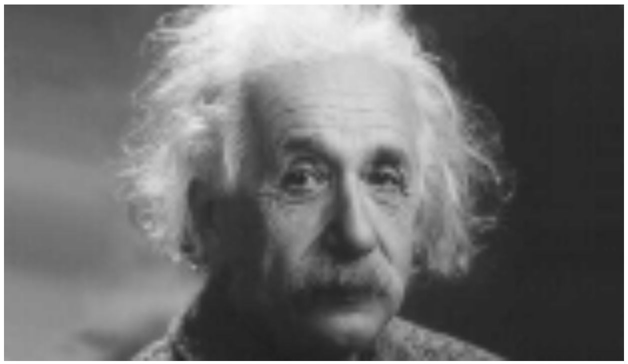
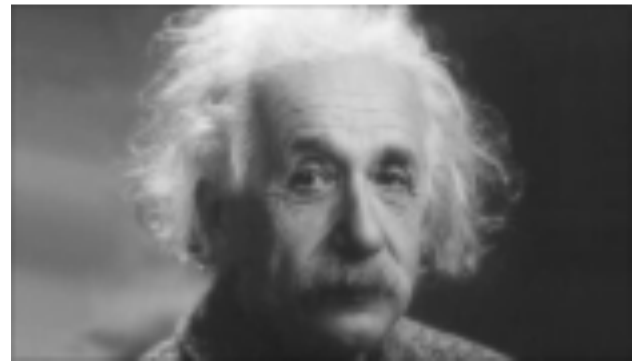
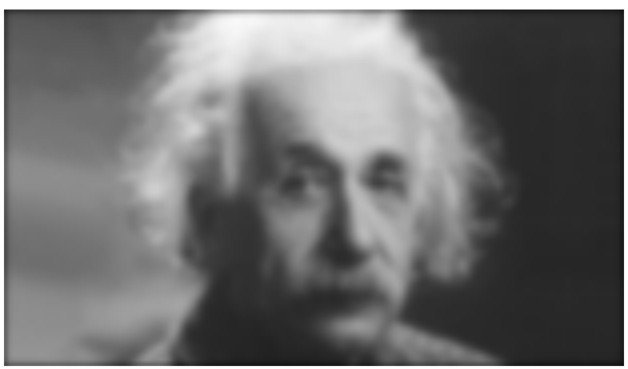
Fourier transforms

Gaussian filter

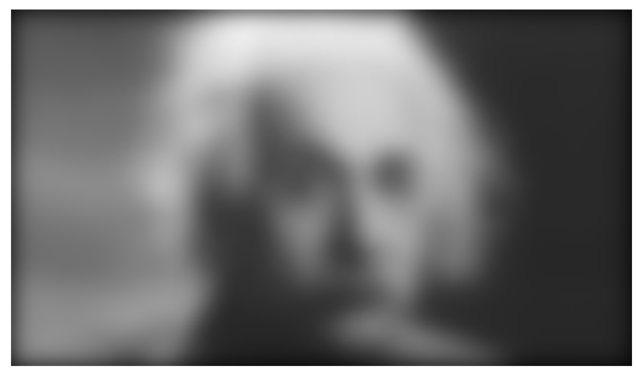




21x21, σ=0.5



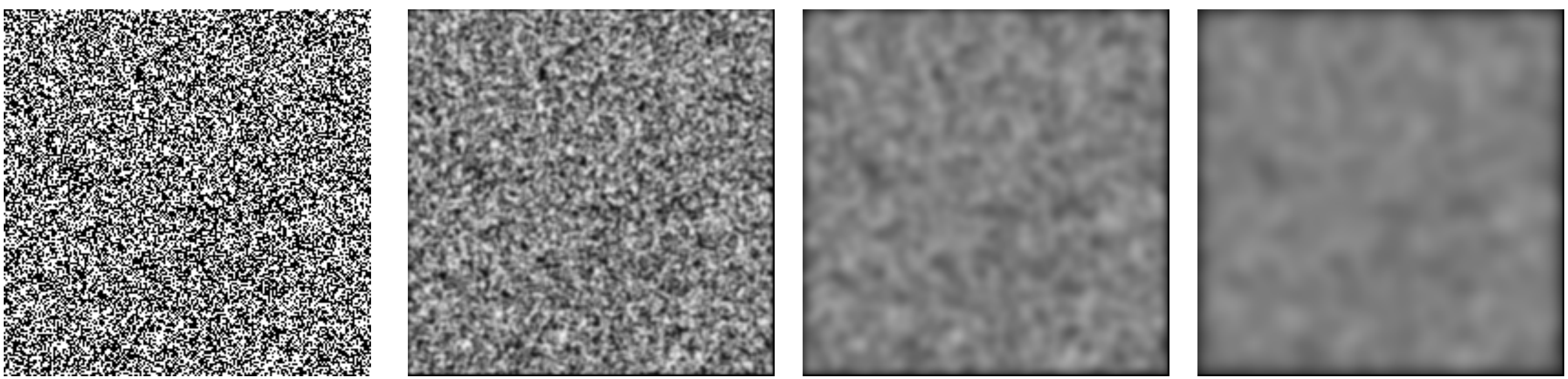
21x21, σ=1



21x21, σ=3

“Blurriness” / “Smoothness”

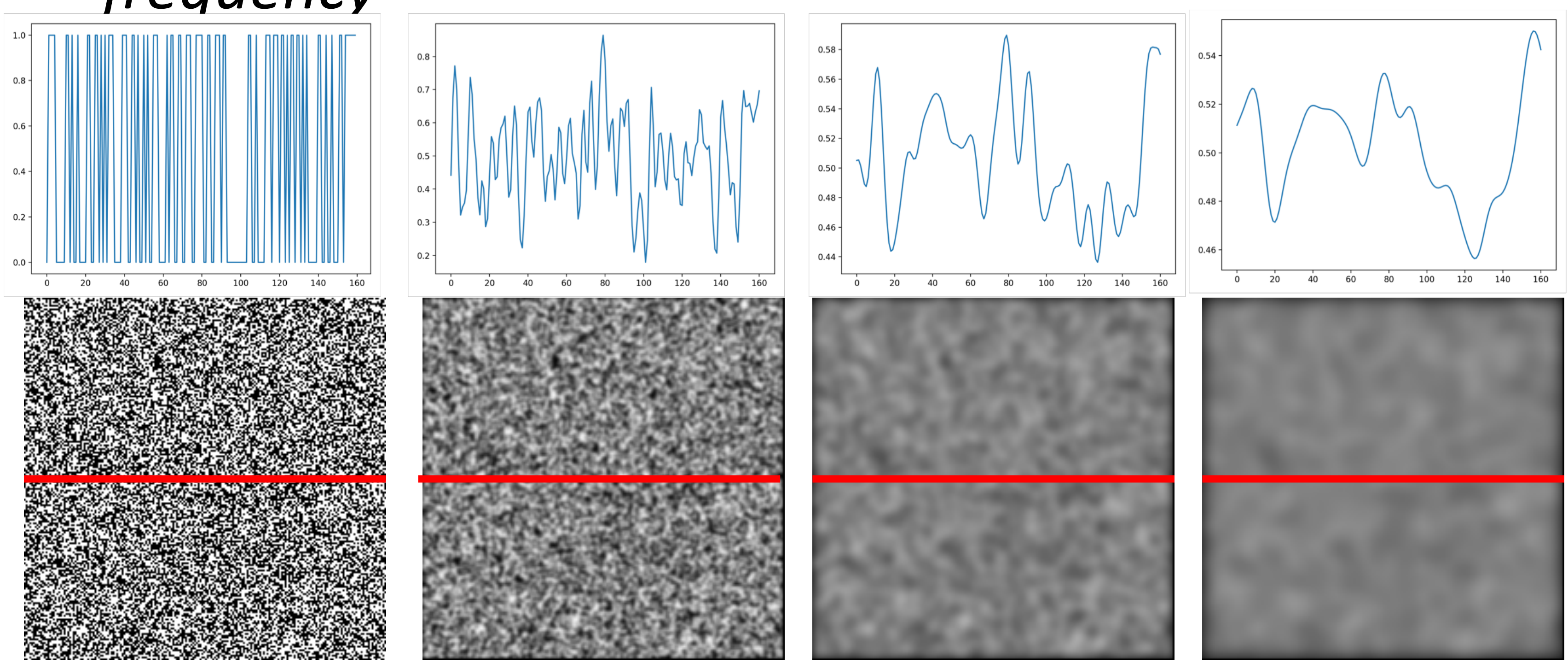
- Gaussian filters with increasing sigma make image “blurrier”



Input $\sigma = 1$ $\sigma = 3$ $\sigma = 5$

Blurriness/smoothness

- In blurrier images, intensity changes more slowly with position
- An important concept in image processing: *frequency*



Input $\sigma = 1$ $\sigma = 3$ $\sigma = 5$

Frequency and the Fourier transform

Images as vector space

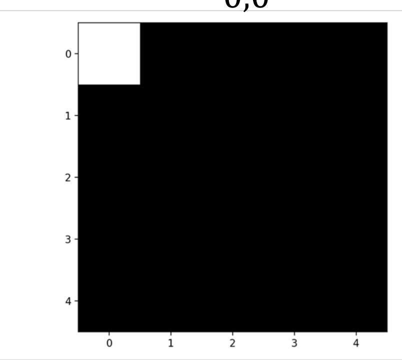
- Images of given size are a vector space
  - Usual definitions of addition and scalar multiplication
  - If I_1 and I_2 are images, so is $\alpha I_1 + \beta I_2$

- The dimensionality of the vector space of $m \times n$ images is mn
- This vector space has a *basis:* images B_i such that for any image I:
  - $I = \alpha_1 B_1 + \alpha_2 B_2 + \cdots + \alpha_{mn} B_{mn}$

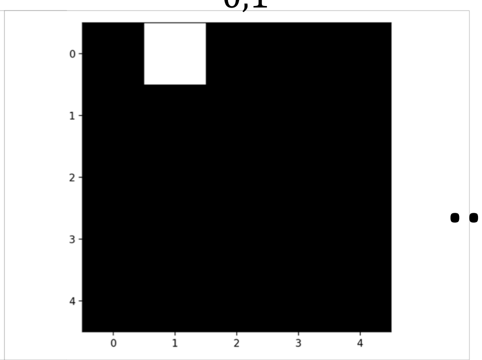
Basis for images

- Canonical basis:
  - Each canonical basis vector is an image with single pixel as 1
  - $\boldsymbol{f} = \sum_{x,y} f(x,y)\boldsymbol{B}_{x,y}$
  - Basis vectors for 5 x 5 images:

$B_{0,0}$

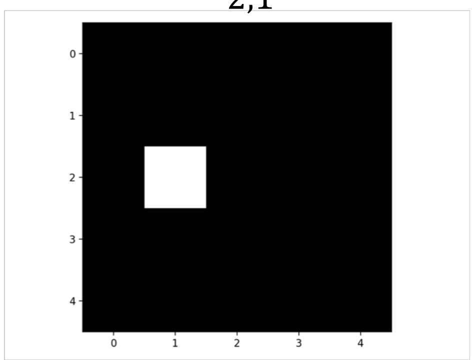


$B_{0,1}$



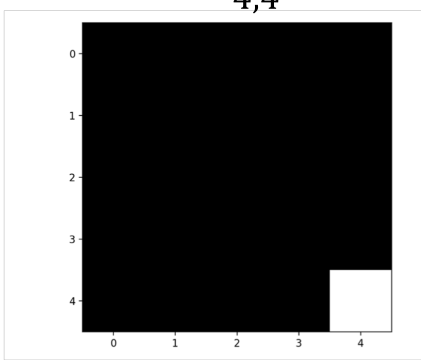
…

$B_{2,1}$



…

$B_{4,4}$

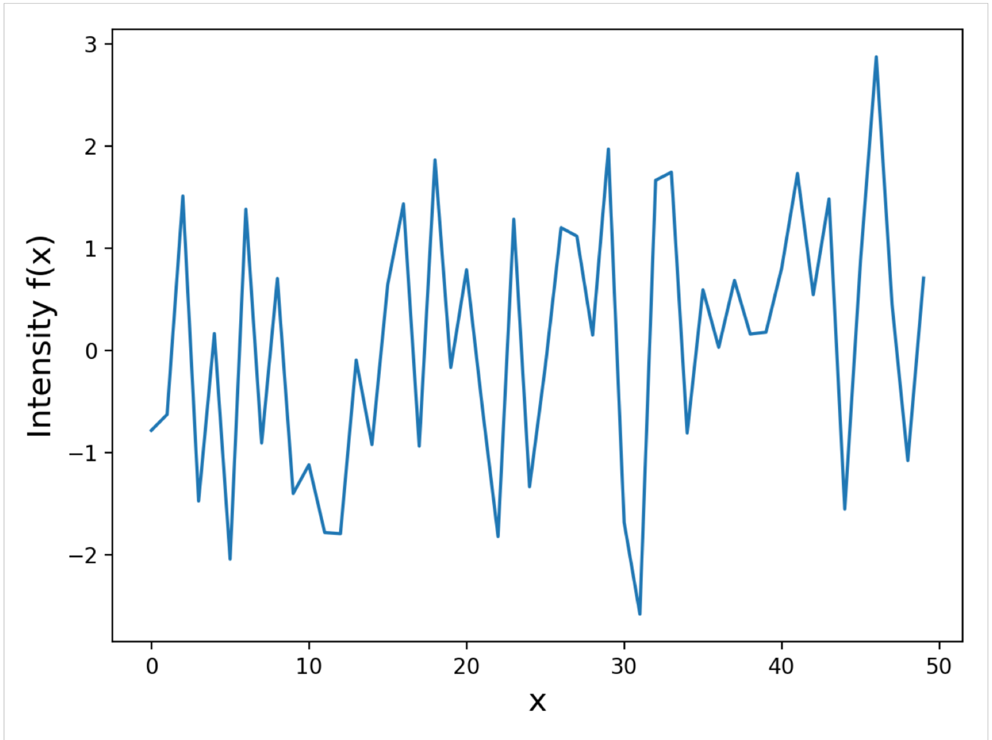


Basis for images

- Images (2D arrays of real numbers) satisfy definition of vector space.
- Canonical basis:
  - Each canonical basis vector is an image with single pixel as 1
- Alternate basis based on *frequency:* Fourier basis

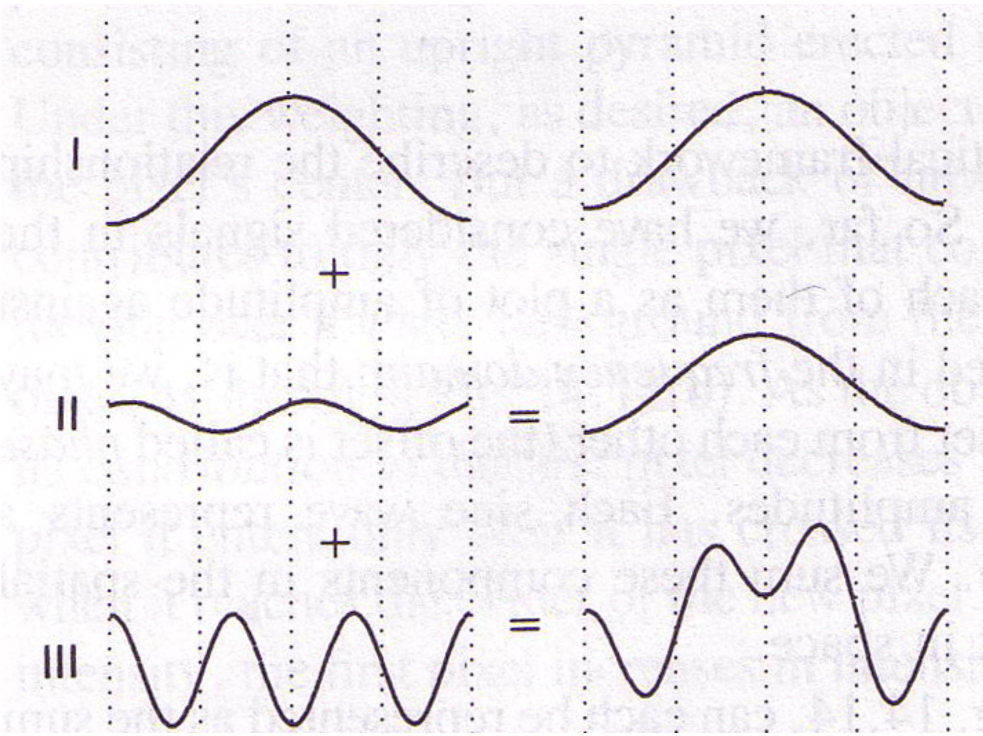
Fourier basis for $1 \times N$ images

- Consider images that are a single row of pixels
- Random image below:



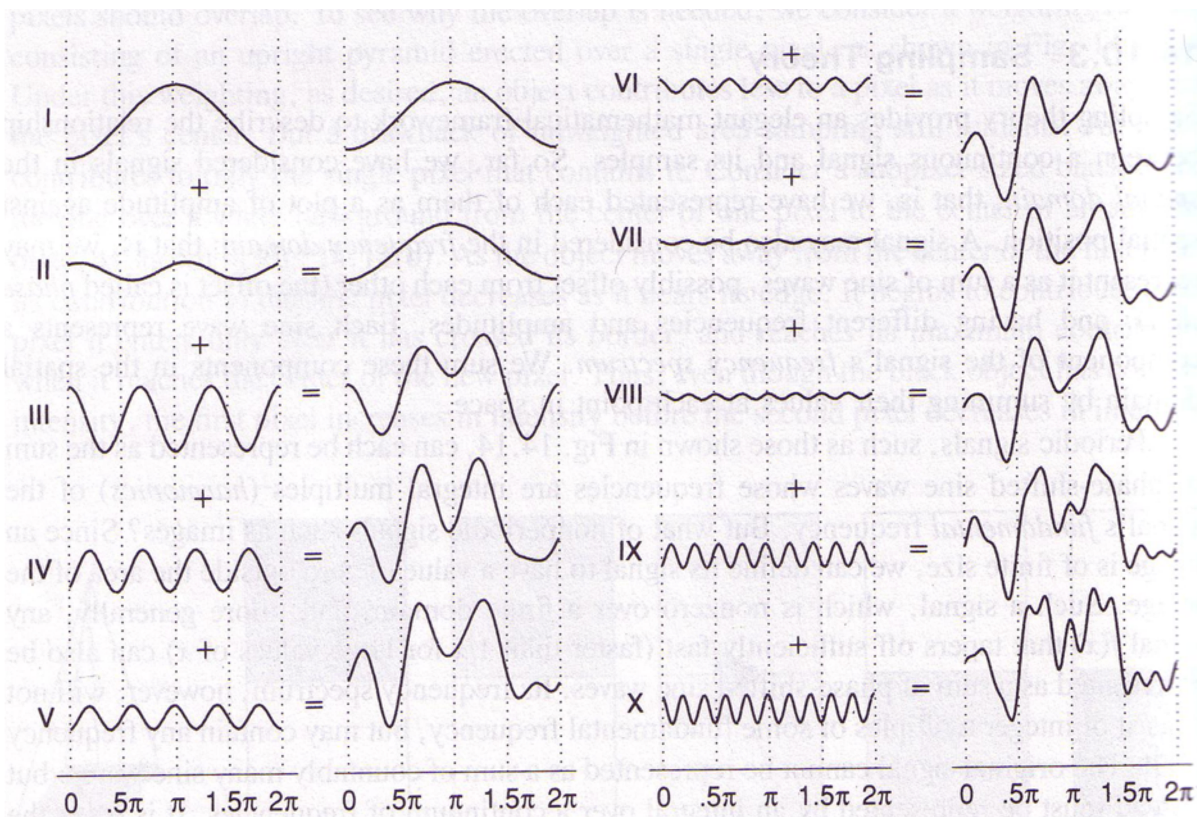
Key idea of Fourier basis

- Every such image (doesn't matter what it is) is a linear combination of sine/cosine waves



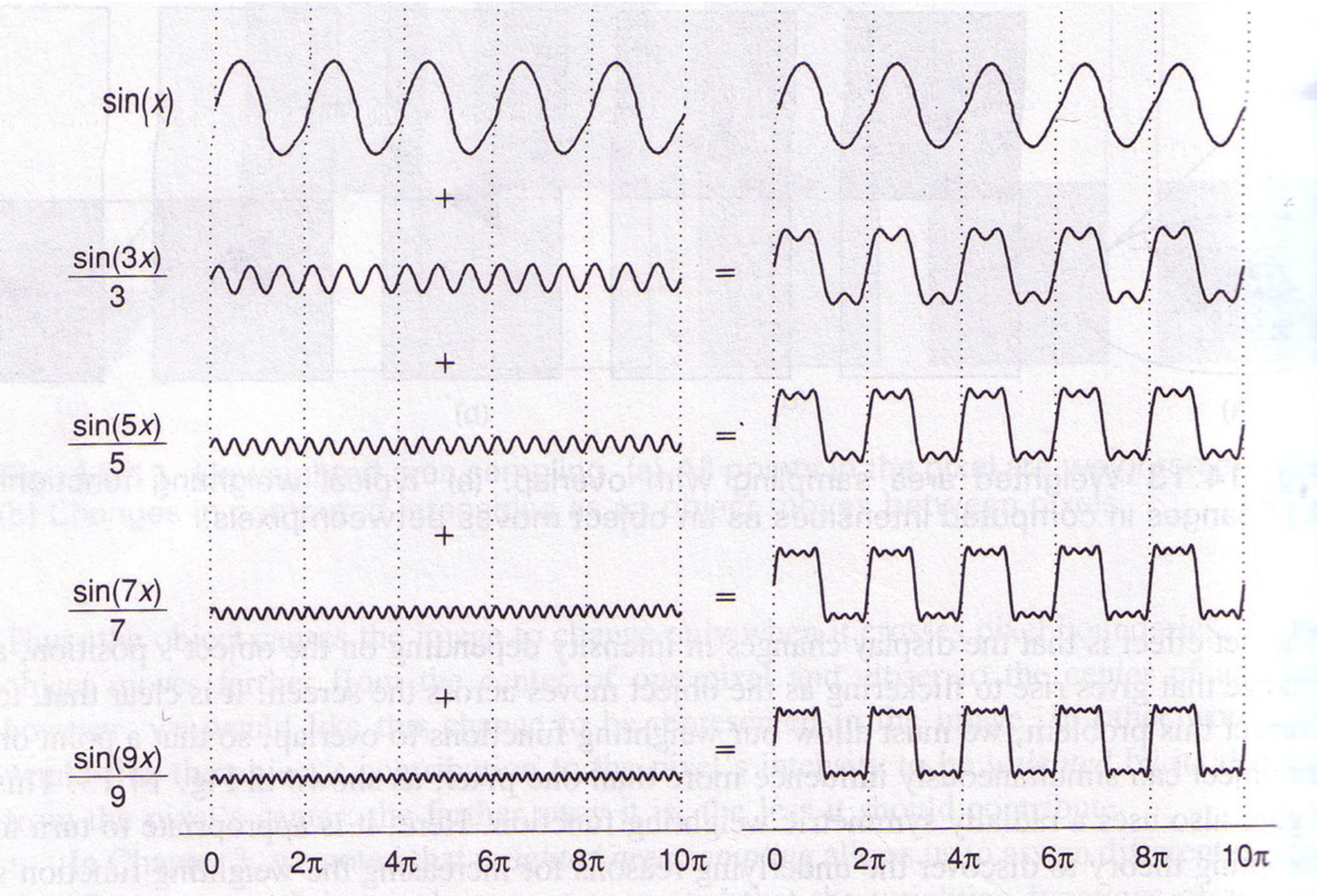
Key idea of Fourier basis

- Every such image (doesn't matter what it is) is a linear combination of sine/cosine waves



(b)

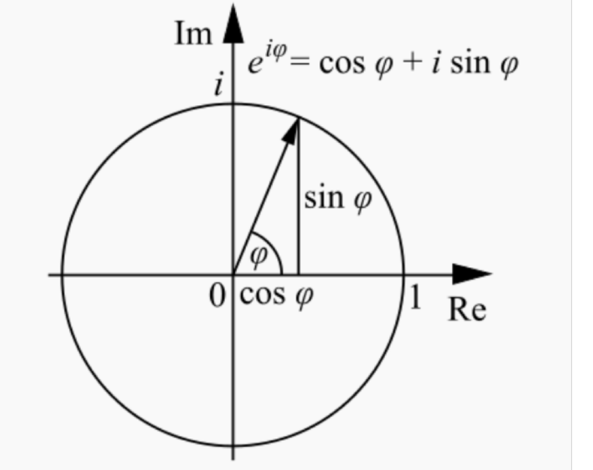
A box-like example



(a)

Fourier basis for $1 \times N$ images

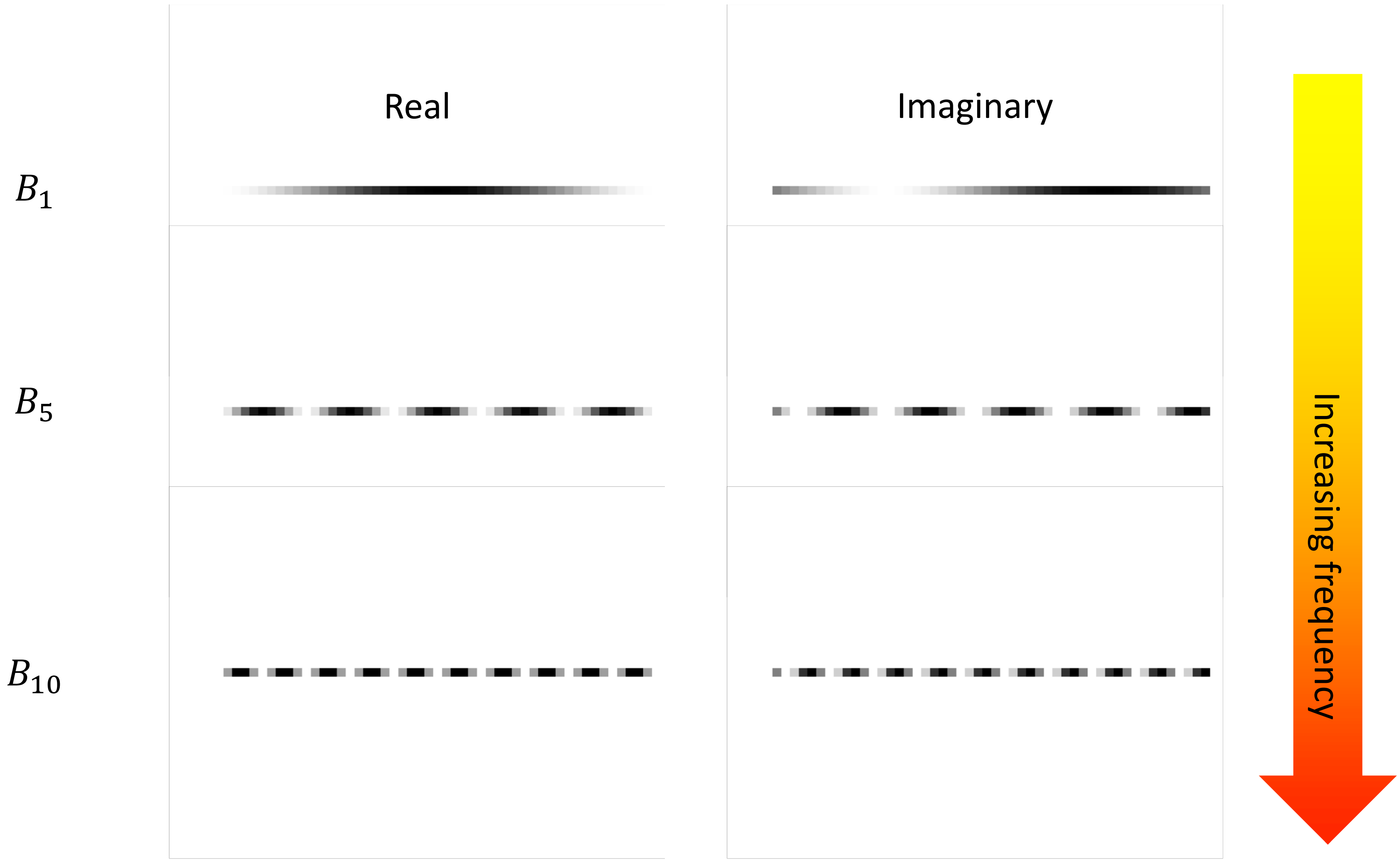
- Not exactly sines and cosines, but *complex* variants
- Euler's formula:
  - $e^{i\phi} = \cos\phi + i\sin\phi$
- k-th basis vector B_k is a $1 \times N$ image
- $B_k(x) = e^{\frac{i2\pi kx}{N}} = \cos\frac{2\pi kx}{N} + i\sin\frac{2\pi kx}{N}$



Understanding the k-th basis

- $B_k(x) = e^{\frac{i2\pi kx}{N}} = \cos\frac{2\pi kx}{N} + i\sin\frac{2\pi kx}{N}$
- Real part: $\cos\frac{2\pi kx}{N}$
- *Periodic in x. Period?*
- $B_k\left(\frac{N}{k}\right) = e^{\frac{i2\pi k}{N}\frac{N}{k}} = e^{i2\pi} = 1 = e^{i0} = B_k(0)$
- B_k *repeats every N/k pixels*

Understanding the Fourier basis



Understanding the Fourier basis

- Different basis elements have different frequencies
- Any image combines these with different coefficients
  - $I_1 = 0.5\, B_1 + 0.8\, B_5$

  - $I_2 = 0.5\, B_1 + 0.8\, B_5 + 0.5\, B_{10}$

  - I_2 is said to have *more high-frequency components*

Fourier transform

- Consider any 1D signal x with N entries
- It can be expressed as a combination of Fourier basis elements:
- $x = a_0(x)B_0 + a_1(x)B_1 + \; \ldots + a_{N-1}(x)B_{N-1}$
- x can be represented using N **Fourier coefficients:**
    - X = $[a_0(x), a_1(x), \ldots, a_{N-1}(x)]$
- Fourier transform of x is X
- Inverse Fourier transform of X is x

Fourier transform

- Problem: basis is complex, but signal is real?
- Combine a pair of conjugate basis elements
- $B_{N-k}(\mathrm{n}) = e^{\frac{2\pi i(N-k)n}{N}} = e^{2\pi in - \frac{2\pi ikn}{N}} = e^{-\frac{2\pi ikn}{N}} = B_{-k}(n)$
- Consider $B_{-N/2}$ to $B_{N/2}$ as basis elements
- Real signals will have same coefficients for B_k and B_{-k}

Fourier transform

- What is B_0?
- $B_0 = e^{2\pi i 0 n/N} = 1$
- Coefficient at 0 acts as a "constant bias"
- All other basis elements average out to 0
- So average must come from 0 coefficient
- "DC component"