Fourier Transforms

Fourier transform for 1D images

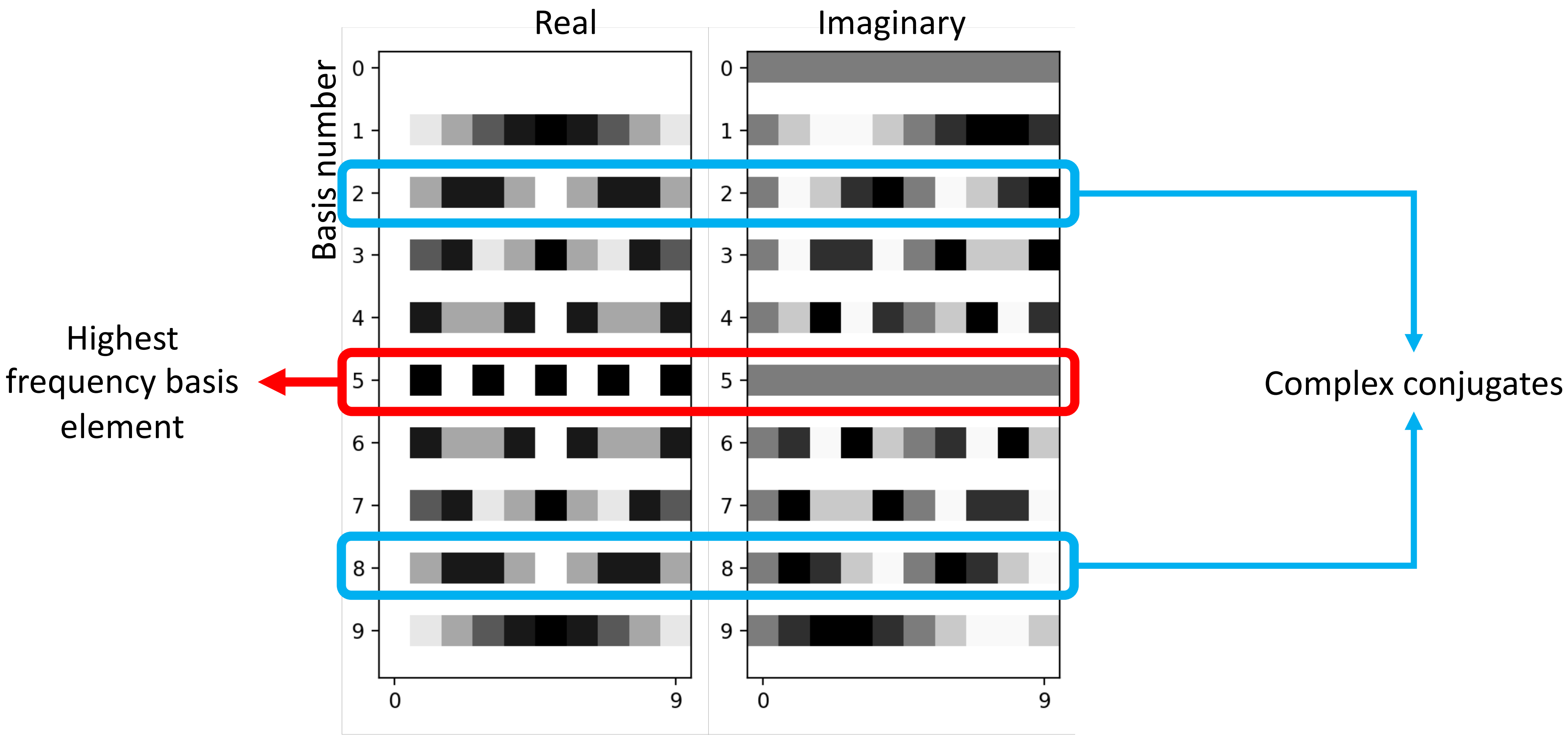
- A 1D image with N pixels is a vector of size N
- Every basis has N pixels
- There must be N basis elements
- n-th element of k-th basis in standard basis
  - $E_k(n) = \begin{cases} 1 & if\ k = n \\ 0 & otherwise \end{cases}$
- n-th element of k-th basis in Fourier basis
  - $B_k(n) = e^{\frac{i2\pi kn}{N}}$

A nuance

- $B_k(n) = e^{\frac{i2\pi kn}{N}}$
- We previously said as k increases, frequency increases
  - i.e., more cycles within N

- $B_{N-k}(\mathrm{n}) = e^{\frac{i2\pi(N-k)n}{N}} = e^{i2\pi n - \frac{i2\pi kn}{N}} = e^{-\frac{i2\pi kn}{N}} = \overline{B_k(n)}$ (complex conjugate)
- Frequency increases till N/2, subsequent basis elements are *complex conjugates of previous elements*

The full Fourier basis (N=10)



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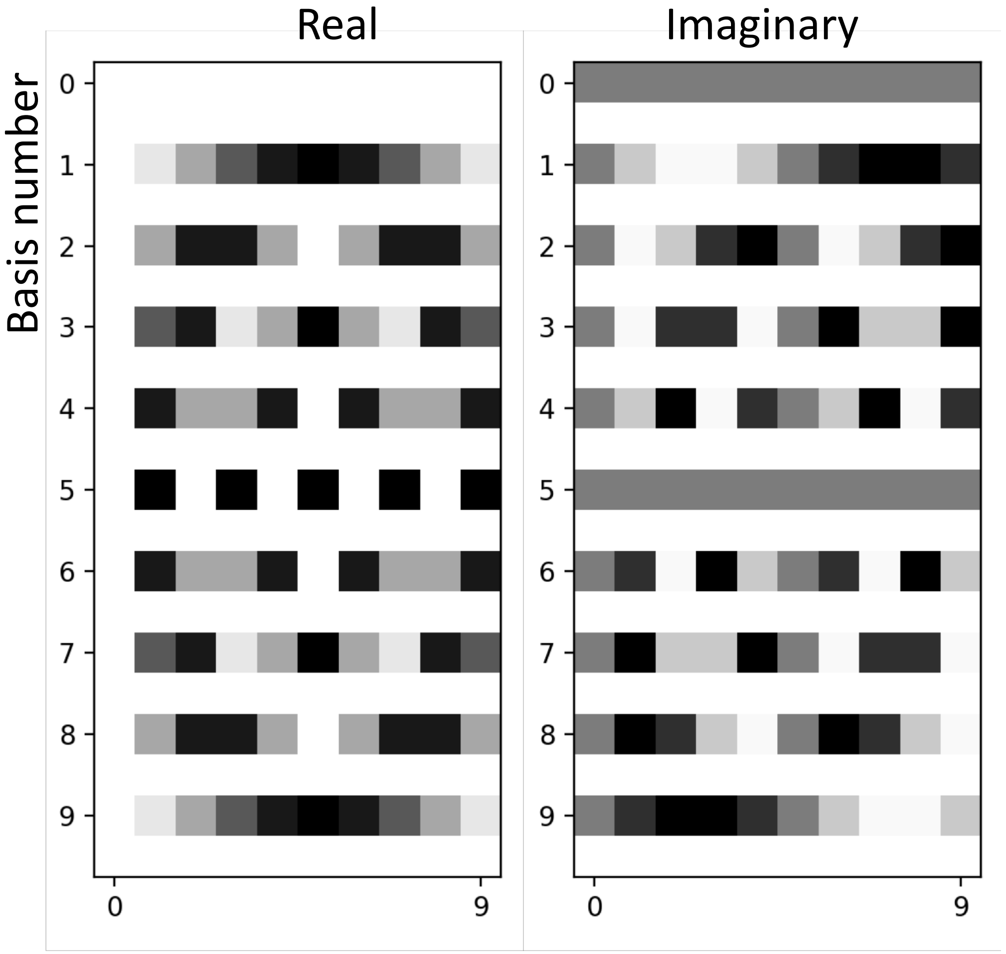

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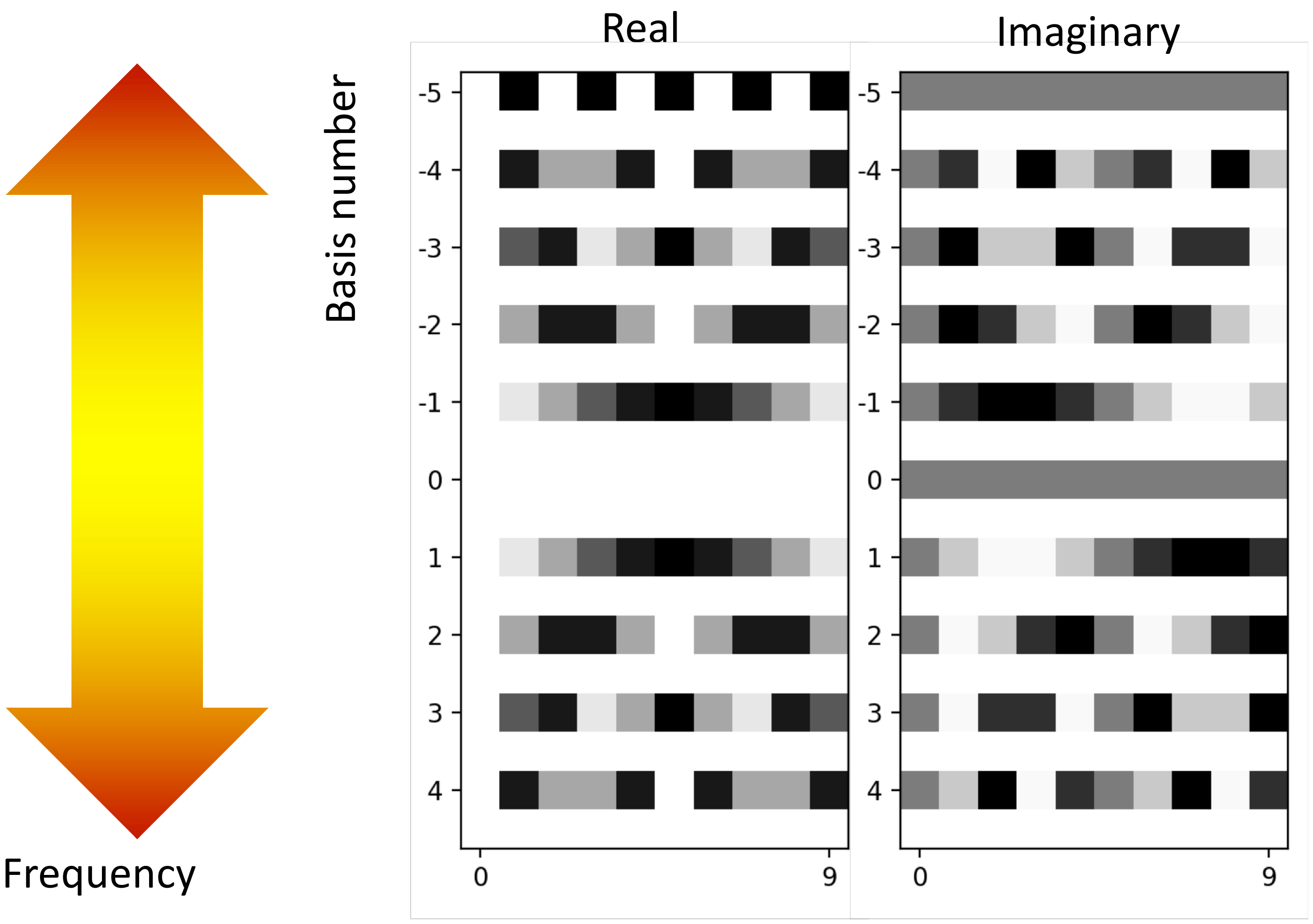
- Instead of considering B_0 to B_{N-1} as basis, often consider $B_{-\frac{N}{2}}$ to $B_{\frac{N}{2}-1}$ as the basis
  - For odd N, use $B_{-(N-1)/2}$ to $B_{(N-1)/2}$

The full Fourier basis (N=10)



Fourier transform for 1D images

- Fourier transform converts from standard basis to Fourier basis
- If $\boldsymbol{x}$ is image in standard basis, and $\boldsymbol{X}$ is representation in Fourier basis, then:
  - $X(k) = \sum_n x(n)\, e^{-\frac{i2\pi kn}{N}}$
  - Or $\boldsymbol{X} = \boldsymbol{B}^*\boldsymbol{x}$ where $\boldsymbol{B}^*$ is a matrix with entries $B^*(k,n) = e^{-\frac{i2\pi kn}{N}}$

Inverse Fourier transform

- Given Fourier transform of image how should we get back image?
- In other words how should we change the basis back to the original coordinates?
- $\boldsymbol{x} = \sum_k X(k)\boldsymbol{B}_k$
- $x(n) = \sum_k X(k) B_k(n) = \sum_k X(k) e^{\frac{i2\pi kn}{N}}$

Real images in the Fourier basis

- Basis is complex but images are real
- Combine a pair of conjugate basis elements
- Coefficients for B_k and B_{-k} will be the same
  - $X(k) = X(-k)$
- So only need to know first $\frac{N}{2}$ Fourier coefficients!

The 0-th basis

- $B_k(n) = e^{\frac{i2\pi kn}{N}}$
- $B_0(n) = e^{\frac{i2\pi 0n}{N}} = 1$
- The zero-th basis element is a constant image
- Every other basis element varies between 1 and -1, so averages out to 0
- Thus, *any image with a non-zero average intensity must have a high zero-th coefficient*

Fourier transform for 2D images

- Images are 2D arrays
- Fourier basis for 1D array indexed by frequencies
- Fourier basis for 2D arrays are indexed by 2 spatial frequencies
- (i,j)th Fourier basis for N x N image
  - Has period N/i along x
  - Has period N/j along y
- $B_{k,l}(x,y) = e^{\frac{2\pi ikx}{N} + \frac{2\pi ily}{N}}$

$$= \cos\left(\frac{2\pi kx}{N} + \frac{2\pi ly}{N}\right) + i\sin\left(\frac{2\pi kx}{N} + \frac{2\pi ly}{N}\right)$$

Visualizing the Fourier basis for images

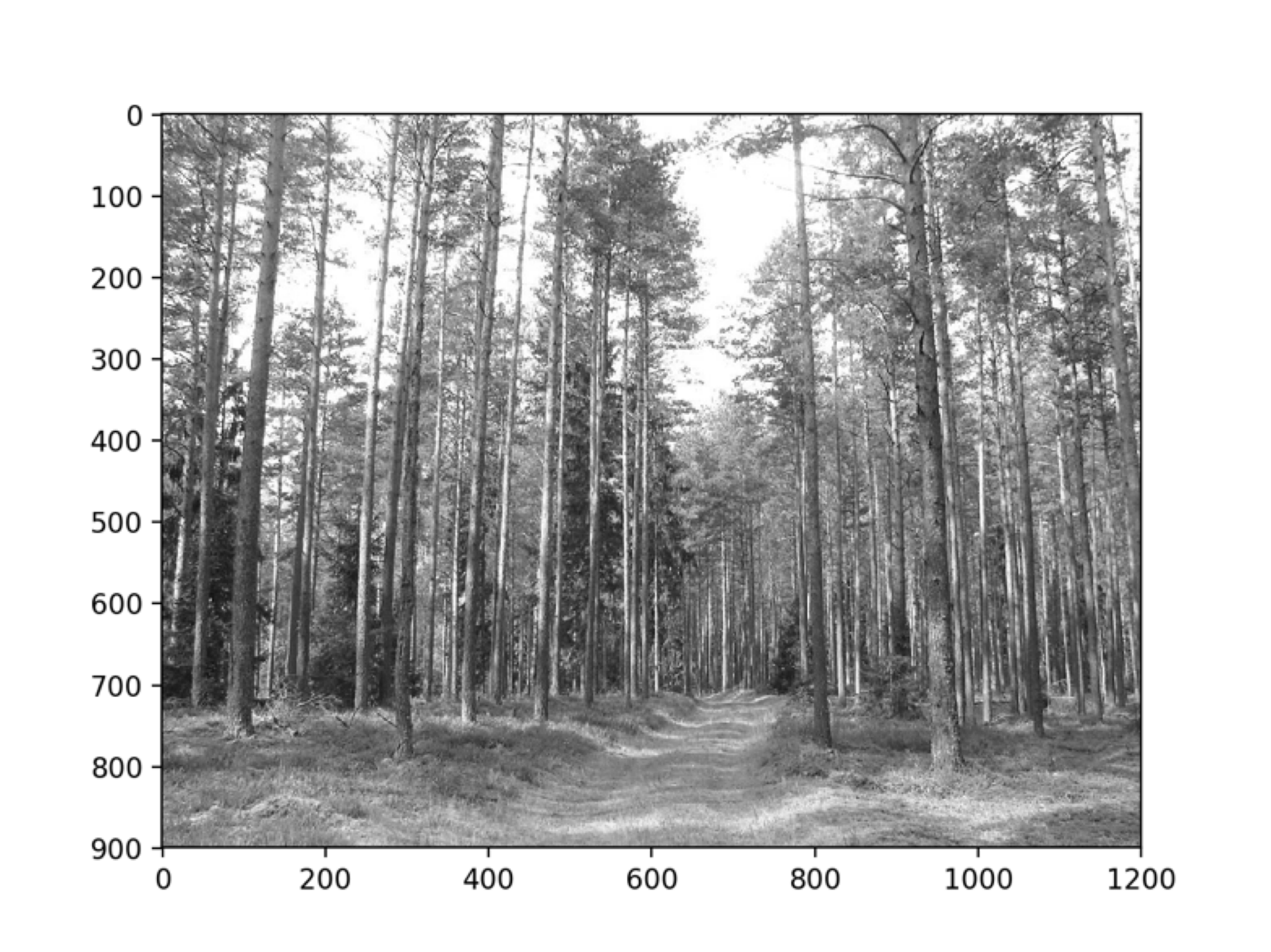
$B_{1,1}$

$B_{3,20}$

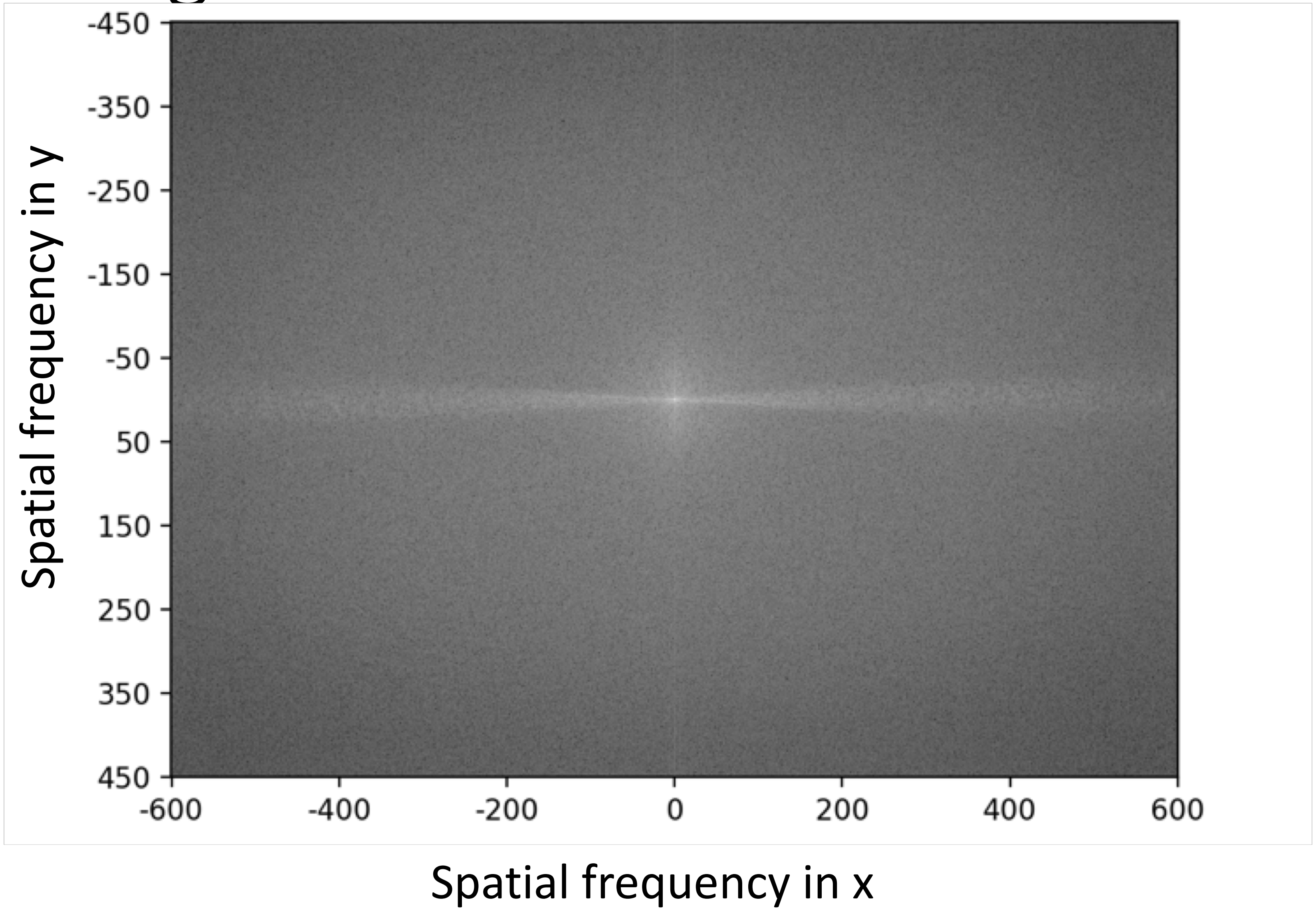
Visualizing the Fourier transform

- Fourier coefficients are *complex*
- Instead of visualizing complex numbers we look at the squared absolute value $|X(k,l)|^2$
- This is called the *power spectrum*
- There are $N \times N$ Fourier coefficients, so we can show this as an $N \times N$ image.
- Because of complex conjugates only the first $\frac{N}{2} \times \frac{N}{2}$ coefficients are unique
- Because of very high values display using log

Visualizing the Fourier transform

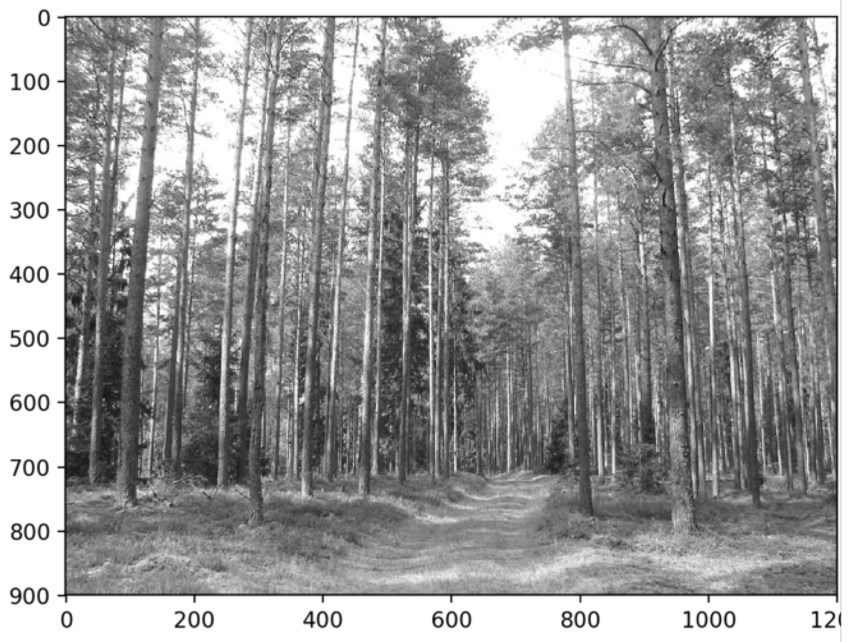


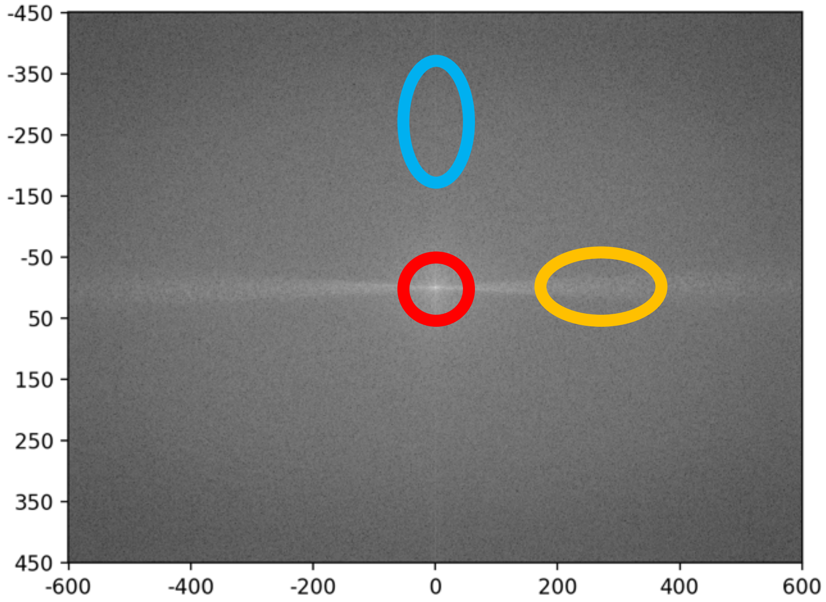
Visualizing the Fourier transform



Visualizing the Fourier transform

- High peak for $B_{0,0}$. Why?
- High values on the X axis and low values on the Y axis. Why?

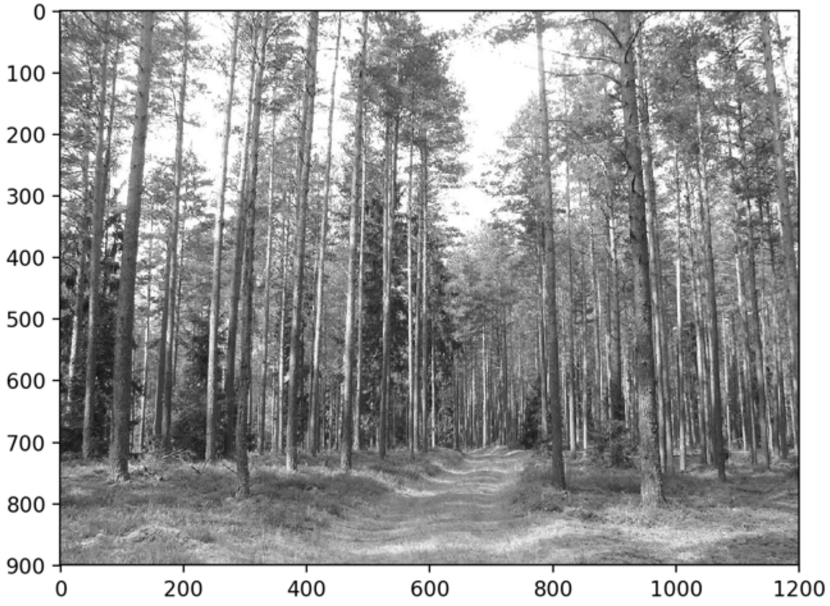


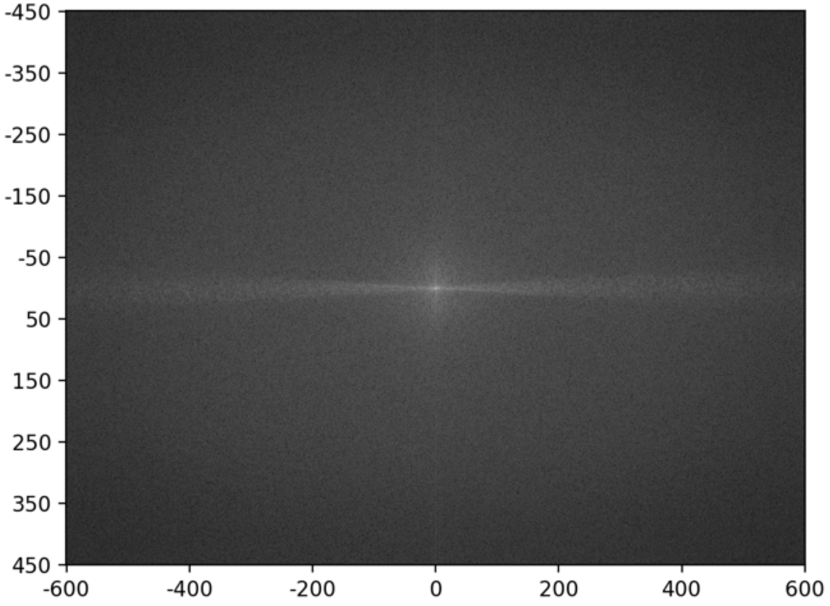


Why Fourier transforms?

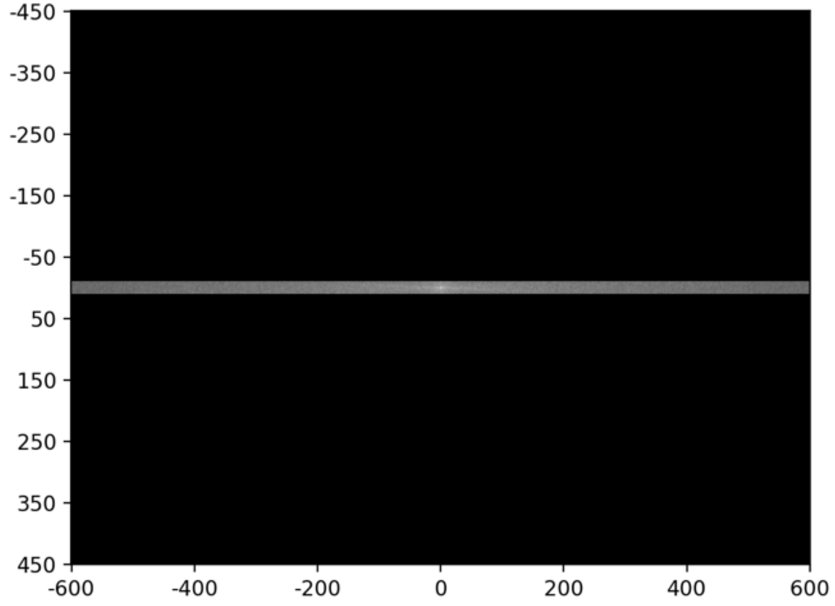
- Think of image in terms of low and high frequency information
- Low frequency: large scale structure, no details
- High frequency: fine structure

Why Fourier transforms?

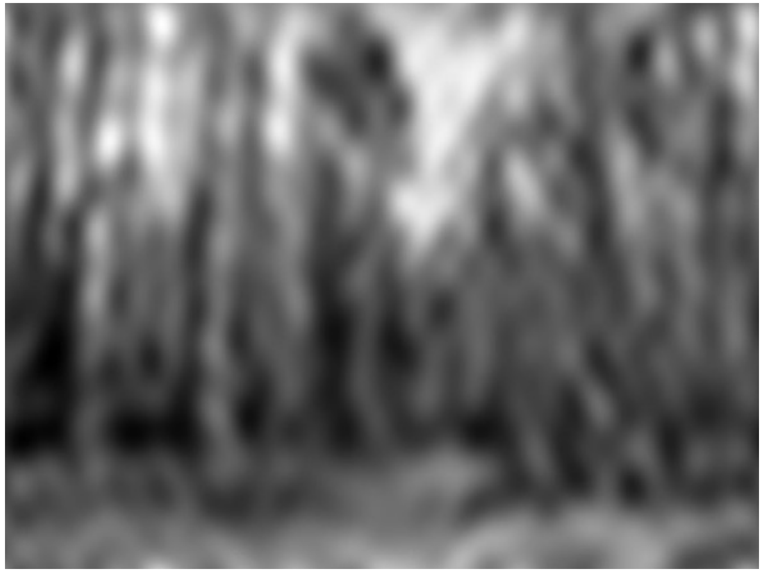


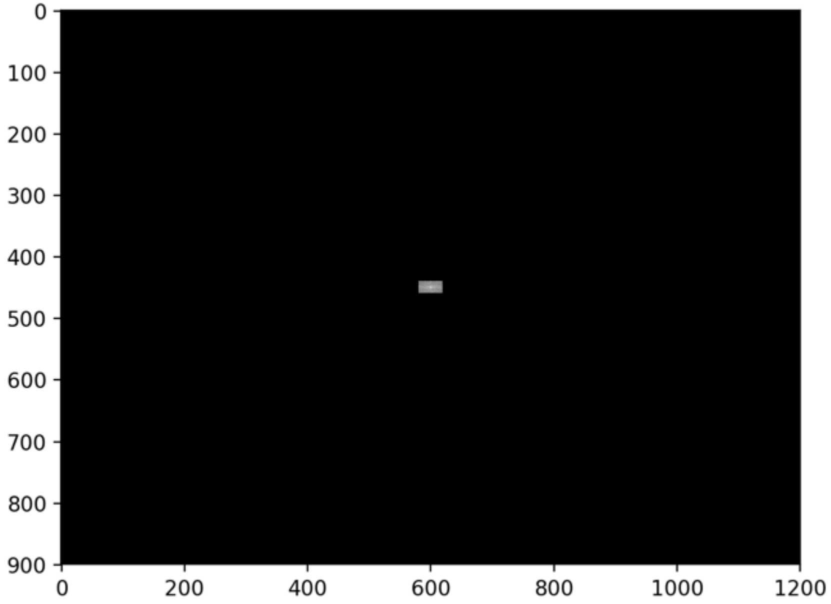


What if we zero out all high y-frequency components?



What if we zero out *all* high frequencies?





Removing high frequency components looks like Gaussian / mean filtering. Is there more to this relationship?

Dual domains

- Image: Spatial domain

- Fourier Transform: Frequency domain
  - Amplitudes are called spectrum

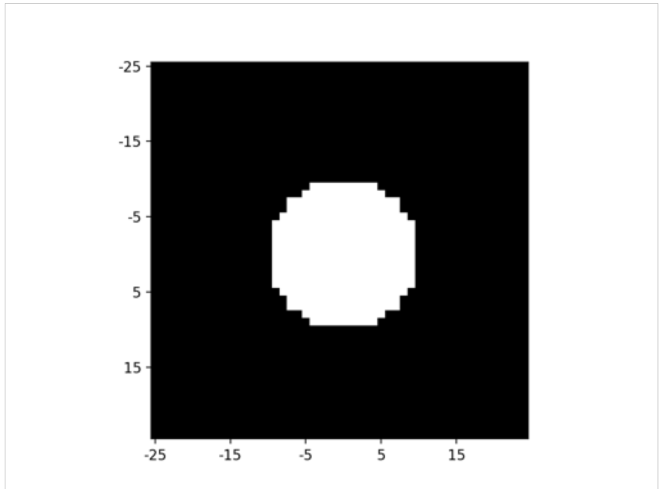
- For any transformations we do in spatial domain, there are corresponding transformations we can do in the frequency domain
- *And vice-versa*

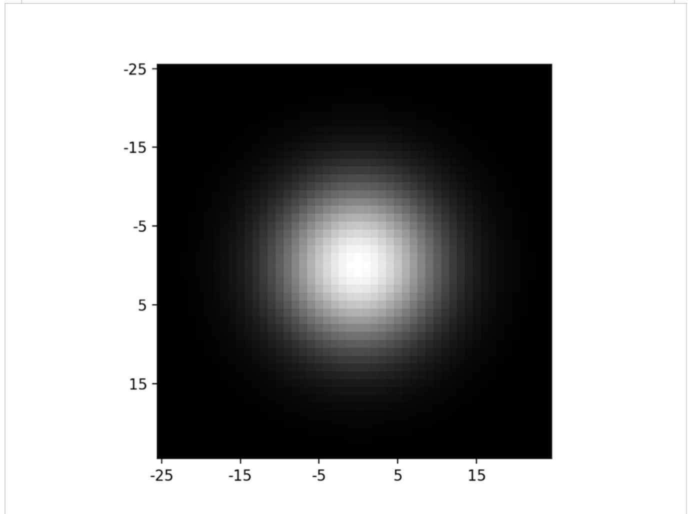
Dual domains

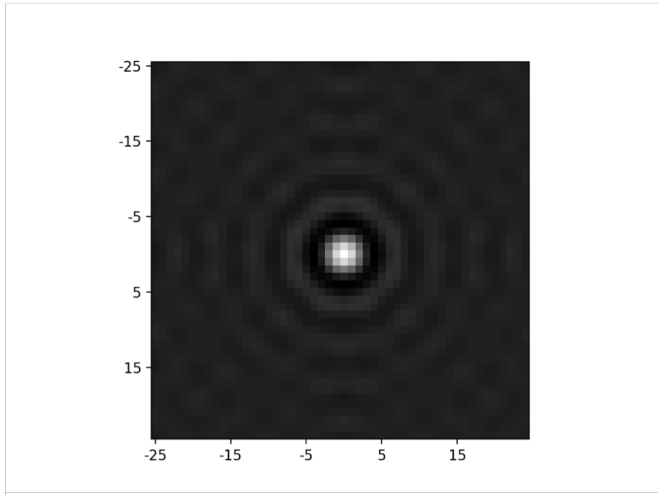
- *Convolution* in spatial domain = *Point-wise multiplication* in frequency domain
  - Suppose $h = f * g$
  - Suppose H is the Fourier transform of h, similarly F and G
  - Then $H(k,l) = F(k,l)G(k,l)$
- *Convolution* in frequency domain = *Point-wise multiplication* in spatial domain
- To understand action of a filter, look at its *Fourier transform*

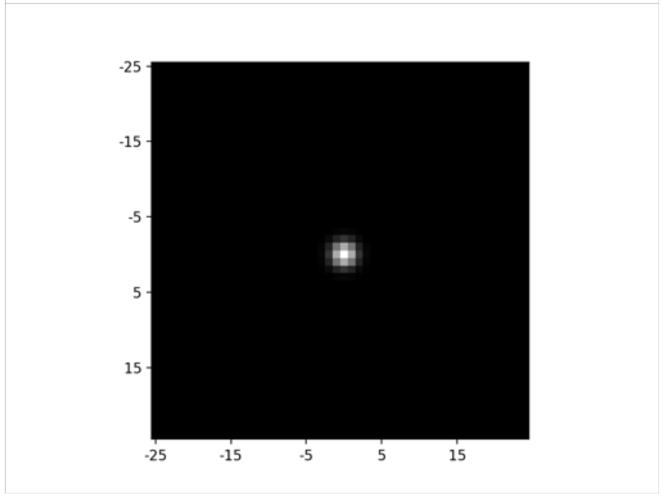
Filter

Fourier transform (Power spectrum)









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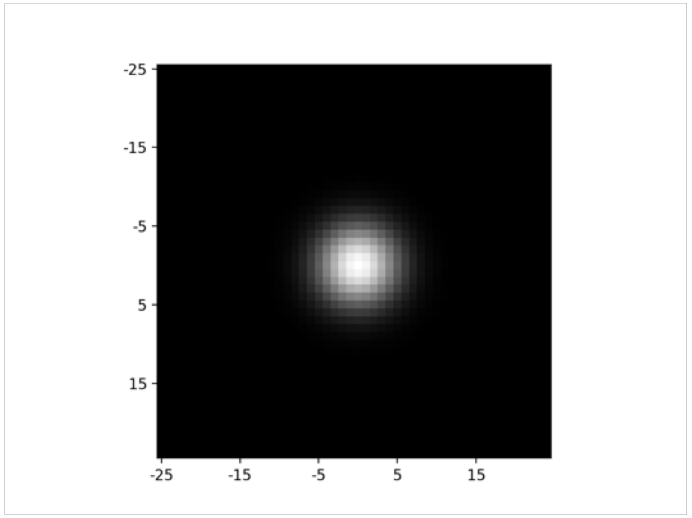

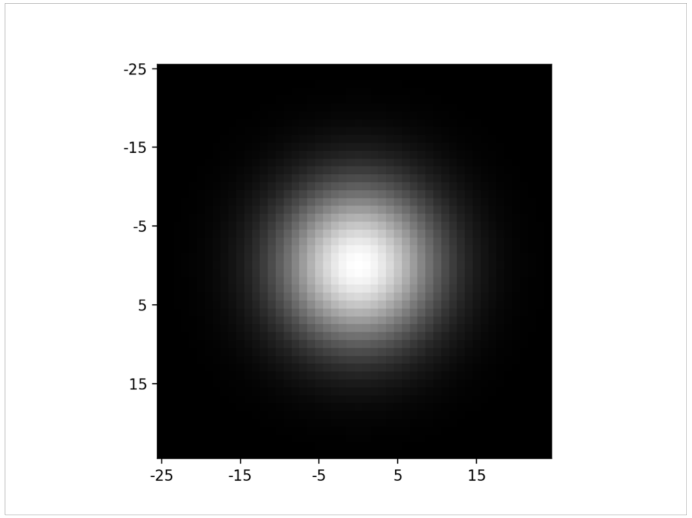

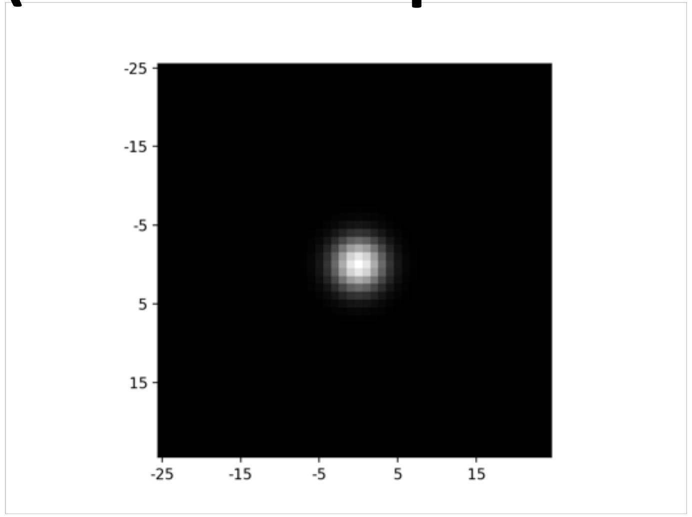

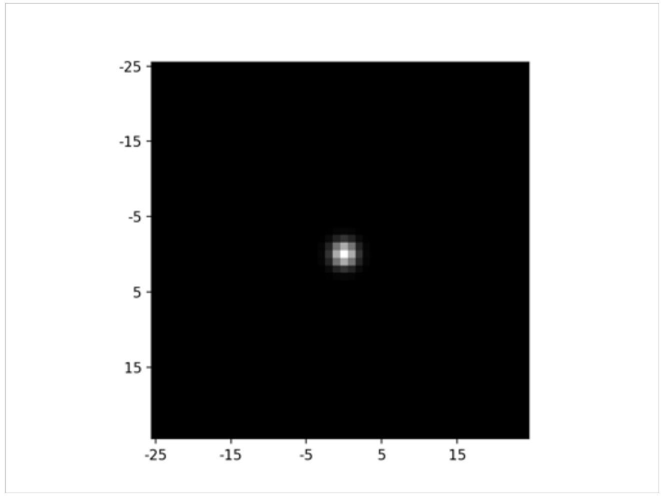

Gaussian filtering

- Fourier transform of a Gaussian *is another Gaussian!*
- So convolving with a Gaussian in spatial domain = multiplying with Gaussian in frequency domain
  - High frequencies get zeroed out
- Higher the standard deviation in spatial domain = lower the std in frequency domain
  - More frequencies get zeroed out.