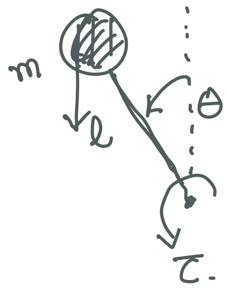
INVERTED PENDULUM.



$$\langle S, A, T, C \rangle$$

$$x = \begin{bmatrix} \theta \\ \dot\theta \end{bmatrix} \quad u = [\tau]$$

$$x_t, u_t \longrightarrow x_{t+1}.$$

DYNAMICS

$$\tau = I\ddot\theta$$

$$mgl\sin\theta + \tau = ml^2\ddot\theta$$

$$\ddot\theta = \frac{g}{l}\sin\theta + \frac{\tau}{ml^2} \approx \frac{g}{l}\theta + \frac{\tau}{ml^2}$$

$$x_{t+1} = f(x_t, u_t)$$

$$\ddot\theta = \frac{g}{l}\theta + \frac{\tau}{ml^2}$$

$$\dot\theta_{t+1} = \dot\theta_t + \ddot\theta_t \cdot \Delta = \dot\theta_t + \Delta\left(\frac{g}{l}\theta + \frac{\tau}{ml^2}\right)$$

$$\theta_{t+1} = \theta_t + \dot\theta_t \cdot \Delta$$

$$\underset{2\times1}{\begin{bmatrix} \theta_{t+1} \\ \dot\theta_{t+1} \end{bmatrix}} = \underset{2\times2}{\begin{bmatrix} 1 & \Delta \\ \frac{\Delta g}{l} & 1 \end{bmatrix}} \underset{2\times1}{\begin{bmatrix} \theta_t \\ \dot\theta_t \end{bmatrix}} + \underset{2\times1}{\begin{bmatrix} 0 \\ \frac{\Delta}{ml^2} \end{bmatrix}} \tau$$

$$\boxed{x_{t+1} = A x_t + B u_t}$$

LINEAR

COST

$$w_1\theta^2 + w_2\dot\theta^2 + w_3\tau^2 \quad c(x_t, u_t)$$

$$c(x_t, u_t) = \boxed{x_t^T Q x_t + u_t^T R u_t}$$

QUADRATIC

$$[\theta \ \dot{\theta}] \begin{bmatrix} w_1 & 0 \\ 0 & w_2 \end{bmatrix} \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} + \tau \, w_3 \, \tau$$

GOAL: ANALYTIC VALUE ITERATION

$$V^*(x_t) = \min_{u_t} \left[c(x_t, u_t) + V^*(x_{t+1}) \right]$$

START FROM LAST TIMESTEP $T-1$

$$V^*(x_{T-1}) = \min_{u_{T-1}} \left[c(x_{T-1}, u_{T-1}) + 0 \right]$$

$$= \min_{u_{T-1}} \left[x_{T-1}^T Q x_{T-1} + u_{T-1}^T R u_{T-1} \right]$$

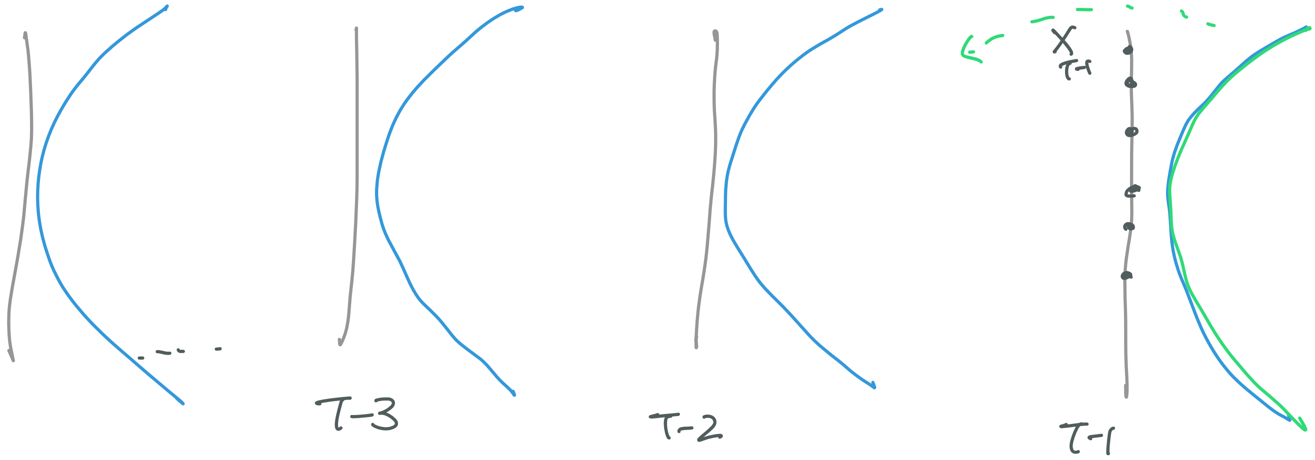
$$\frac{\partial}{\partial u_{T-1}} (\cdot) = 0 \Rightarrow 2 u_{T-1}^T R = 0 \Rightarrow u_{T-1} = 0$$

$$= x_{T-1}^T Q x_{T-1}$$ → quadratic!

LQR TRICK

① Show that value function $V^*(\cdot)$ is a quadratic at timestep $T-1$ ✓

② If $V^*(x_{t+1})$ is a quadratic, then show $V^*(x_t)$ must also be a quadratic.



$$V^*(x_t) = \min_{u_t} \Big[c(x_t, u_t) + \underbrace{V^*(x_{t+1})}_{\text{A QUADRATIC}} \Big]$$

$$V^*(x_{t+1}) := x_{t+1}^T V_{t+1} x_{t+1}$$

$$= \min_{u_t} \Big[x_t^T Q x_t + u_t^T R u_t + x_{t+1}^T V_{t+1} x_{t+1} \Big]$$

$$x_{t+1} = Ax_t + Bu_t$$

$$= \min_{u_t} \Big[x_t^T Q x_t + u_t^T R u_t + (Ax_t + Bu_t)^T V_{t+1} (Ax_t + Bu_t) \Big]$$

$$\frac{\partial}{\partial u_t}(\cdot) = 0 \Rightarrow \boxed{\cancel{2} u_t^T R + \cancel{2}(Ax_t + Bu_t)^T V_{t+1} B = 0}$$

$$R^T u_t + B^T V_{t+1}^T (Ax_t + Bu_t) = 0$$

$$\left(R^T + B^T V_{t+1}^T B\right) u_t = -B^T V_{t+1}^T A x_t$$

$$u_t = -\left(R^T + B^T V_{t+1}^T B\right)^{-1} B^T V_{t+1}^T A x_t$$

$$= \underbrace{-\left(R + B^T V_{t+1} B\right)^{-1} B^T V_{t+1} A}_{} \; x_t$$

$$= K_t \cdot x_t$$

$$V^*(x_t) = x_t^T \Big(\underbrace{Q + K_t^T R K_t + (A + B K_t)^T V_{t+1} (A + B K_t)}_{V_t} \Big) x_t$$