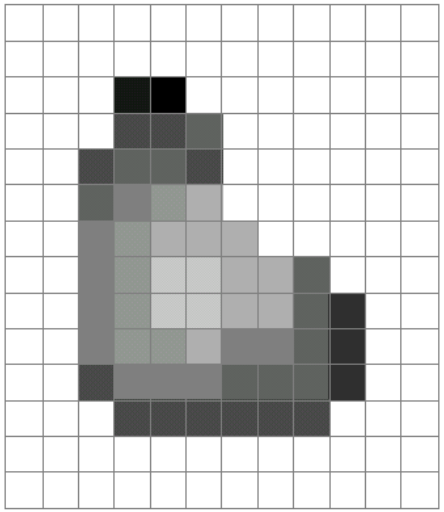
Lecture 2: Image filtering

What is an image?

- A grid (matrix) of intensity values



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255	255	255	255	255	255	255	255	255	255	255	255
255	255	255	255	255	255	255	255	255	255	255	255
255	255	255	20	0	255	255	255	255	255	255	255
255	255	255	75	75	75	255	255	255	255	255	255
255	255	75	95	95	75	255	255	255	255	255	255
255	255	96	127	145	175	255	255	255	255	255	255
255	255	127	145	175	175	175	255	255	255	255	255
255	255	127	145	200	200	175	175	95	255	255	255
255	255	127	145	200	200	175	175	95	47	255	255
255	255	127	145	145	175	127	127	95	47	255	255
255	255	74	127	127	127	95	95	95	47	255	255
255	255	255	74	74	74	74	74	74	255	255	255
255	255	255	255	255	255	255	255	255	255	255	255
255	255	255	255	255	255	255	255	255	255	255	255

(common to use one byte per value: 0 = black, 255 = white)

Images as functions

- An image contains discrete numbers of pixels

- Pixel value
  - grayscale/intensity
    - [0,255] or [0,1] or any real
  - Color
    - RGB [R, G, B]
    - Lab [L, a, b]: Lightness, a and b are color-opponent dimensions
    - HSV [H, S, V]: Hue, saturation, value

Images as functions

- Can think of image as a **function**, f, from R^2 to R or R^M:
  - Grayscale: $f(x,y)$ gives **intensity** at position (x,y)
    - f: [a,b] x [c,d] ➔R
  - Color: $f(x,y) = [r(x,y), g(x,y), b(x,y)]$

What is an image?

A **digital** image is a discrete (**sampled**, **quantized**) version of this function

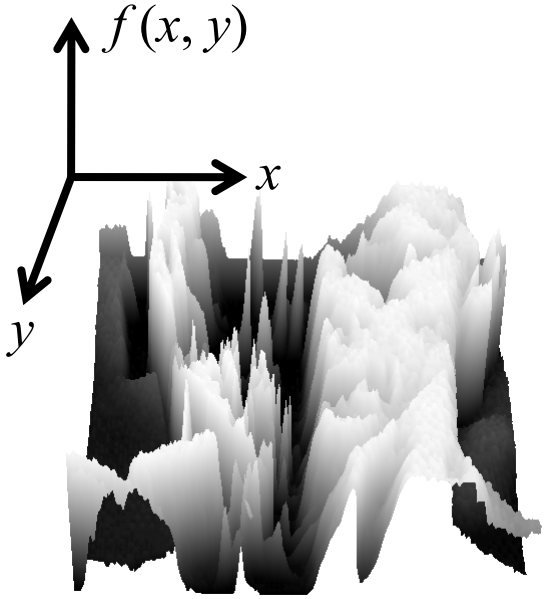
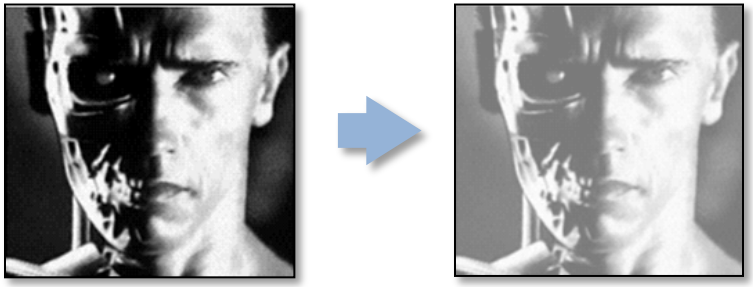


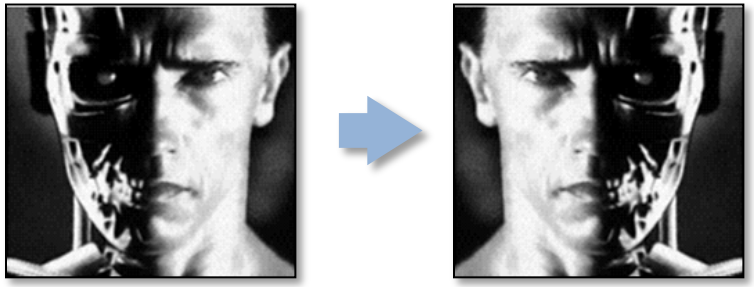
Image transformations

Appearance transformations



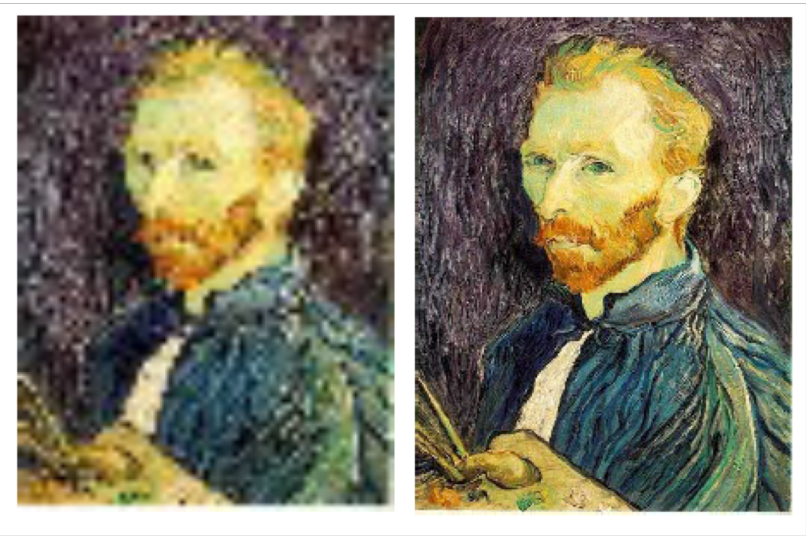
$$g(x,y) = f(x,y) + 20$$

Geometric transformations



$$g(x,y) = f(-x,y)$$

Example applications



Super-resolution: Increase image size

Noise reduction: Remove noise

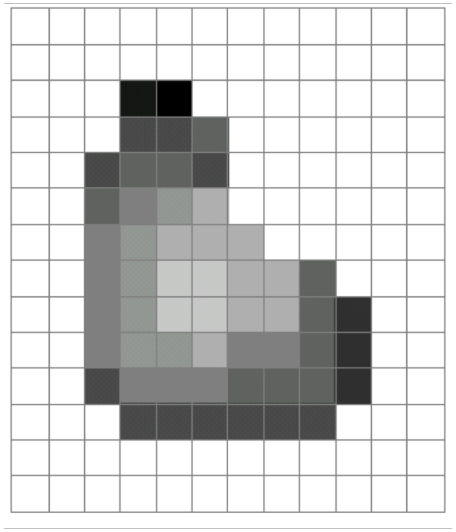
Image denoising

Why would images have noise?

- Sensor noise
  - Sensors count photons: noise in count
- Dead pixels
- Old photographs
- …

What is an image?

- A grid (matrix) of intensity values: 1 color or 3 colors



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255	255	255	255	255	255	255	255	255	255	255	255
255	255	255	255	255	255	255	255	255	255	255	255
255	255	255	20	0	255	255	255	255	255	255	255
255	255	255	75	75	75	255	255	255	255	255	255
255	255	75	95	95	75	255	255	255	255	255	255
255	255	96	127	145	175	255	255	255	255	255	255
255	255	127	145	175	175	175	255	255	255	255	255
255	255	127	145	200	200	175	175	95	255	255	255
255	255	127	145	200	200	175	175	95	47	255	255
255	255	127	145	145	175	127	127	95	47	255	255
255	255	74	127	127	127	95	95	95	47	255	255
255	255	255	74	74	74	74	74	74	255	255	255
255	255	255	255	255	255	255	255	255	255	255	255
255	255	255	255	255	255	255	255	255	255	255	255

(common to use one byte per value: 0 = black, 255 = white)

An assumption about noise

- Let us assume noise at a pixel is
  - independent of other pixels
  - distributed according to a Gaussian distribution
    - i.e., low noise values are more likely than high noise values
    - “grainy images”

Noise reduction

- Nearby pixels are likely to belong to same object
  - thus likely to have similar color
- Replace each pixel by *average of neighbors*

Mean filtering

0	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

$$(0 + 0 + 0 + 10 + 40 + 0 + 10 + 0 + 0)/9 = 6.66$$

Mean filtering

0	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

$(0 + 0 + 0 + 0 + 0 + 10 + 0 + 0 + 0 + 0 + 20 + 10 + 40 + 0 + 0 + 20 + 10 + 0 + 0 + 0 + 30 + 20 + 10 + 0 + 0)/25 = 6.8$

Mean filtering

0	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$(0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 10)/9 = 1.11$$

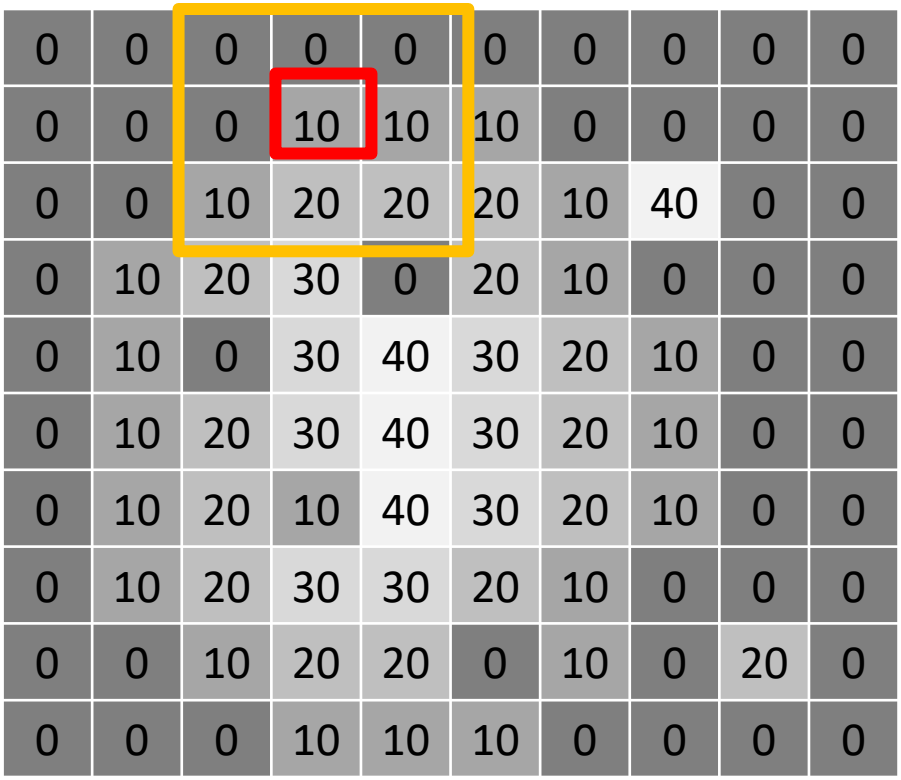
Mean filtering

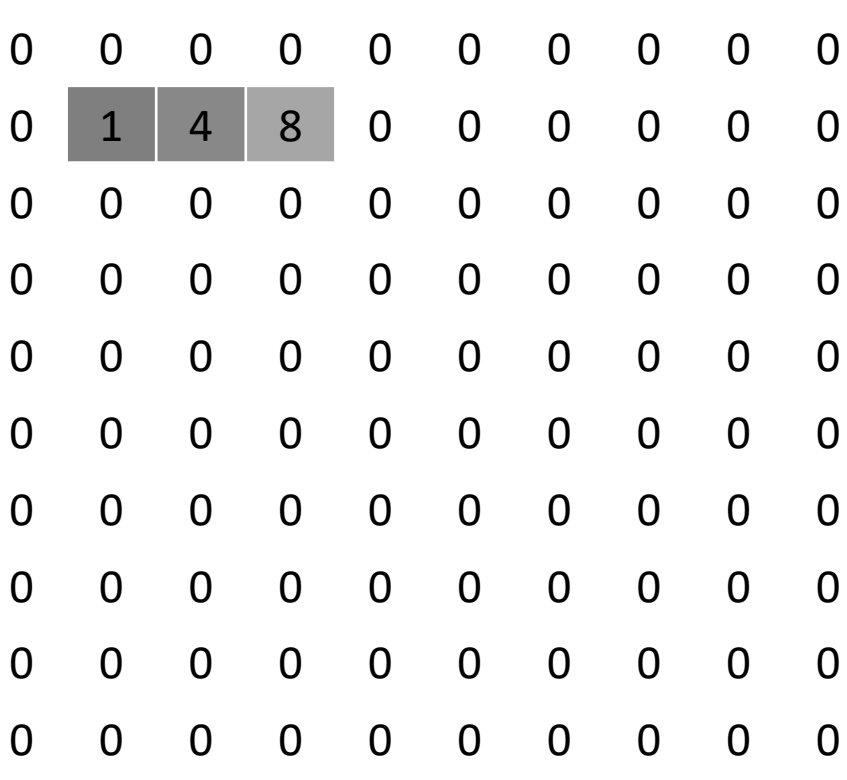
0	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

0	0	0	0	0	0	0	0	0	0
0	1	4	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0

$$(0 + 0 + 0 + 0 + 0 + 10 + 0 + 10 + 20)/9 = 4.44$$

Mean filtering





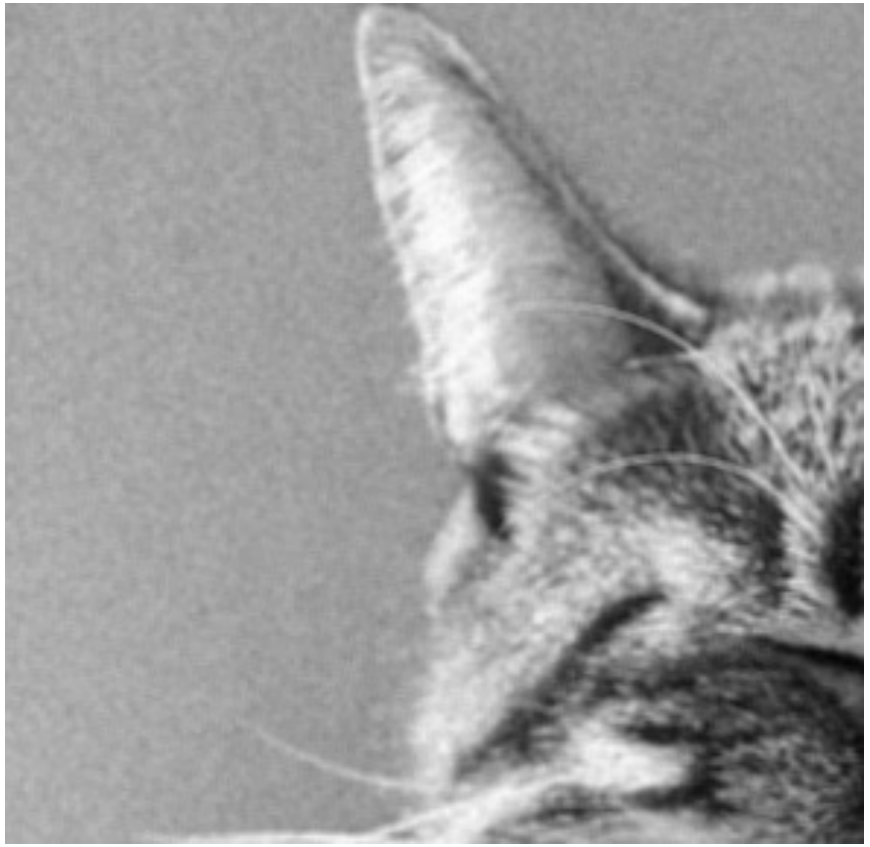
$$(0 + 0 + 0 + 0 + 10 + 10 + 10 + 20 + 20)/9 = 7.77$$

Mean filtering

0	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

0	0	0	0	0	0	0	0	0	0
0	1	4	8	10	8	9	6	4	0
0	4	11	13	16	11	12	7	4	0
0	6	14	19	23	19	18	10	6	0
0	8	18	23	28	23	17	8	2	0
0	8	16	26	31	30	20	10	3	0
0	10	18	27	29	27	17	8	2	0
0	8	14	22	22	20	11	8	3	0
0	4	11	17	17	12	6	4	2	0
0	0	0	0	0	0	0	0	0	0

Noise reduction using mean filtering



Mean filtering

- Replace pixel by mean of neighborhood

10	5	3
4	5	1
1	1	7

Local image data

f

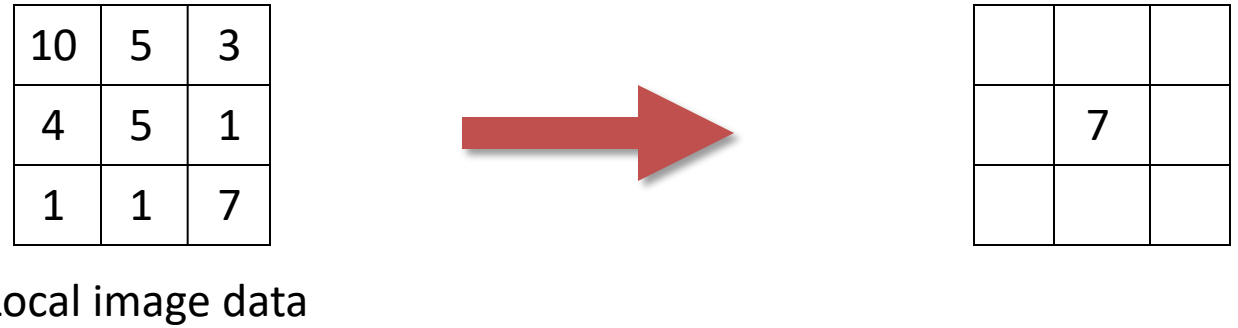
	4.1	

Modified image data

$S[f]$

$$S[f](m,n) = \sum_{i=-1}^{1} \sum_{j=-1}^{1} f(m+i, n+j)/9$$

A more general version



Local image data

Kernel / filter

$$S[f](m,n) = \sum_{i=-1}^{1} \sum_{j=-1}^{1} w(i,j) f(m+i, n+j)$$

A more general version

0	10	5	7	0
5	11	6	8	3
9	22	4	5	1
2	9	14	6	7
3	10	15	12	9

Local image data

		7		

Kernel size = 2k+1

$$S[f](m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i, n+j)$$

A more general version

$$S[f](m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i, n+j)$$

- w(i,j) = $1/(2k+1)^2$ for mean filter
- If w(i,j) >=0 and sum to 1, *weighted mean*
- But w(i,j) can be *arbitrary real numbers!*
- This operation is called *cross-correlation*

Boundary conditions

Clearly valid

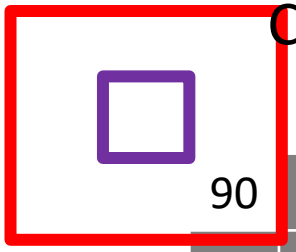
90	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

Boundary conditions

Can do if we assume empty space is 0

90	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

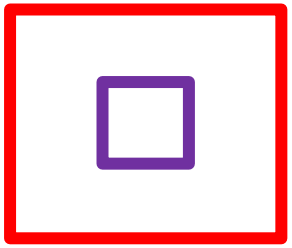
Boundary conditions



Can still potentially compute

90	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

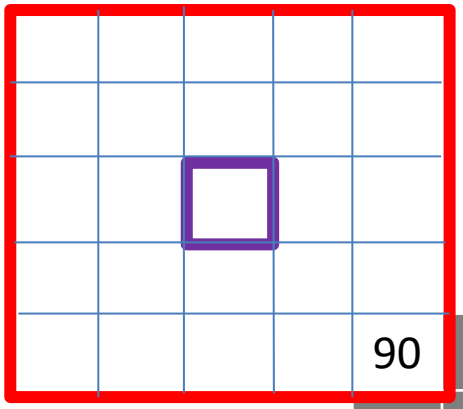
Boundary conditions



Output will be 0 so no point computing

90	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

Boundary conditions



90	0	0	0	0	0	0	0	0	0
0	0	0	10	10	10	0	0	0	0
0	0	10	20	20	20	10	40	0	0
0	10	20	30	0	20	10	0	0	0
0	10	0	30	40	30	20	10	0	0
0	10	20	30	40	30	20	10	0	0
0	10	20	10	40	30	20	10	0	0
0	10	20	30	30	20	10	0	0	0
0	0	10	20	20	0	10	0	20	0
0	0	0	10	10	10	0	0	0	0

Boundary conditions

- Setup
  - m x m image
  - k x k kernel
- Output size?

Boundary conditions in practice

- "Full": compute if *any* part of kernel intersects with image
  - Assumes image is padded with 0's
  - Output size = m+k-1
  - Technically cross-correlation and convolution means this
  - 0 padding can cause artifacts
- "Same convolution": compute if center of kernel is in image
  - Assumes image is padded with 0's
  - output size = m
  - Common in practice
- "Valid convolution": compute only if *all* of kernel is in image
  - no padding
  - output size = m-k+1
  - No artifacts

Properties: Linearity

$$(w \otimes f)(m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i, n+j)$$

$$f'(m,n) = af(m,n)$$

$$(w \otimes f')(m,n) = a(w \otimes f)(m,n)$$

Properties: Linearity

$$(w \otimes f)(m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i, n+j)$$

$$f' = af$$

$$(w \otimes f') = a(w \otimes f)$$

Properties: Linearity

$$(w \otimes f)(m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i, n+j)$$

$$f' = af + bg$$

$$w \otimes f' = a(w \otimes f) + b(w \otimes g)$$

Properties: Linearity

$$(w \otimes f)(m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i, n+j)$$

$$w' = aw + bv$$

$$w' \otimes f = a(w \otimes f) + b(v \otimes f)$$

Properties: Shift equivariance

$$(w \otimes f)(m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i, n+j)$$

$$f'(m,n) = f(m - m_0, n - n_0)$$

f

f'

Shift equivariance

$$(w \otimes f)(m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i, n+j)$$

$$f'(m,n) = f(m - m_0, n - n_0)$$

$$(w \otimes f')(m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f'(m+i, n+j)$$

$$= \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i-m_0, n+j-n_0)$$

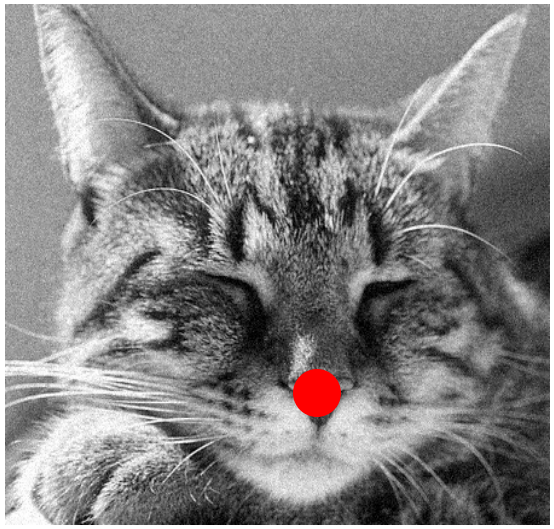
$$= (w \otimes f)(m - m_0, n - n_0)$$

Shift equivariance

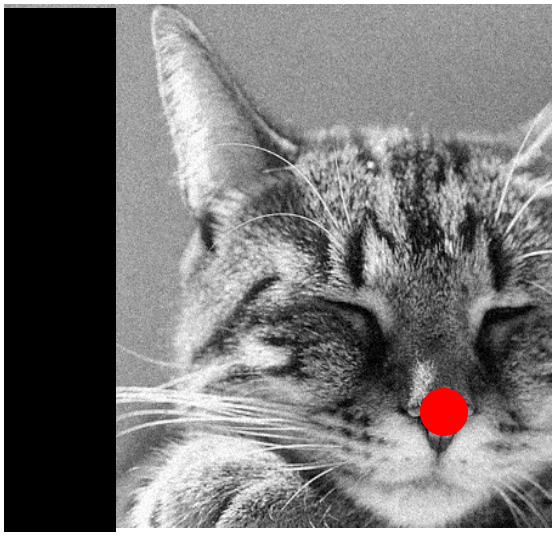
$$f'(m,n) = f(m - m_0, n - n_0)$$

$$(w \otimes f')(m,n) = (w \otimes f)(m - m_0, n - n_0)$$

- Shift, then filter = filter, then shift
- Output of filtering does not depend on where the pixel is



f



f'

Convolution and cross-correlation

- Cross correlation

$$S[f] = w \otimes f$$

$$S[f](m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m+i, n+j)$$

- Convolution

$$S[f] = w * f$$

$$S[f](m,n) = \sum_{i=-k}^{k} \sum_{j=-k}^{k} w(i,j) f(m-i, n-j)$$

Cross-correlation

1	2	3
4	5	6
7	8	9

w

1	2	3
4	5	6
7	8	9

f

1*1 + 2*2 + 3*3 + 4*4 + 5*5 + 6*6 + 7*7 + 8*8 + 9*9

Convolution

1	2	3
4	5	6
7	8	9

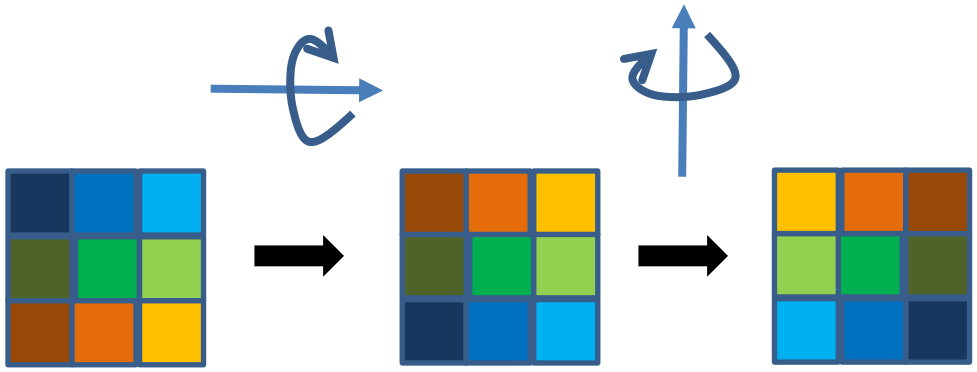
w

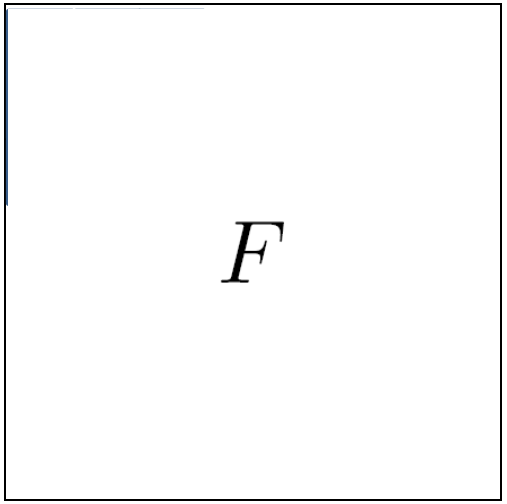
1	2	3
4	5	6
7	8	9

f

1*9 + 2*8 + 3*7 + 4*6 + 5*5 + 6*4 + 7*3 + 8*2 + 9*1

Convolution



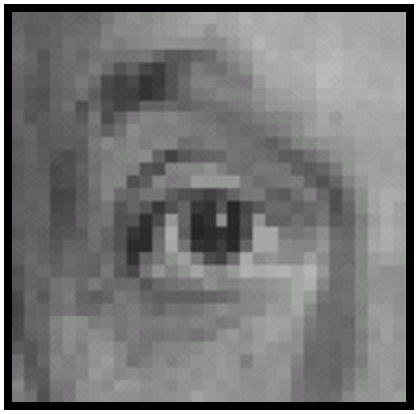


Adapted from F. Durand

Convolution

- Convolution is the more fundamental operation
- ”Full” convolution satisfies associative property:
  - $a * (b * c) = (a * b) * c$

Filters: examples



Original (f)

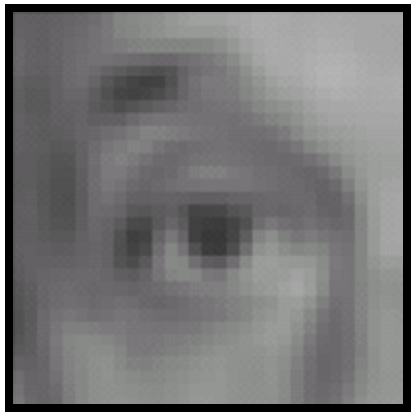
$*$

$\frac{1}{9}$

1	1	1
1	1	1
1	1	1

Kernel (k)

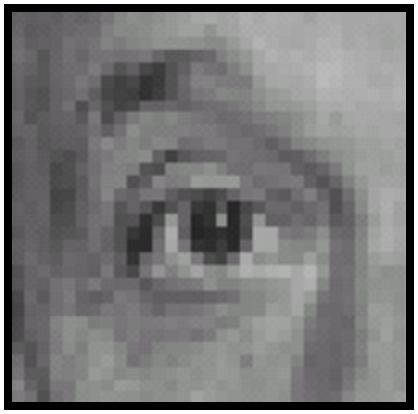
$=$



Blur (with a mean filter) (g)

Source: D. Lowe

Filters: examples



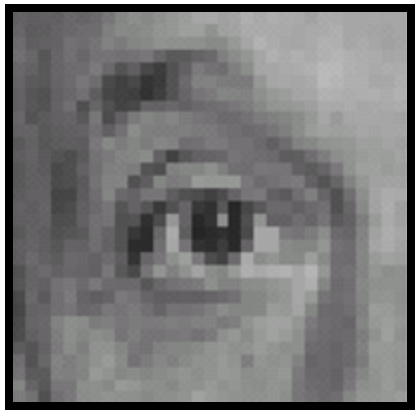
Original (f)

*

0	0	0
0	1	0
0	0	0

Kernel (k)

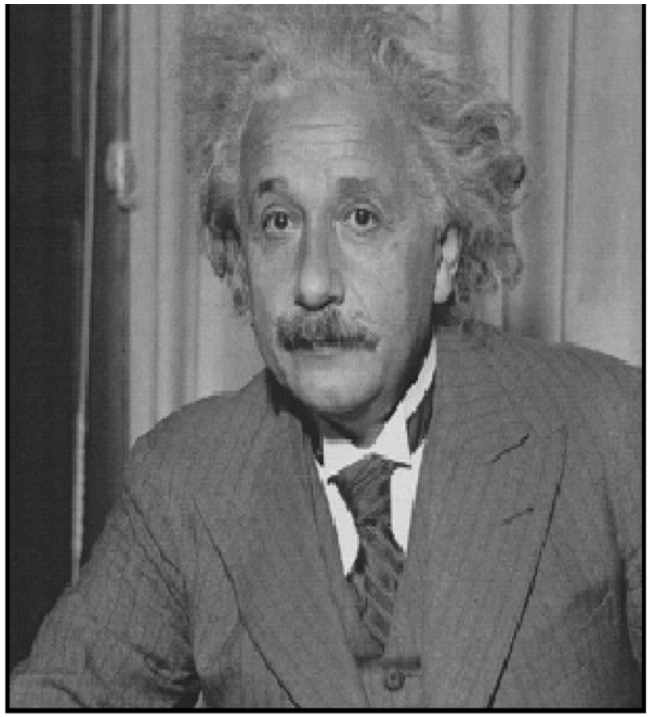
=



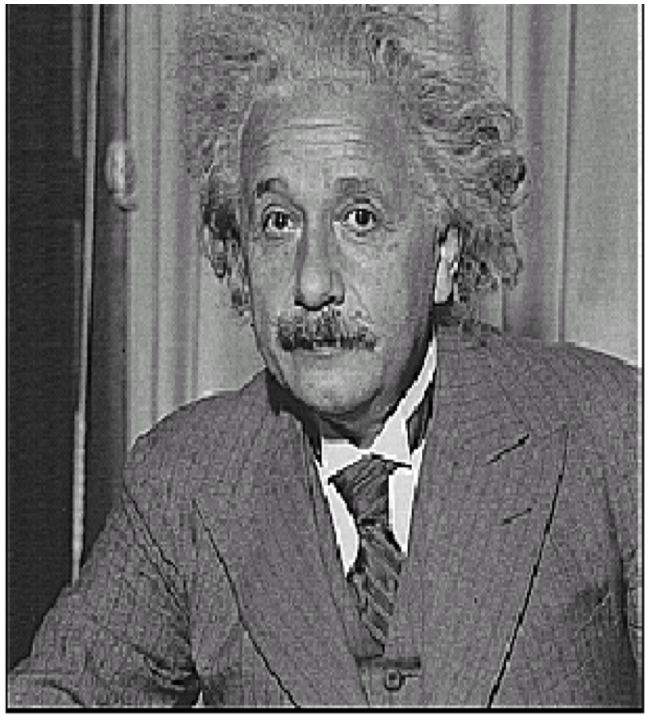
Identical image (g)

Source: D. Lowe

Sharpening



before

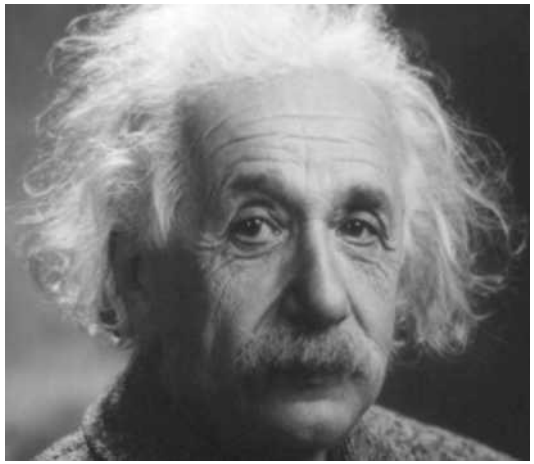
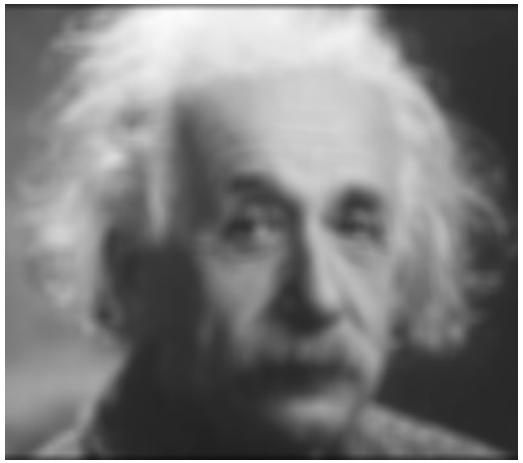
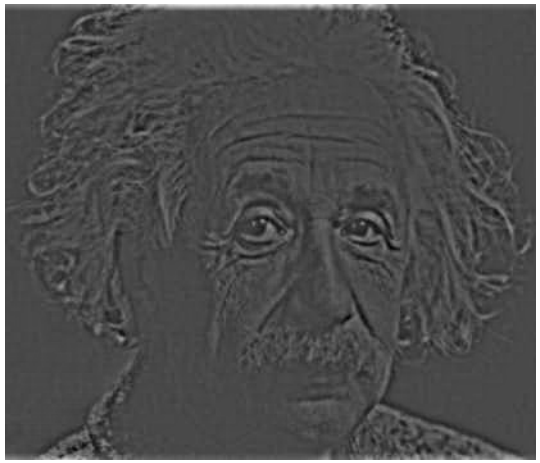


after

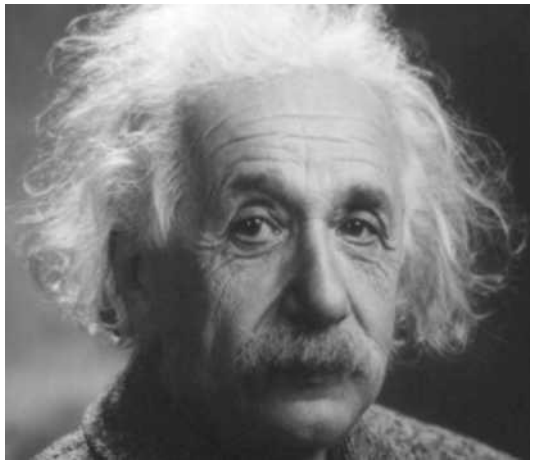
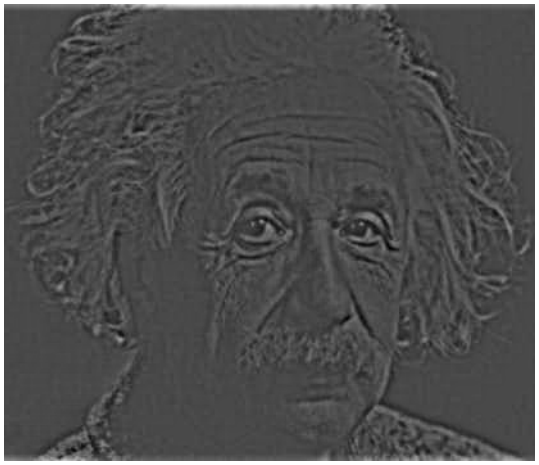
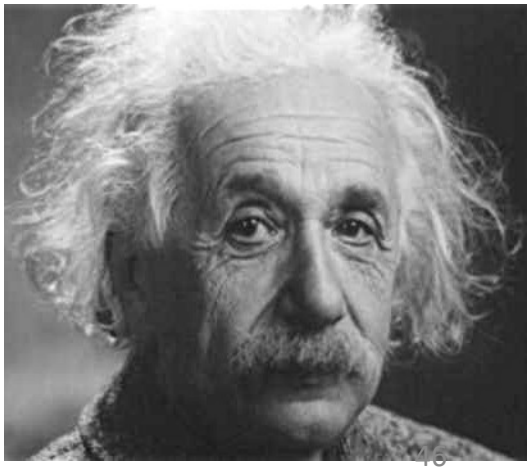
Source: D. Lowe

Sharpening

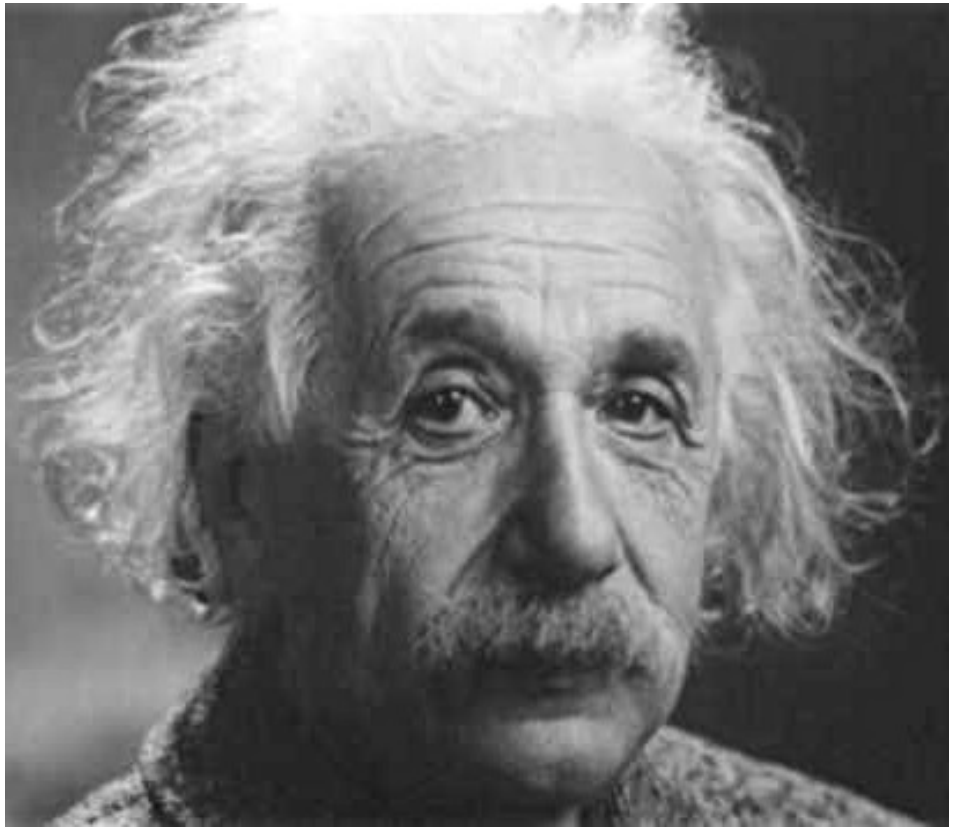
- What does blurring take away?

Let's add it back:

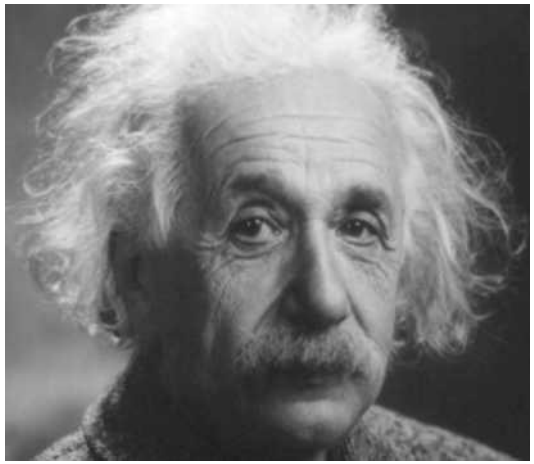
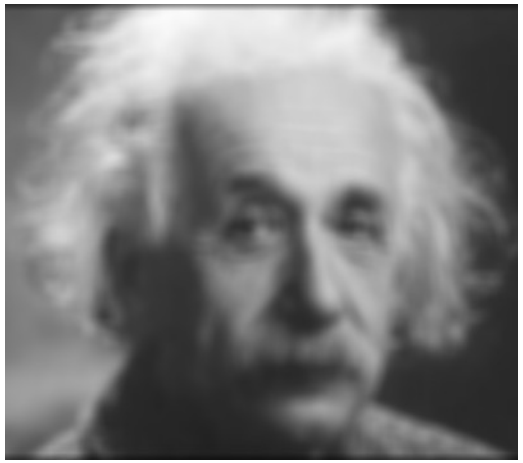
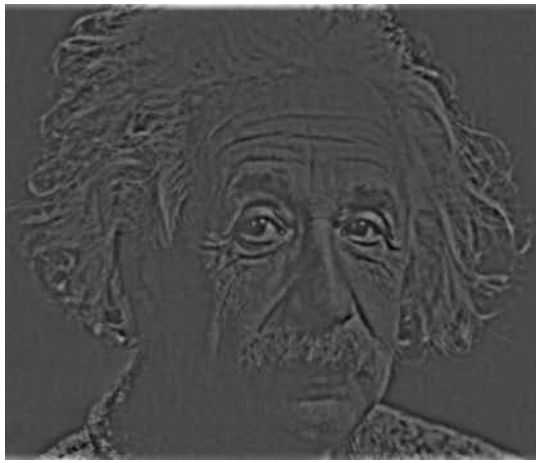




Sharpening

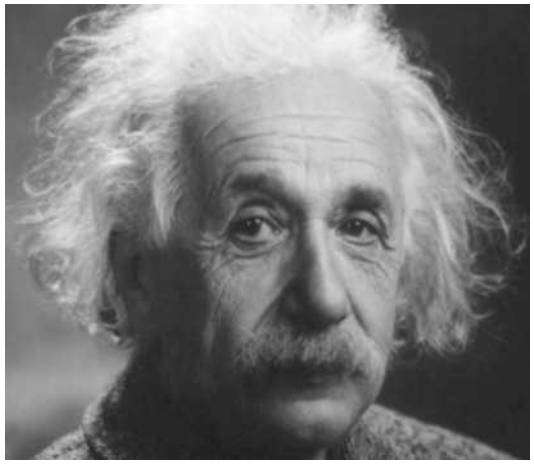
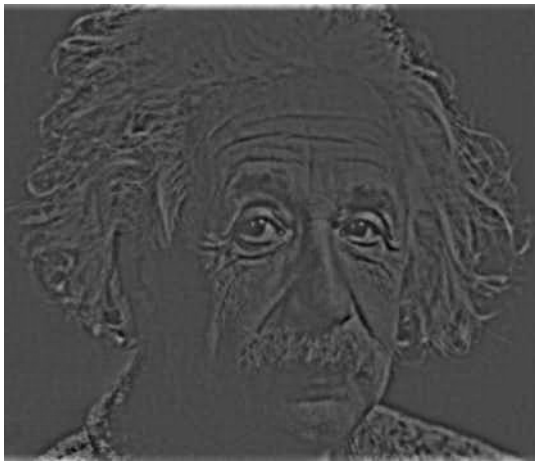
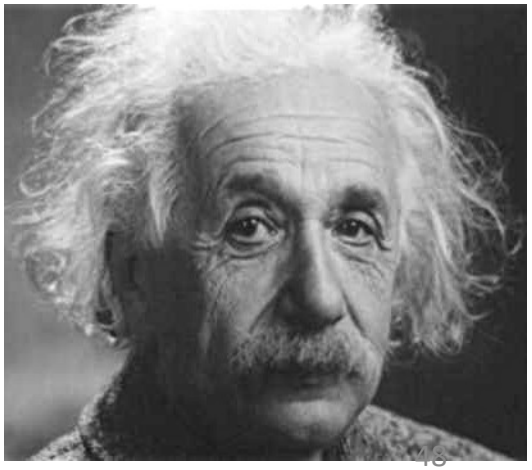


Sharpening

- What does blurring take away?

Let's add it back:

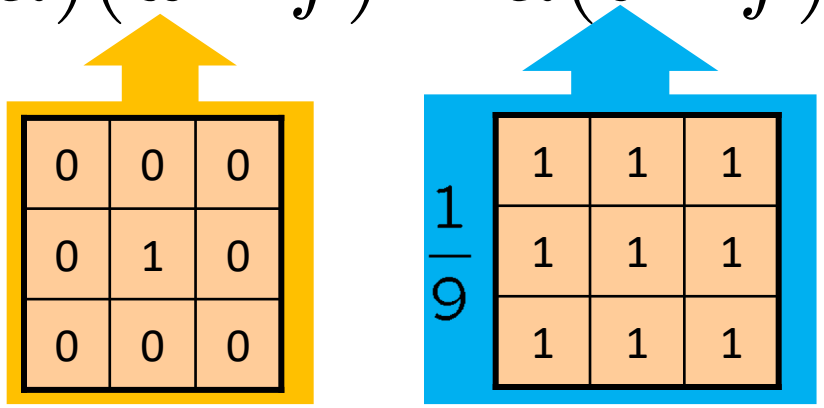




Sharpening

$$f_{sharp} = f + \alpha(f - f_{blur})$$

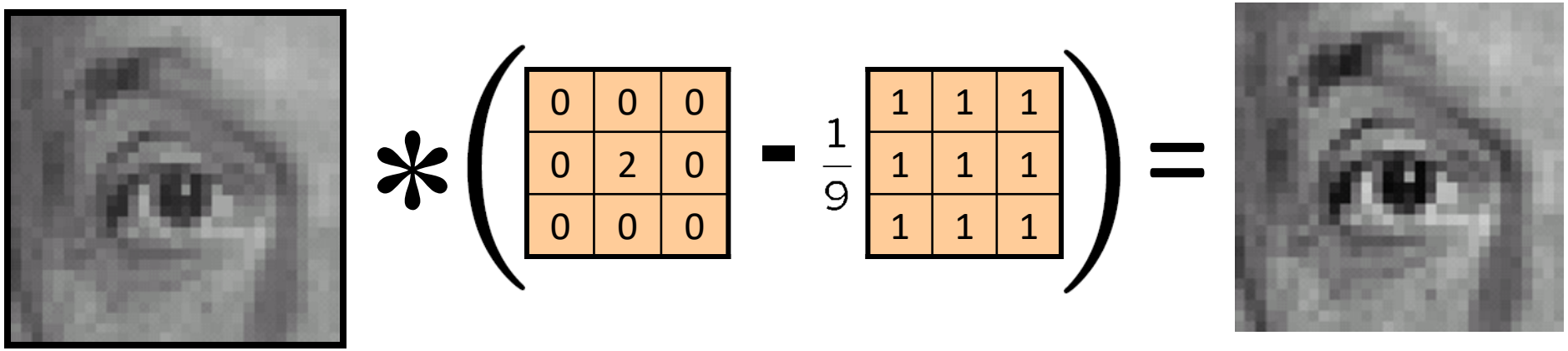
$$= (1+\alpha)f - \alpha f_{blur}$$

$$= (1+\alpha)(w * f) - \alpha(v * f)$$



$$= ((1+\alpha)w - \alpha v) * f$$

Sharpening filter



Original

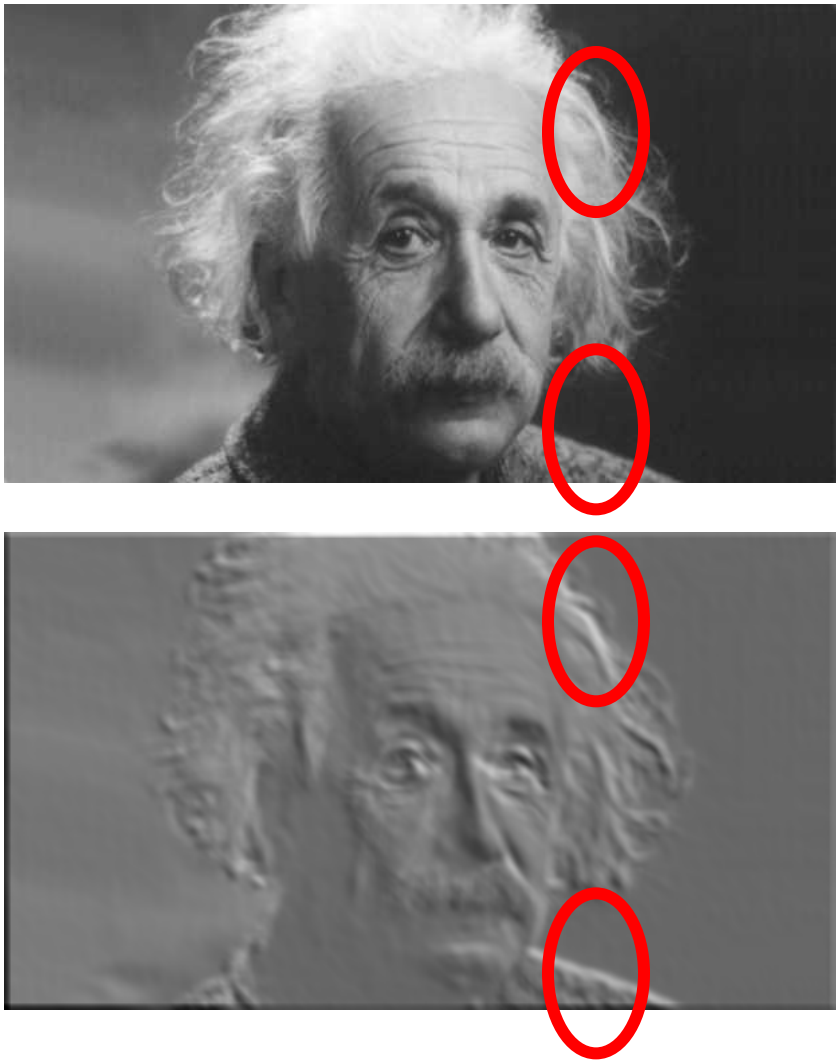
Sharpening filter
(accentuates edges)

Source: D. Lowe

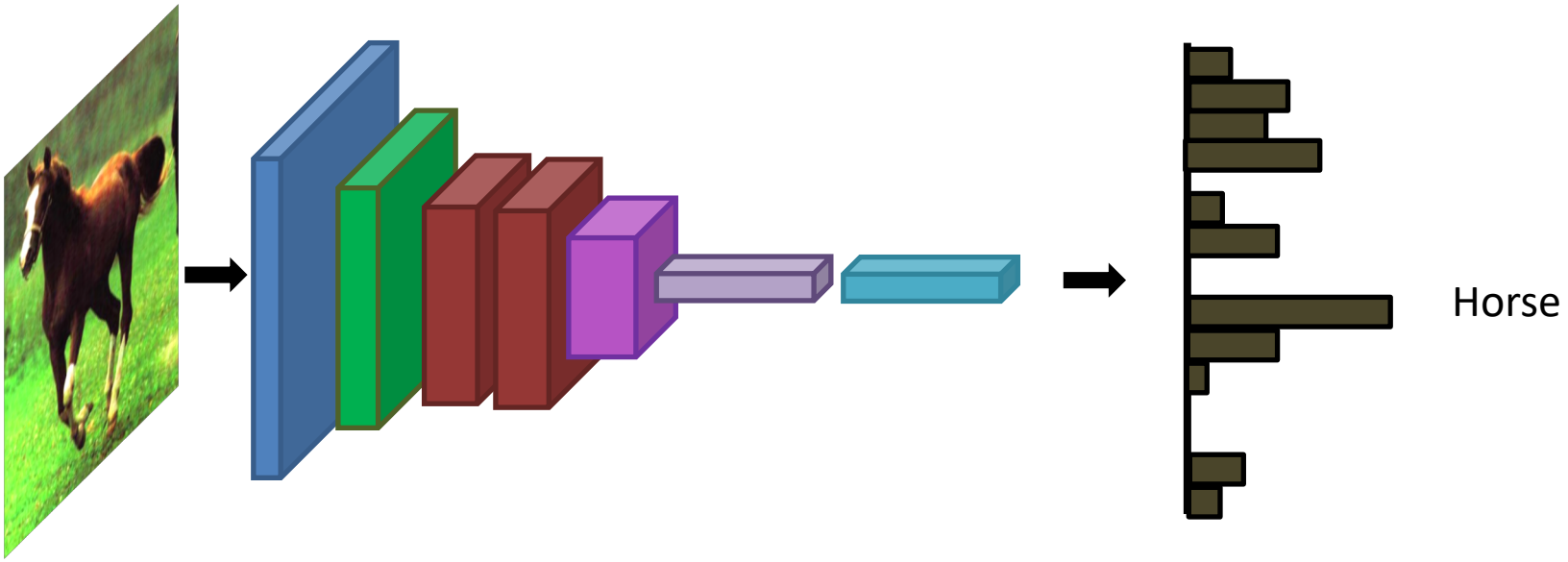
Another example

-1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	-1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	1	-1	-1	-1	-1	-1	-1	-1	-1
1	1	1	1	-1	-1	-1	-1	-1	-1	-1
1	1	1	1	1	-1	-1	-1	-1	-1	-1
1	1	1	1	1	1	-1	-1	-1	-1	-1
1	1	1	1	1	1	1	-1	-1	-1	-1
1	1	1	1	1	1	1	1	-1	-1	-1
1	1	1	1	1	1	1	1	1	-1	-1
1	1	1	1	1	1	1	1	1	1	-1

Another example

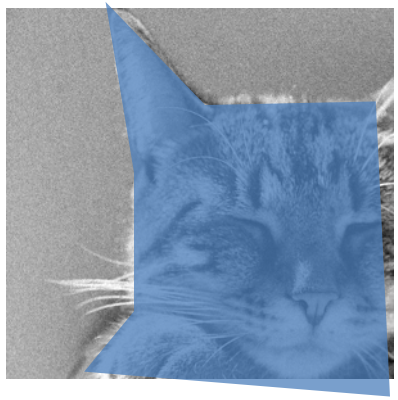


Convolution is everywhere



Why is convolution important?

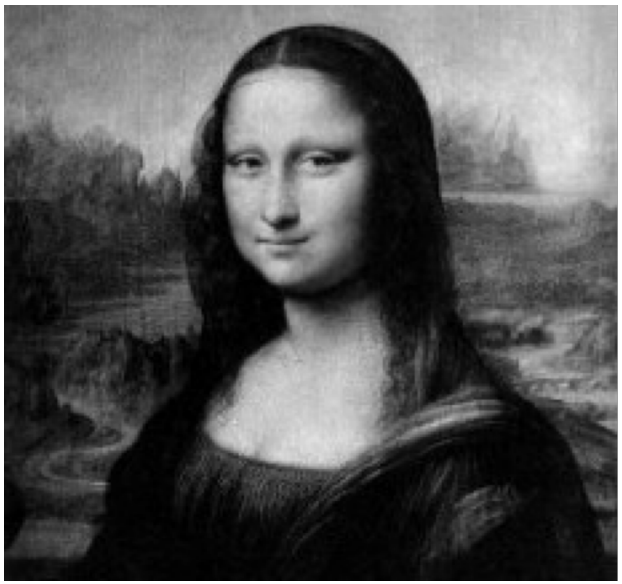
- Shift equivariance is a crucial property



Why is convolution important?

- We *like* linearity
  - Linear functions behave predictably when input changes
  - Lots of theory just easier with linear functions
- *All linear shift-equivariant systems can be expressed as a convolution*

Non-linear filters: Thresholding



$$g(m,n) = \begin{cases} 255, & f(m,n) > A \\ 0 & otherwise \end{cases}$$

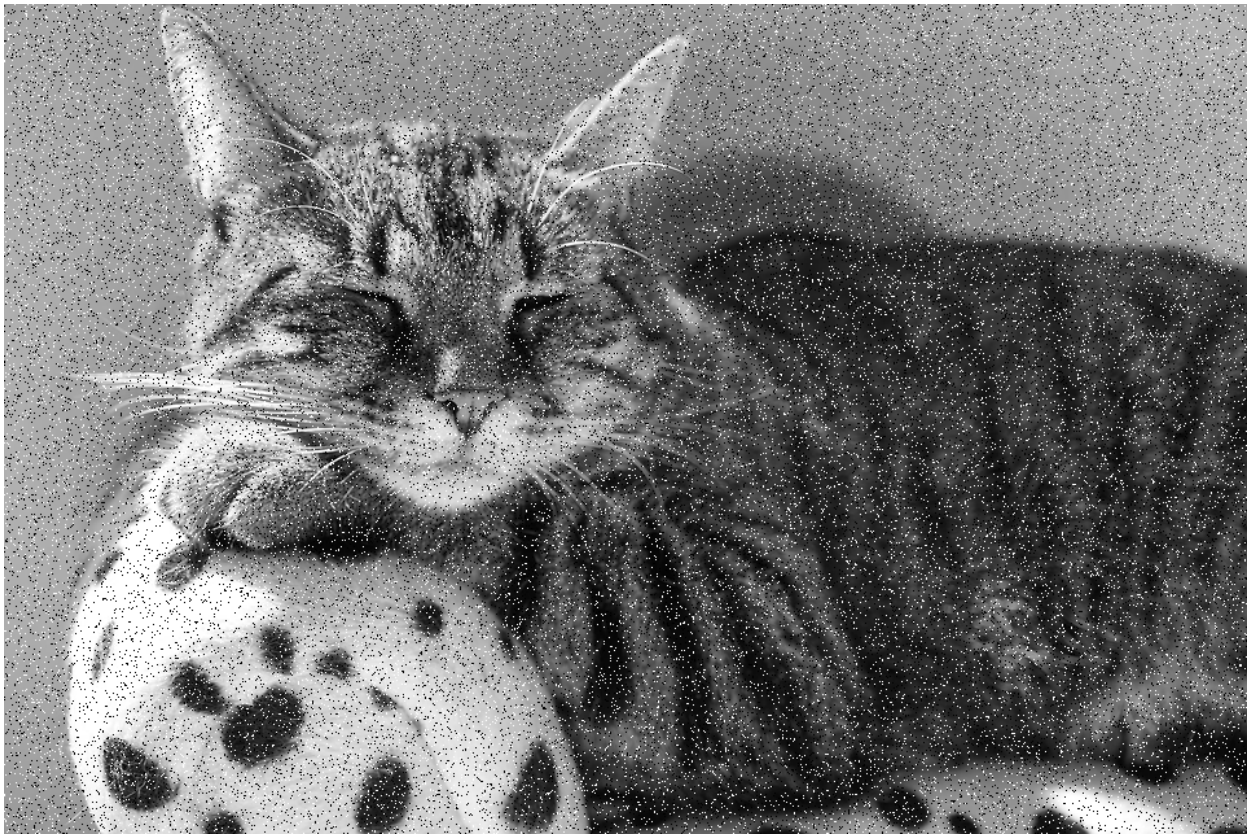
Non-linear filters: Rectification

- g(m,n) = max(f(m,n), 0)

- Crucial component of modern convolutional networks

Non-linear filters

- Sometimes mean filtering does not work



Non-linear filters

- Sometimes mean filtering does not work

Non-linear filters

- Mean is sensitive to outliers
- Median filter: Replace pixel by *median* of neighbors

Non-linear filters

Takeaway

- Two general recipes:
  - convolution
  - cross-correlation
- Properties
  - Shift-equivariant: a sensible thing to require
  - Linearity: convenient
- Can be used for smoothing, sharpening
- Also main component of CNNs

Next up

- Back to linear filters
- Signal processing view of filtering
- Filtering for detecting edges etc