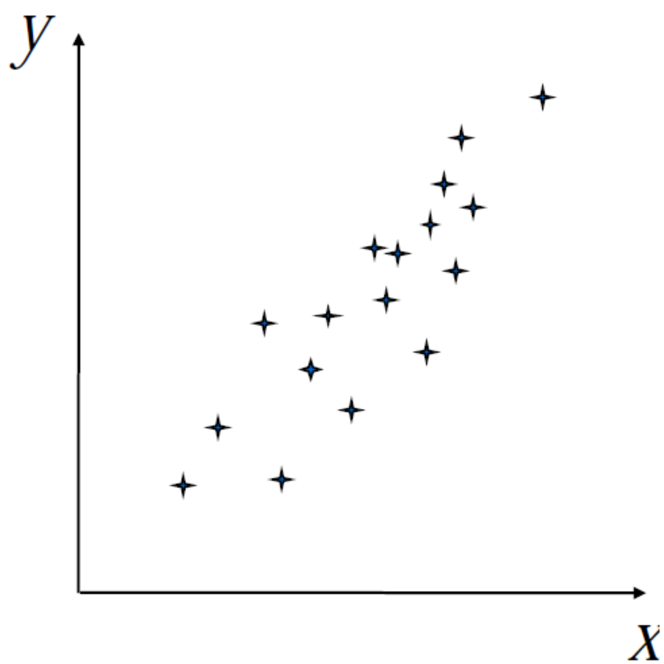
Linear Regression: One-Dimensional Case



- **Given:** a set of N input-response pairs
- The inputs (x) and the responses (y) are one dimensional scalars
- **Goal:** Model the relationship between x and y

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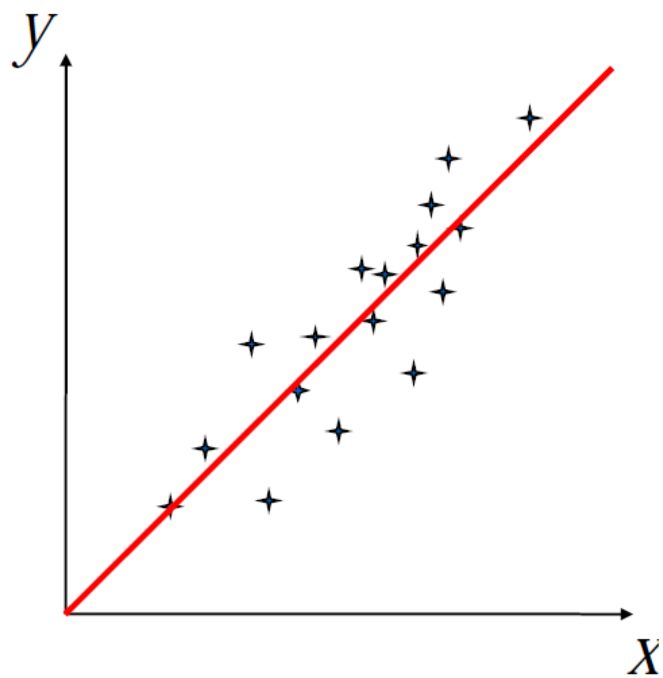

- Let's assume the relationship between x and y is linear

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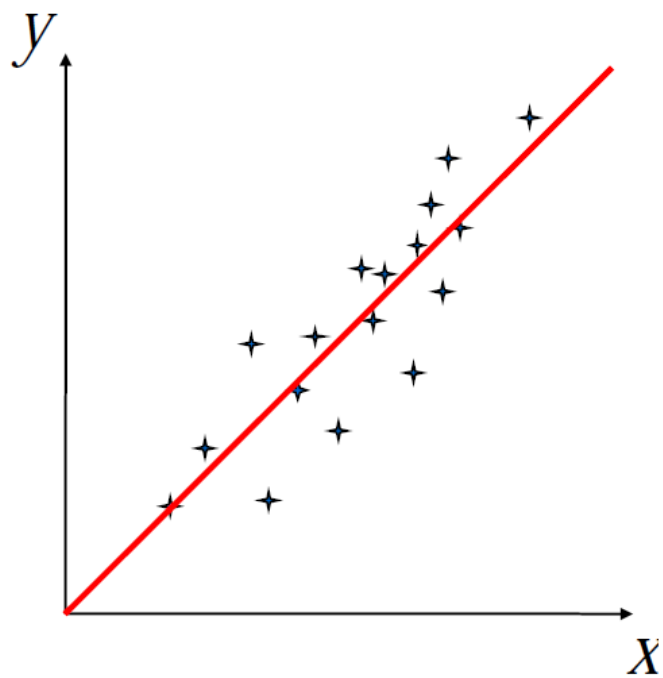

- Let's assume the relationship between x and y is linear
- Linear relationship can be defined by a straight line with *parameter* w
- Equation of the straight line: $y = wx$

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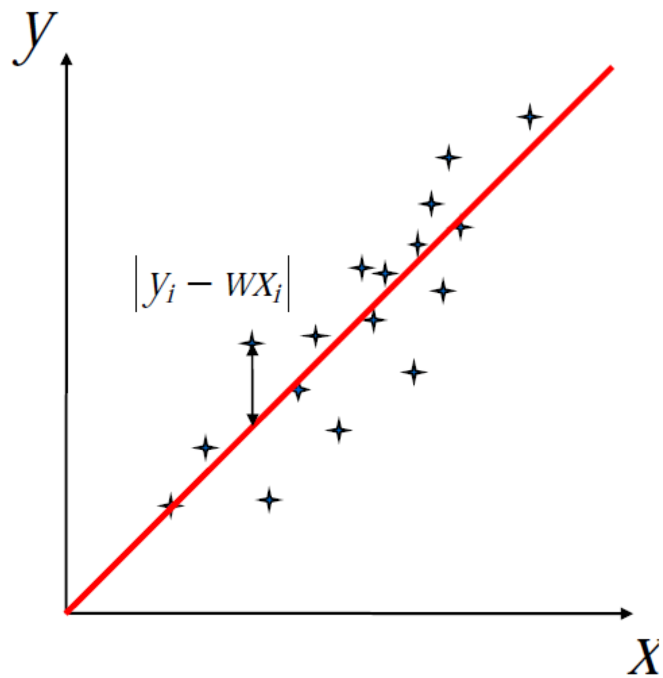

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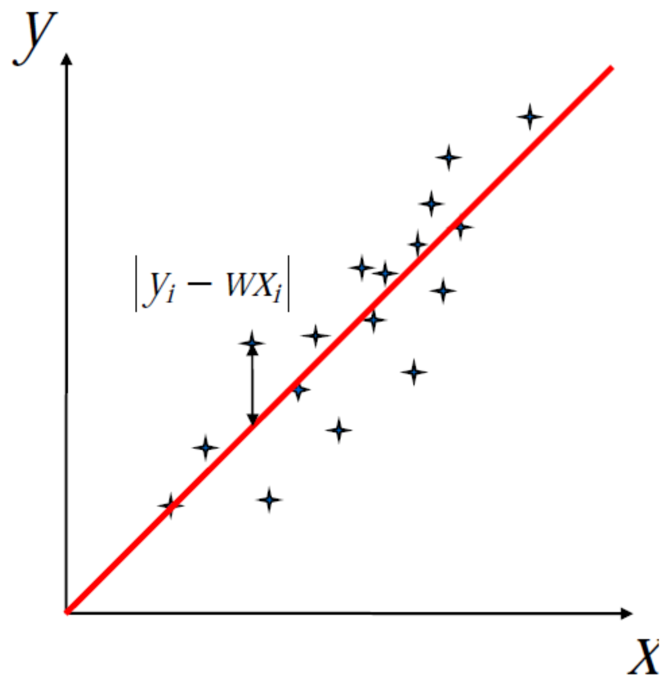

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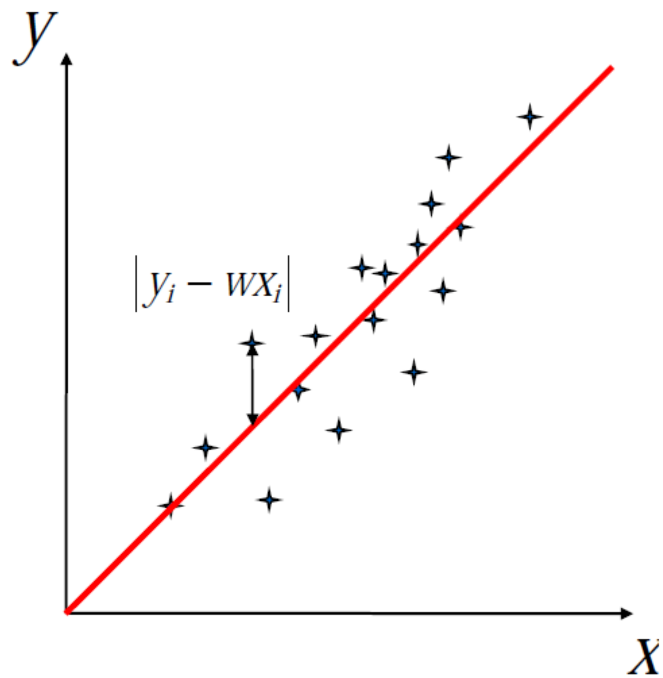

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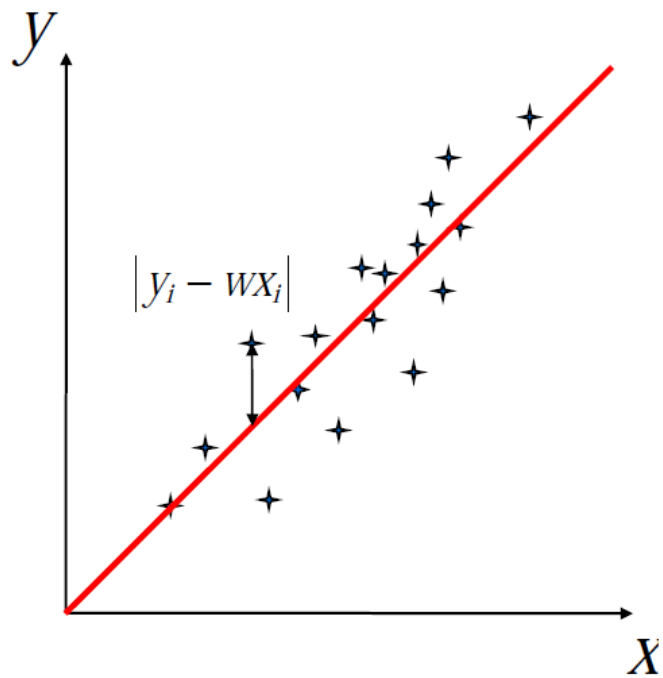

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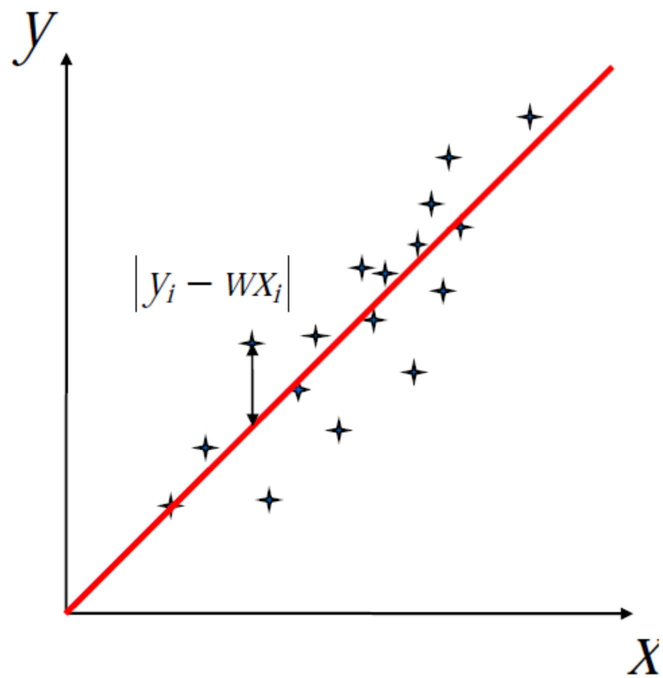

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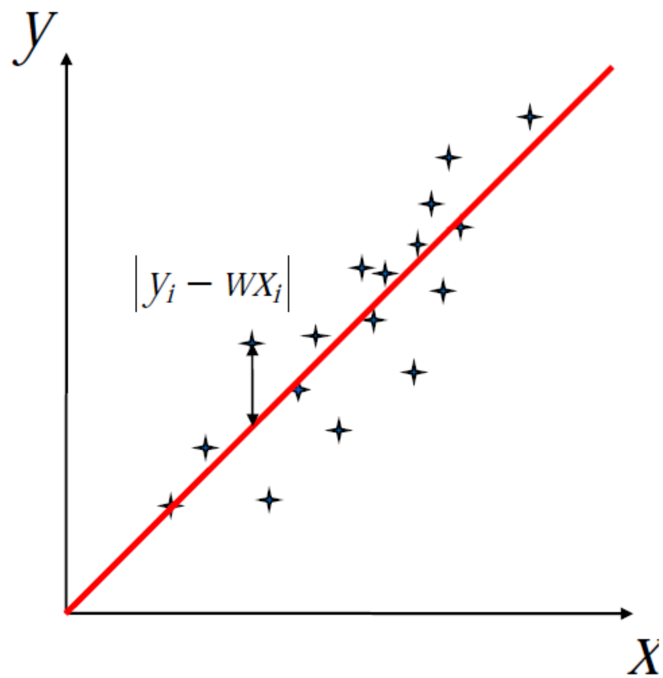

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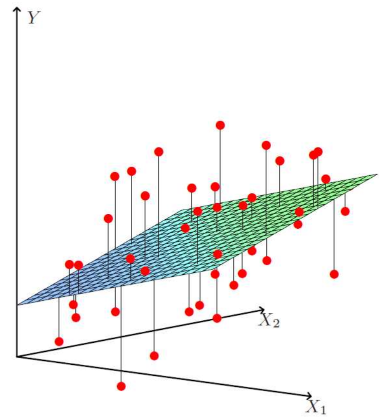
- Just requires a little bit of calculus to find it (take derivative, equate to zero..)

Linear Regression: In Higher Dimensions

- **Analogy to line fitting:** In higher dimensions, we will fit hyperplanes
- For 2-dim. inputs, linear regression fits a 2-dim. plane to the data



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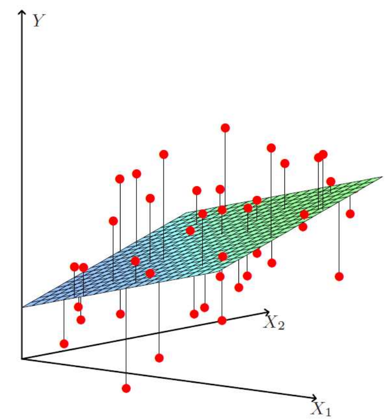

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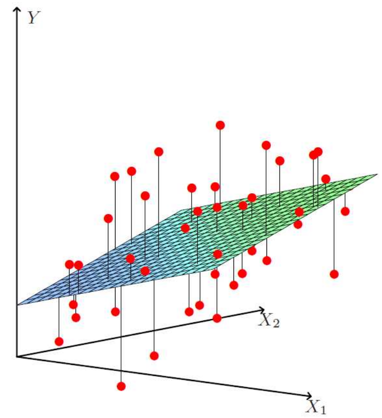

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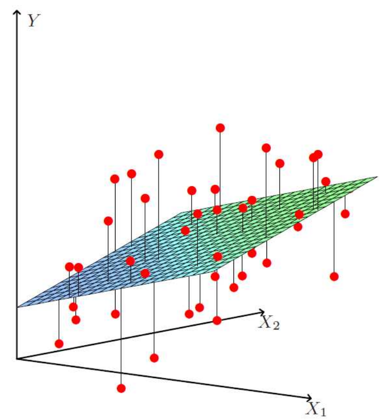

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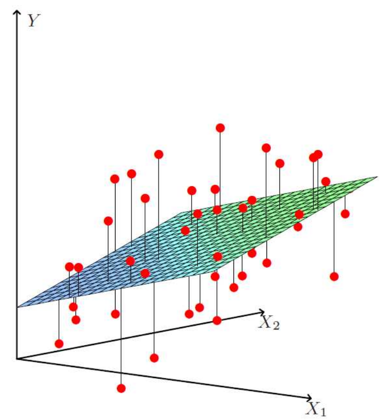

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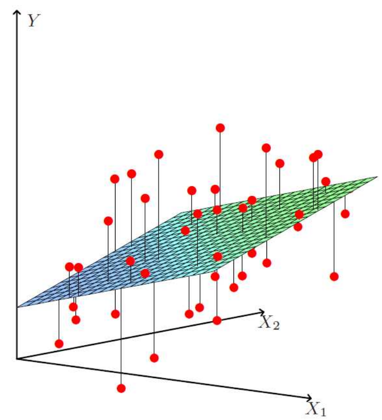

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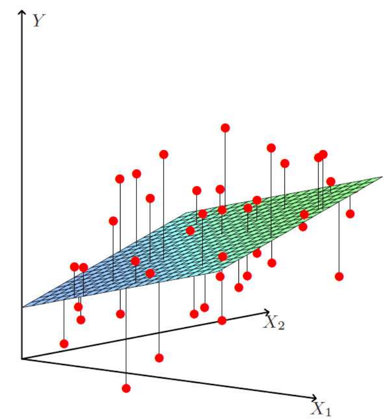

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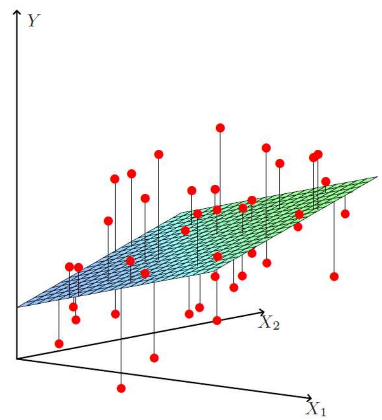

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- The hyperplane is defined by parameters $\mathbf{w}$ (a $D \times 1$ weight vector)

Linear Regression: In Higher Dimensions (Formally)

- Given training data $\mathcal{D} = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_N, y_N)\}$
- Inputs $\mathbf{x}_i$: D-dimensional vectors ($\mathbb{R}^D$), responses y_i: scalars ($\mathbb{R}$)

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- Each ϕ_j is called a basis function
  - Allows change of representation of the input $\mathbf{x}$ (often desired)

Linear Regression: In Higher Dimensions

The linear model:

$$y = b + \sum_{j=1}^{M} w_j \phi_j(\mathbf{x}) = b + \mathbf{w}^T \phi(\mathbf{x})$$

- $\phi = [\phi_1, \ldots . \phi_M]$
- $\mathbf{w} = [w_1, \ldots, w_M]$, the **weight vector** (to learn using the training data)

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- **Note:** Nonlinear relationships between $\mathbf{x}$ and y can be modeled using suitably chosen ϕ_j's (more when we cover Kernel Methods)

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- Given training data $\mathcal{D} = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_N, y_N)\}$
- Fit each training example $(\mathbf{x}_i, y_i)$ using the linear model

$$y_i = b + \mathbf{w}^T \mathbf{x}_i$$

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- A bit of notation abuse: write $\mathbf{w} = [b, \mathbf{w}]$, write $\mathbf{x}_i = [1, \mathbf{x}_i]$

$$y_i = \mathbf{w}^T \mathbf{x}_i$$