Homogenous coordinates

Putting everything together

- Change coordinate system so that center of the coordinate system is at pinhole and Z axis is along viewing direction

$$\mathbf{x}'_w = R\mathbf{x}_w + \mathbf{t}$$

- Perspective projection

$$\mathbf{x}'_w \equiv (X, Y, Z)$$

$$\mathbf{x}'_{img} \equiv (x, y)$$

$$x = \frac{X}{Z}$$

$$y = \frac{Y}{Z}$$

The projection equation

$$x = \frac{X}{Z}$$
$$y = \frac{Y}{Z}$$

- Is this equation linear?
- Can this equation be represented by a matrix multiplication?

Is projection linear?

$$X' = aX + b$$
$$Y' = aY + b$$
$$Z' = aZ + b$$

$$x' = \frac{aX + b}{aZ + b}$$

$$y' = \frac{aY + b}{aZ + b}$$

Can projection be represented as a matrix multiplication?

Matrix multiplication

$$\begin{bmatrix} a & b & c \\ p & q & r \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} aX + bY + cZ \\ pX + qY + rZ \end{bmatrix}$$

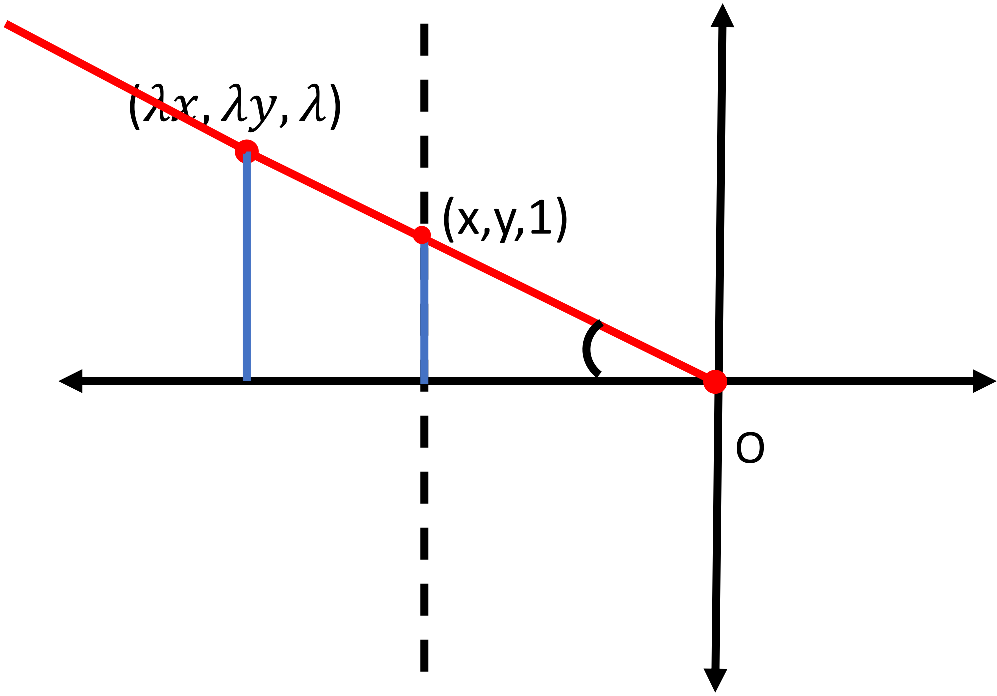
Perspective projection

$$x = \frac{X}{Z}$$

$$y = \frac{Y}{Z}$$

The space of rays

- Every point on a ray maps it to a point on image plane
- Perspective projection maps rays to points
- All points $(\lambda x, \lambda y, \lambda)$ map to the same image point (x,y,1)



Projective space

- Standard 2D space (plane) $\mathbb{R}^2$: Each point represented by 2 coordinates (x,y)
- Projective 2D space (plane) $\mathbb{P}^2$: Each "point" represented by *3* coordinates (x,y,z), BUT:
  - $(\lambda x, \lambda y, \lambda z) \equiv (x, y, z)$

- Mapping $\mathbb{R}^2$ to $\mathbb{P}^2$ (points to rays):

$$(x, y) \rightarrow (x, y, 1)$$

- Mapping $\mathbb{P}^2$ to $\mathbb{R}^2$ (rays to points):

$$(x, y, z) \rightarrow (\frac{x}{z}, \frac{y}{z})$$

Projective space and homogenous coordinates

- Mapping $\mathbb{R}^2$ to $\mathbb{P}^2$ (points to rays):

$$(x, y) \rightarrow (x, y, 1)$$

- Mapping $\mathbb{P}^2$ to $\mathbb{R}^2$ (rays to points):

$$(x, y, z) \rightarrow (\frac{x}{z}, \frac{y}{z})$$

- A change of coordinates
- Also called *homogenous coordinates*

Homogenous coordinates

- In standard Euclidean coordinates
  - 2D points : (x,y)
  - 3D points : (x,y,z)
- In homogenous coordinates
  - 2D points : (x,y,1)
  - 3D points : (x,y,z,1)

Why homogenous coordinates?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \equiv \begin{bmatrix} \frac{X}{Z} \\ \frac{Y}{Z} \\ 1 \end{bmatrix}$$

Homogenous coordinates of world point

Homogenous coordinates of image point

Why homogenous coordinates?

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \equiv \begin{bmatrix} \frac{X}{Z} \\ \frac{Y}{Z} \\ 1 \end{bmatrix}$$

$$P\vec{\mathbf{x}}_w \equiv \vec{\mathbf{x}}_{img}$$

- Perspective projection is matrix multiplication in homogenous coordinates!

Why homogenous coordinates?

$$\begin{bmatrix} 1 & 0 & 0 & t_x \\ 0 & 1 & 0 & t_y \\ 0 & 0 & 1 & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

- Translation is matrix multiplication in homogenous coordinates!

Homogenous coordinates

$$\begin{bmatrix} a & b & c & t_x \\ d & e & f & t_y \\ g & h & i & t_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} aX + bY + cZ + t_x \\ dX + eY + fZ + t_y \\ gX + hY + iZ + t_z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} = \begin{bmatrix} M\mathbf{x}_w + \mathbf{t} \\ 1 \end{bmatrix}$$

Homogenous coordinates

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} \equiv \begin{bmatrix} \frac{X}{Z} \\ \frac{Y}{Z} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I & \mathbf{0} \end{bmatrix}$$

Perspective projection in homogenous coordinates

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

More about matrix transformations

$\begin{bmatrix} I & \mathbf{0} \end{bmatrix}$ 3 x 4 : Perspective projection

$\begin{bmatrix} I & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$ 4 x 4 : Translation

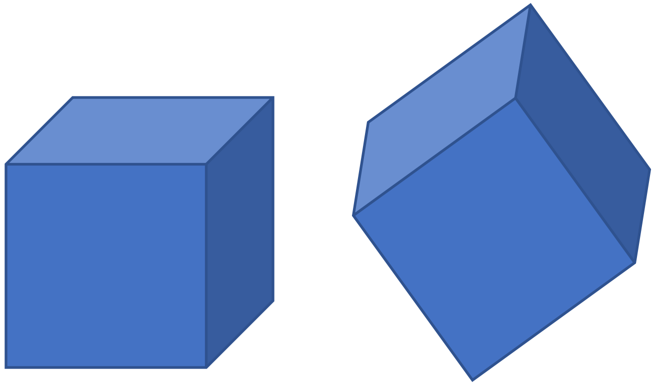
$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$ 4 x 4 : Affine transformation (linear transformation + translation)

More about matrix transformations

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$M^T M = I$$

Euclidean



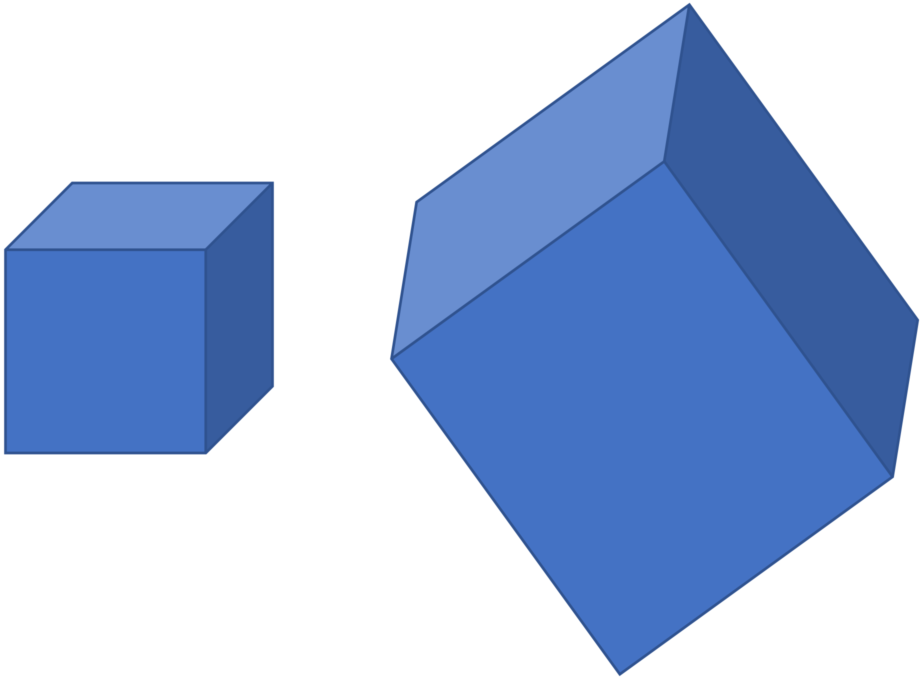
More about matrix transformations

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$M = sR$$

$$R^T R = I$$

Similarity transformation

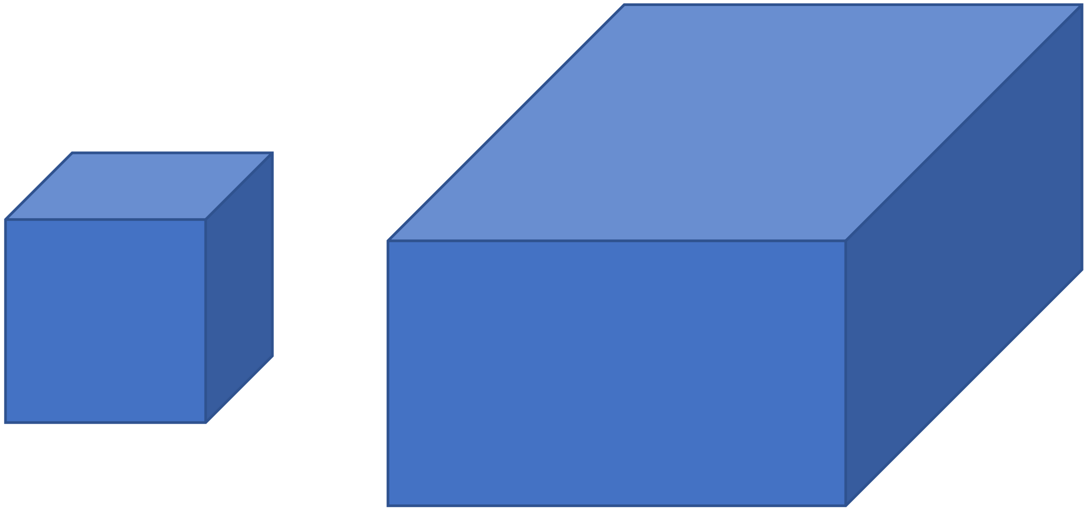


More about matrix transformations

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

$$M = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix}$$

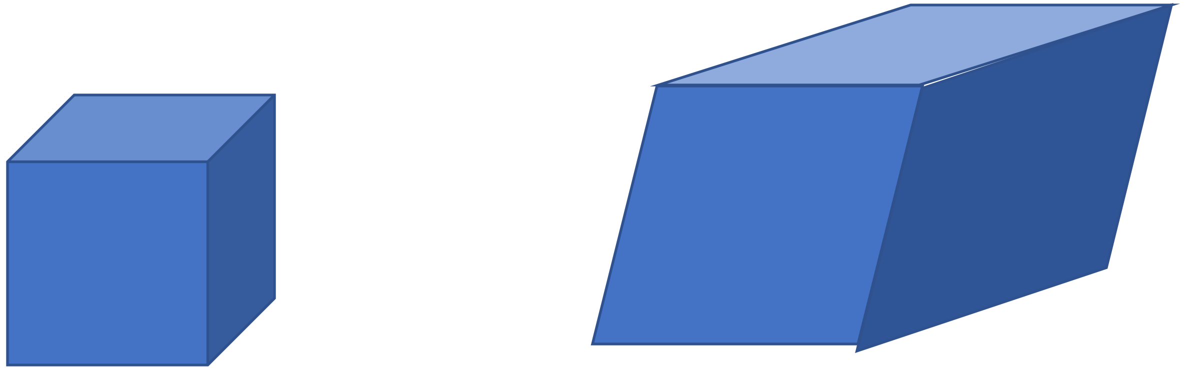
Anisotropic scaling and translation



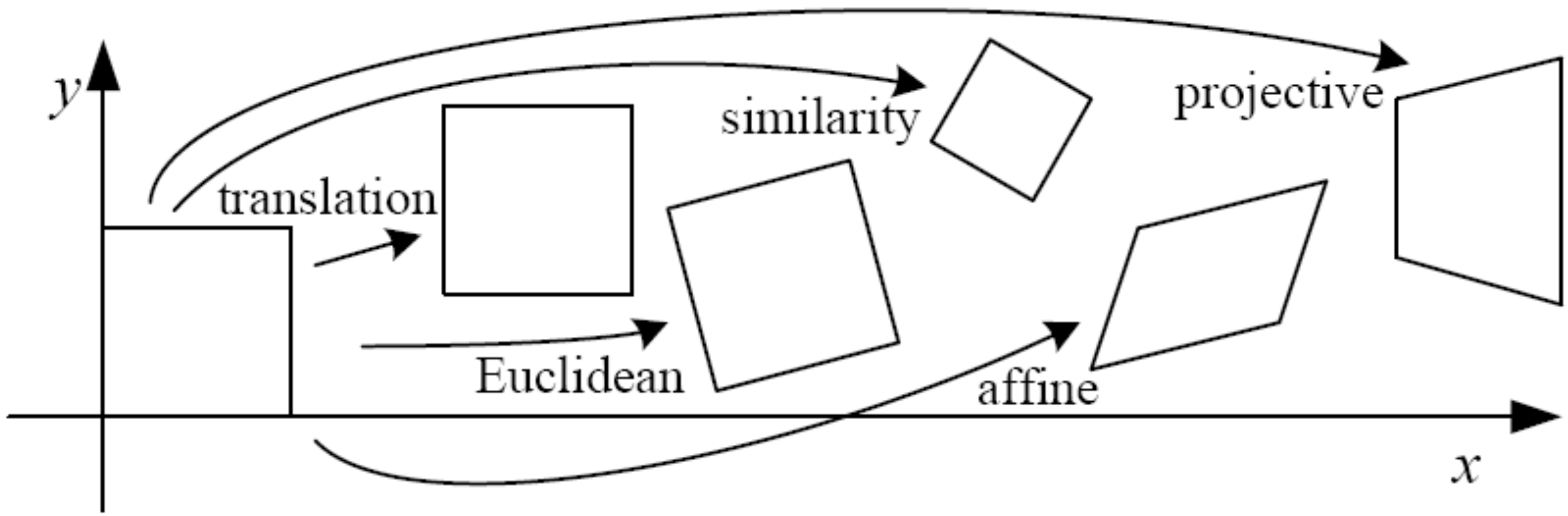
More about matrix transformations

$$\begin{bmatrix} M & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}$$

General affine transformation



Matrix transformations in 2D



Perspective projection in homogenous coordinates

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Matrix transformations in 2D

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

$$K = \begin{bmatrix} 1 & 0 & t_u \\ 0 & 1 & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

$$K = \begin{bmatrix} s_x & 0 & t_u \\ 0 & s_y & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling of Image x and y (conversion from "meters" to "pixels")

$$K = \begin{bmatrix} s_x & \alpha & t_u \\ 0 & s_y & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Added skew if image x and y axes are not perpendicular

Final perspective projection

Camera extrinsics: where your camera is relative to the world. Changes if you move the camera

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Camera intrinsics: how your camera handles pixel. Changes if you change your camera

$$\vec{\mathbf{x}}_{img} \equiv P \vec{\mathbf{x}}_w$$

Final perspective projection

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Camera parameters

$$\vec{\mathbf{x}}_{img} \equiv P \vec{\mathbf{x}}_w$$

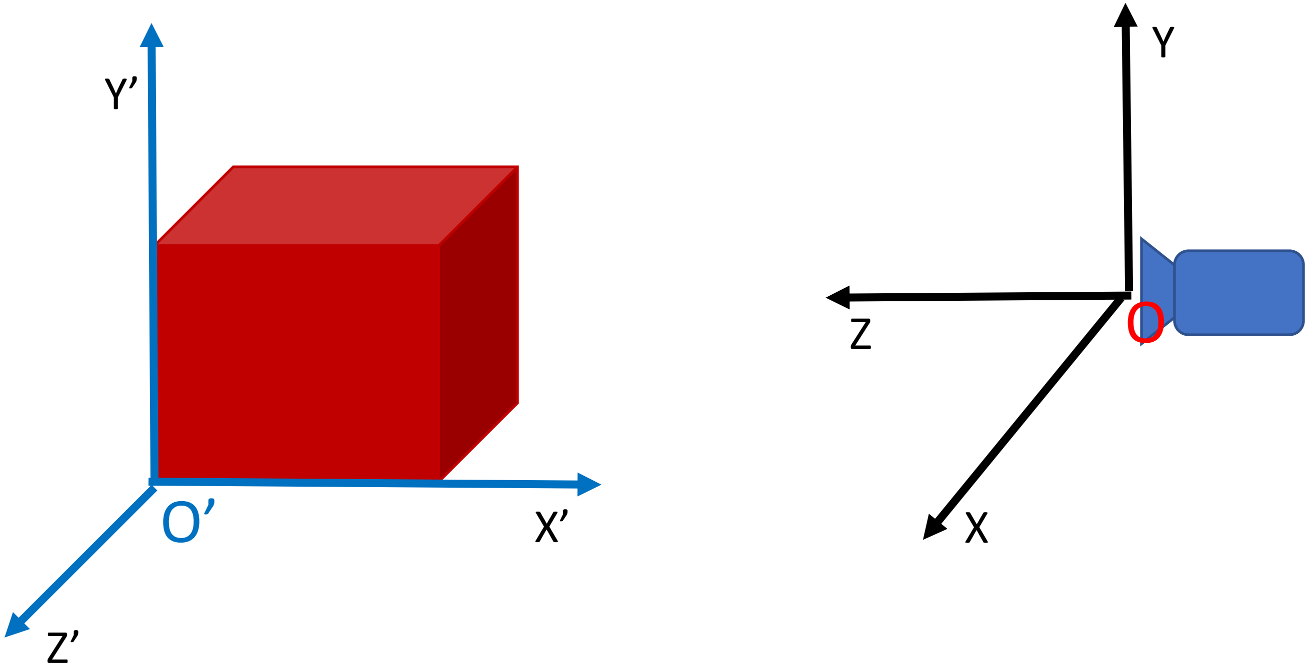
Camera calibration

- Goal: find the parameters of the camera

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

- Why?
  - Tells you where the camera is relative to the world/particular objects
  - Equivalently, tells you where objects are relative to the camera
  - Can allow you to ”render” new objects into the scene

Camera calibration



Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Need to estimate P
- How many parameters does P have?
  - Size of P : 3 x 4
  - But: $\lambda P\vec{\mathbf{x}}_w \equiv P\vec{\mathbf{x}}_w$
  - P can only be known *upto a scale*
  - *3*4 - 1 = 11 parameters*

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \equiv P \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Need to convert equivalence into equality.

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

Note: λ is unknown

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = P \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

$$\lambda x = P_{11}X + P_{12}Y + P_{13}Z + P_{14}$$

$$\lambda y = P_{21}X + P_{22}Y + P_{23}Z + P_{24}$$

$$\lambda = P_{31}X + P_{32}Y + P_{33}Z + P_{34}$$

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

$$(P_{31}X + P_{32}Y + P_{33}Z + P_{34})x = P_{11}X + P_{12}Y + P_{13}Z + P_{14}$$
$$(P_{31}X + P_{32}Y + P_{33}Z + P_{34})y = P_{21}X + P_{22}Y + P_{23}Z + P_{24}$$

- 2 equations!
- Are the equations linear in the parameters?
- How many equations do we need?

Camera calibration

$$(P_{31}X + P_{32}Y + P_{33}Z + P_{34})x = P_{11}X + P_{12}Y + P_{13}Z + P_{14}$$

$$XxP_{31} + YxP_{32} + ZxP_{33} + xP_{34} - XP_{11} - YP_{12} - ZP_{13} - P_{14} = 0$$

- In matrix vector form: Ap = 0
- 6 points give 12 equations, 12 variables to solve for
- But can only solve upto scale

Camera calibration

- In matrix vector form: Ap = 0
- We want non-trivial solutions
- If p is a solution, αp is a solution too
- Let's just search for a solution with unit norm

$$A\mathbf{p} = 0$$

s.t

$$\|\mathbf{p}\| = 1$$

- How do you solve this?

Camera calibration

- In matrix vector form: Ap = 0
- We want non-trivial solutions
- If p is a solution, αp is a solution too
- Let's just search for a solution with unit norm

$$A\mathbf{p} = 0$$

s.t

$$\|\mathbf{p}\| = 1$$

- How do you solve this? *Eigenvector with 0 eigenvalue!*

Camera calibration

- We need 6 world points for which we know image locations
- Would any 6 points work?
  - What if all 6 points are the same?
- Need at least 6 non-coplanar points!

Camera calibration