $e(t) = x(t) - x_{desired}(t)$

Proportional Control in Action



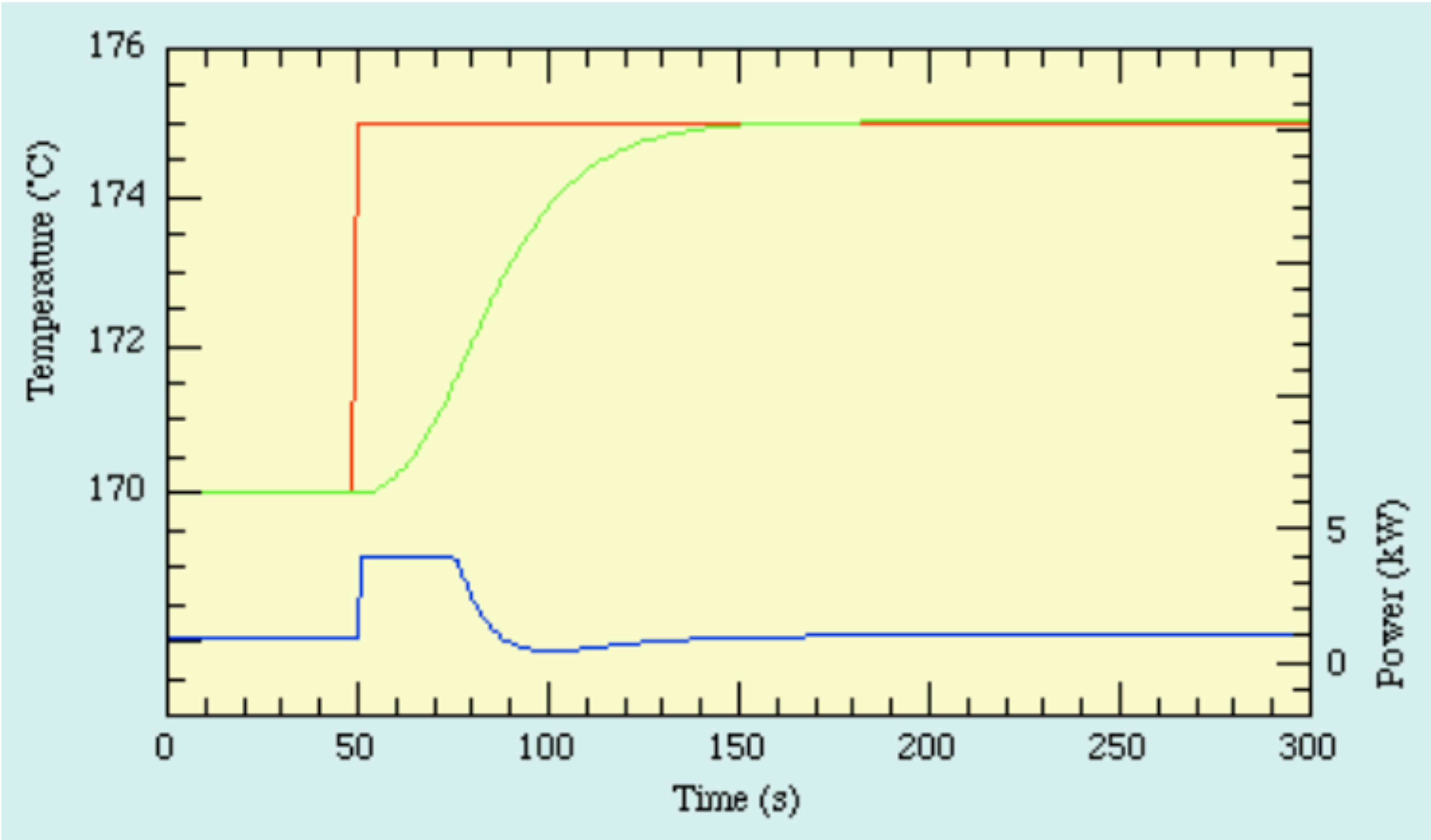
- Increasing gain approaches setpoint faster
- Can leads to overshoot, and even instability
- Steady-state offset

Derivative Control in Action



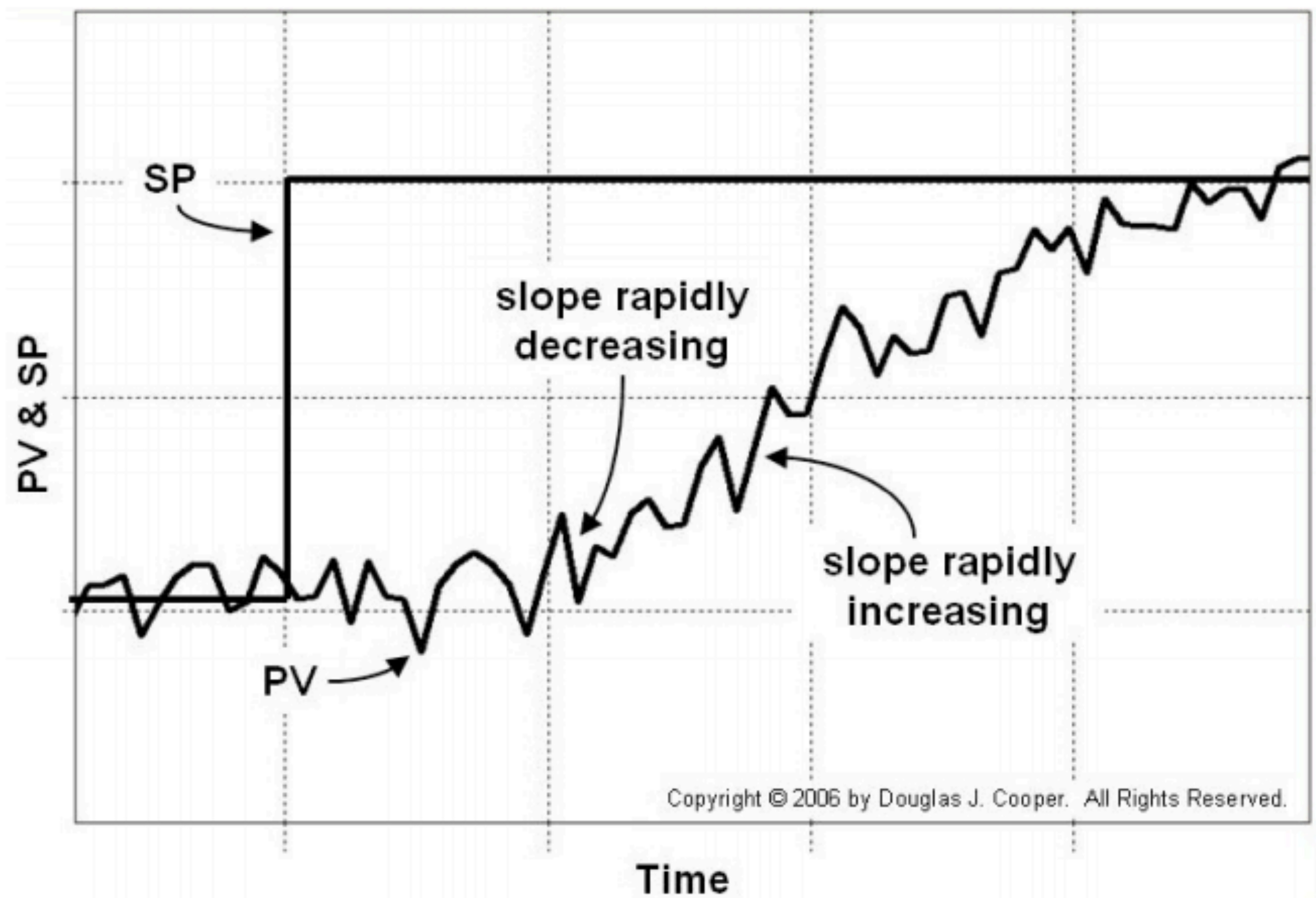
- Damping fights oscillation and overshoot
- But it's vulnerable to noise

PID Control in Action



- But, good behavior depends on good tuning!

Derivatives Amplify Noise



- This is a problem if control output (CO) depends on slope (with a high gain).

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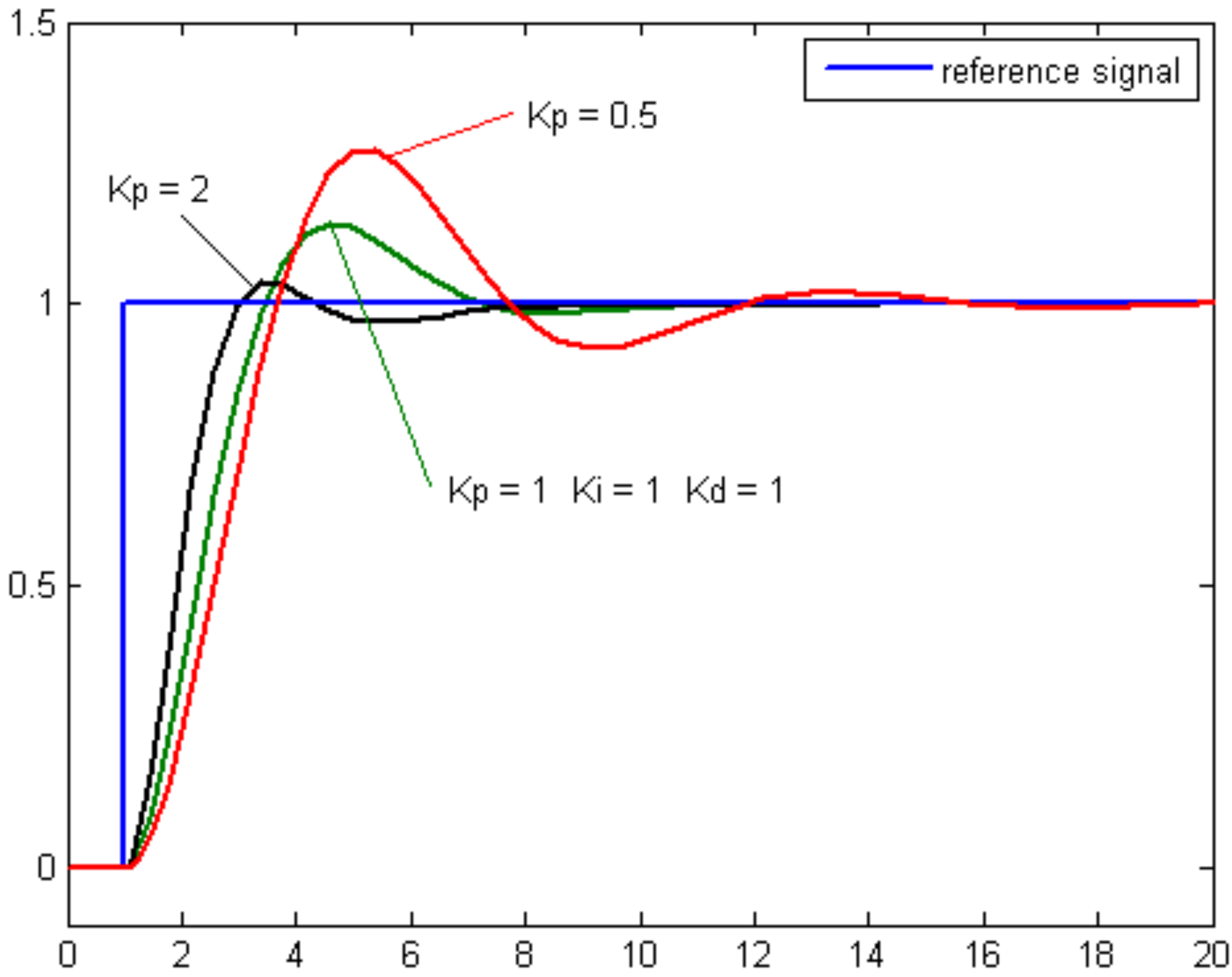
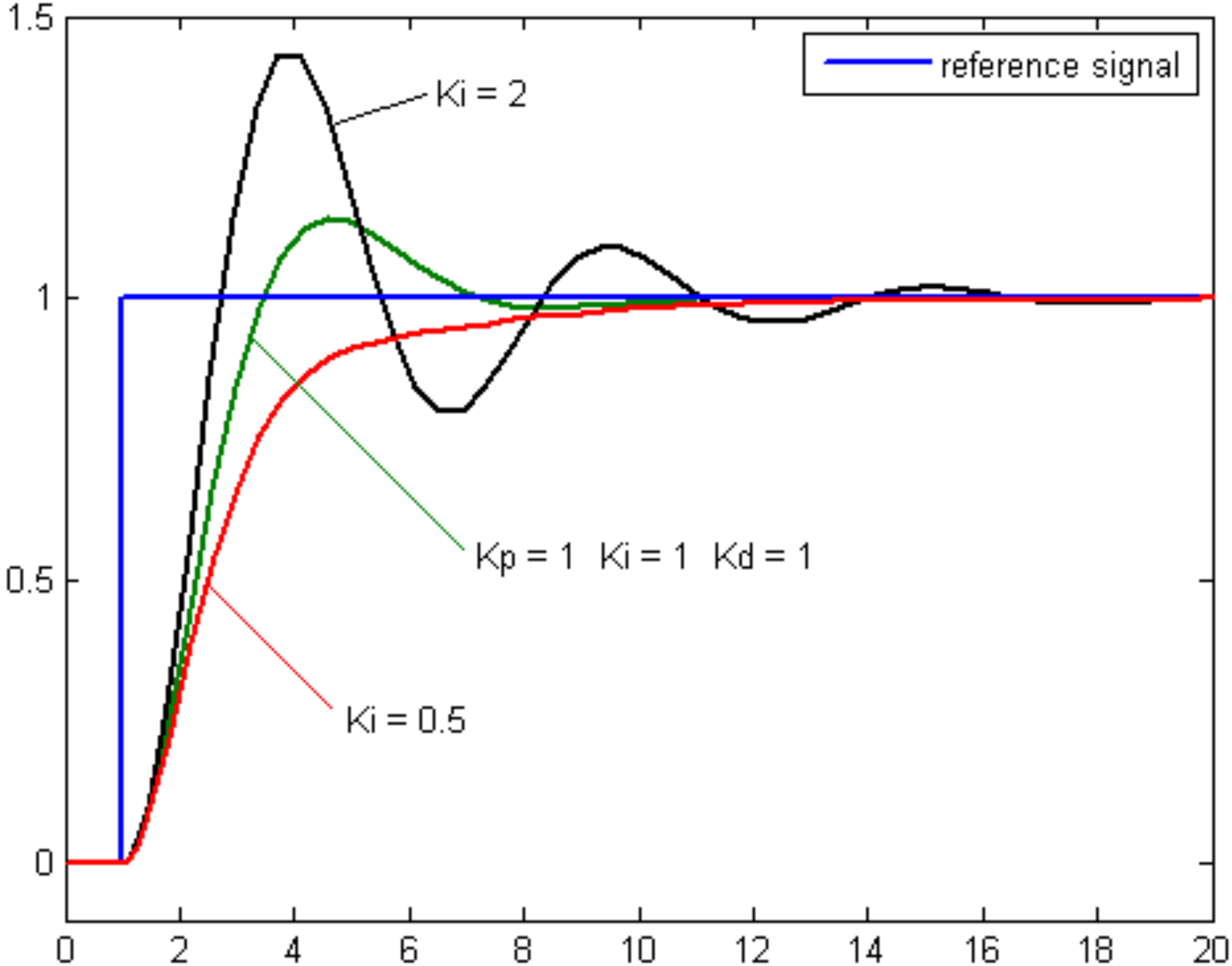
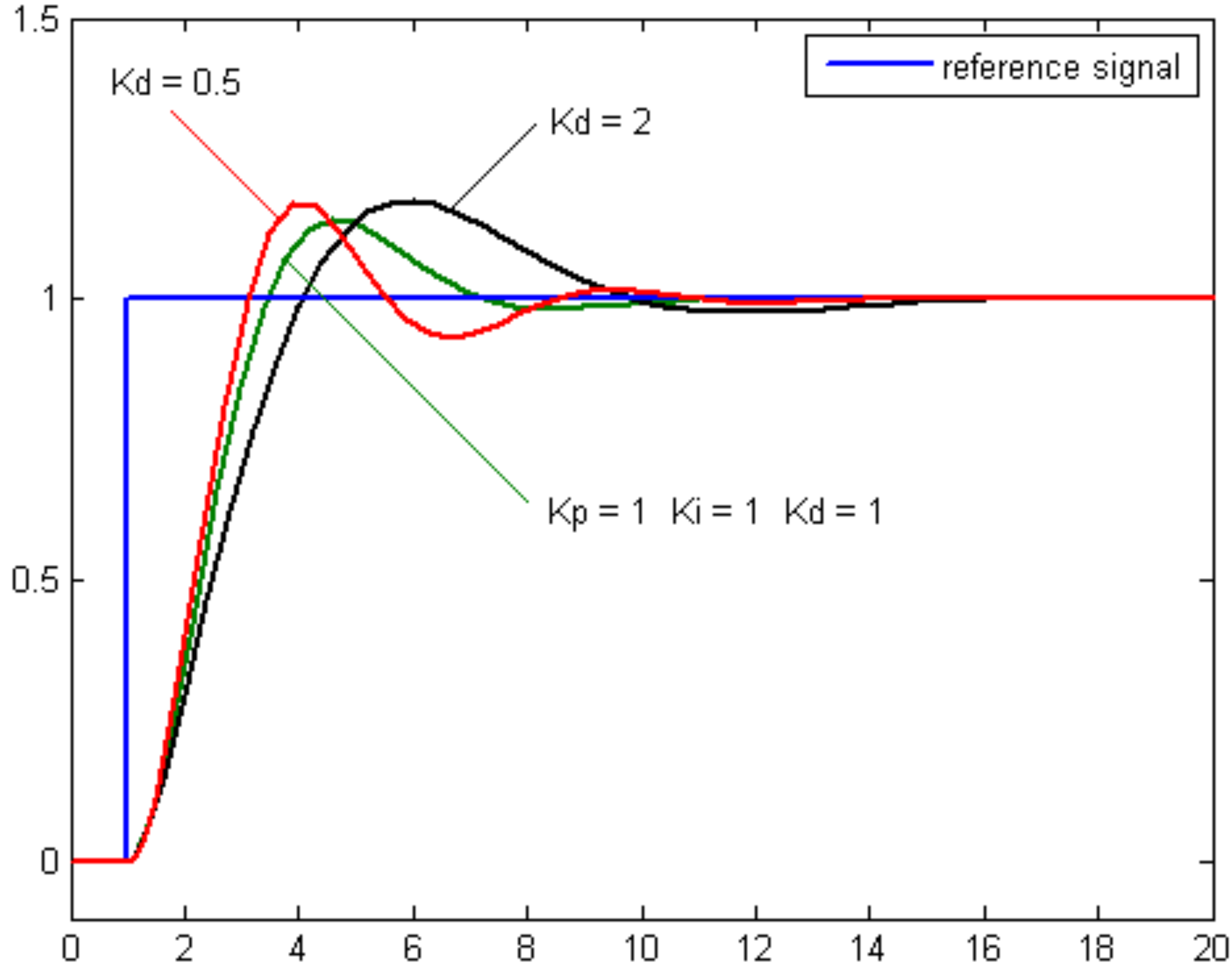

reference signal
Kp = 0.5
Kp = 2
Kp = 1 Ki = 1 Kd = 1
1.5
1
0.5
0
0
2
4
6
8
10
12
14
16
18
20



1.5
1
0.5
0
0
2
4
6
8
10
12
14
16
18
20
reference signal
Ki = 2
Kp = 1 Ki = 1 Kd = 1
Ki = 0.5



reference signal
Kd = 0.5
Kd = 2
Kp = 1 Ki = 1 Kd = 1
1.5
1
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20

Stability of P controller.

For discrete time linear system:

The magnitude of the real part of the eigenvalues of the state transition matrix should be less than 1.

E.g., For A = 1, and B = 1,

$X(t+1) = (1-k_p)\ x(t)$

$|1-k_p| < 1$

=> For $0 < k_p < 2$, the system will be stable.