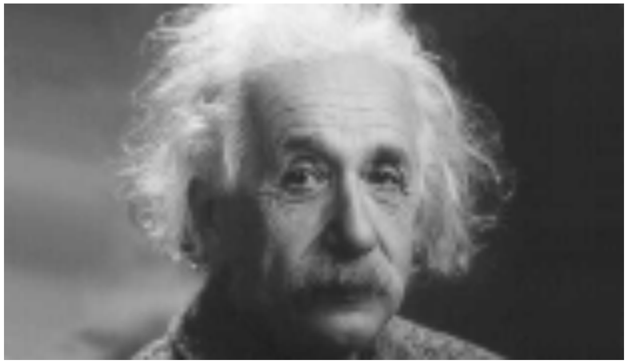
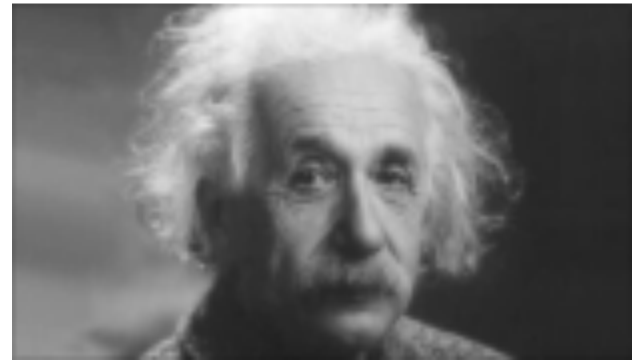
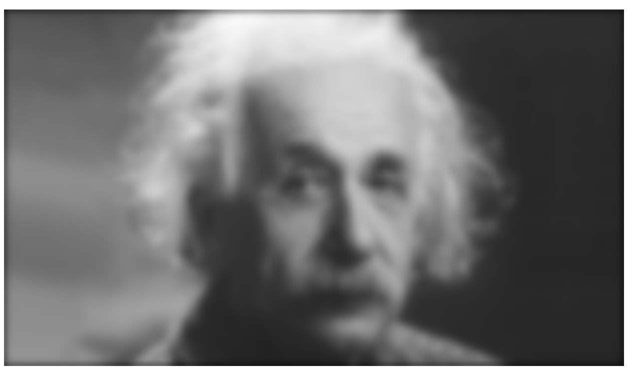
Fourier transforms

Gaussian filter

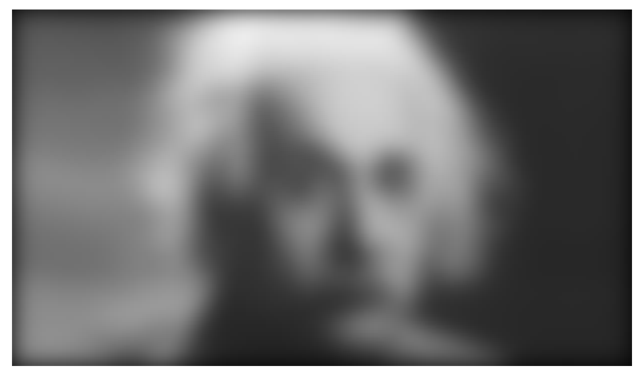




21x21, σ=0.5

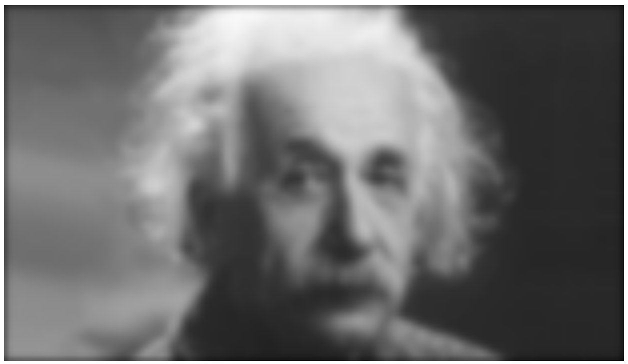


21x21, σ=1

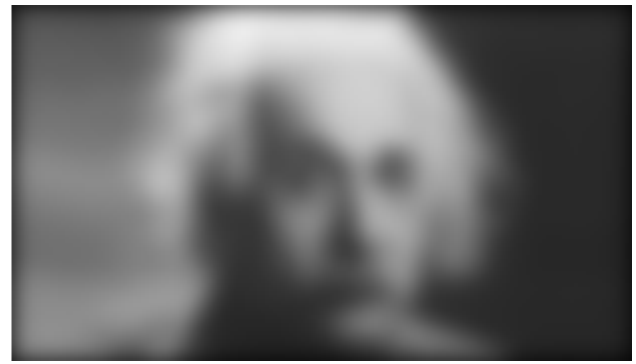


21x21, σ=3

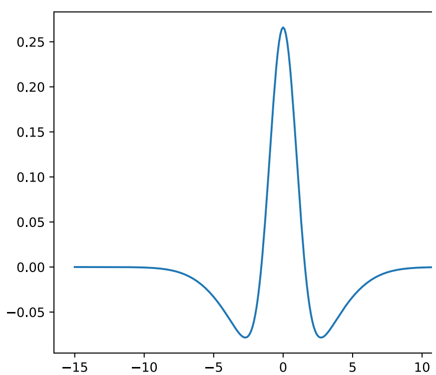
Difference of Gaussians

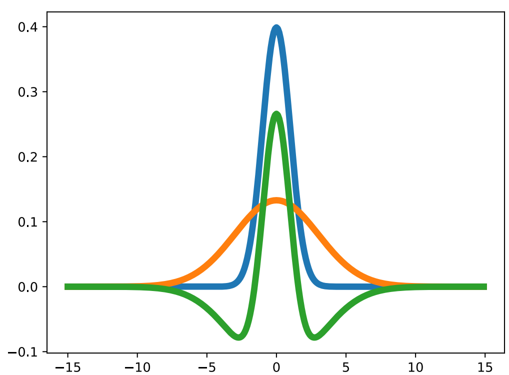


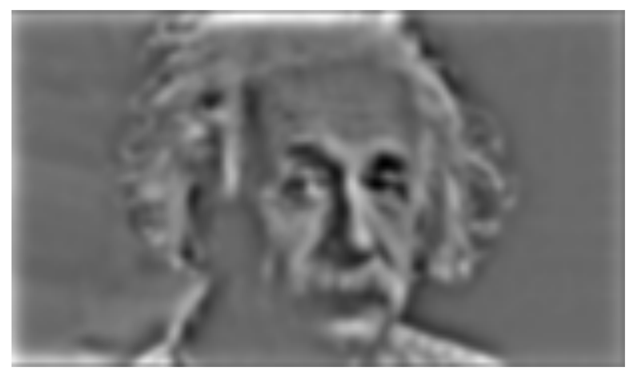
21x21, σ=1



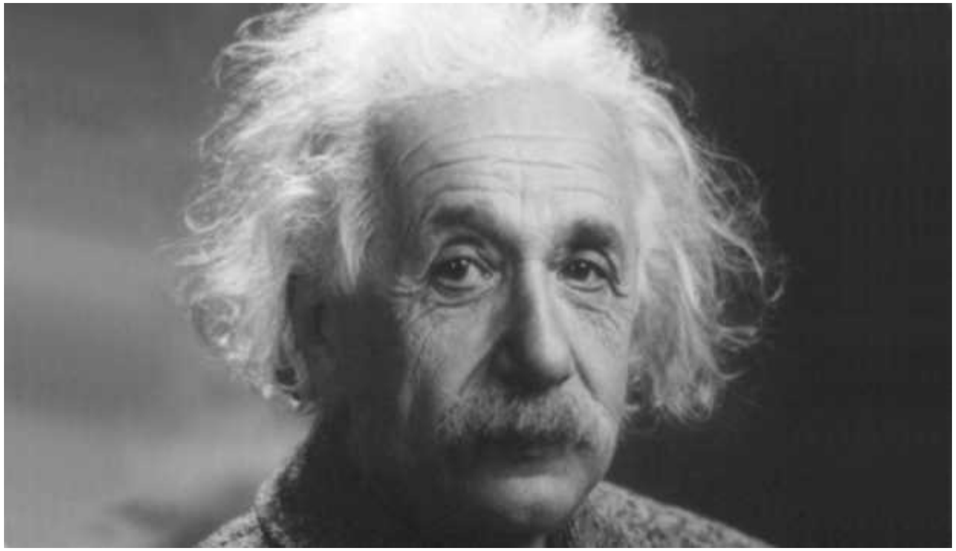
21x21, σ=3

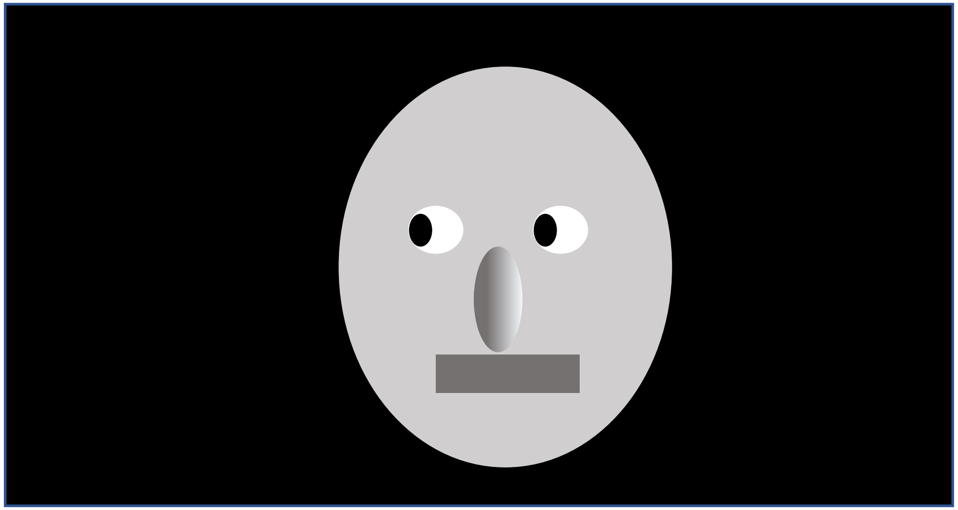






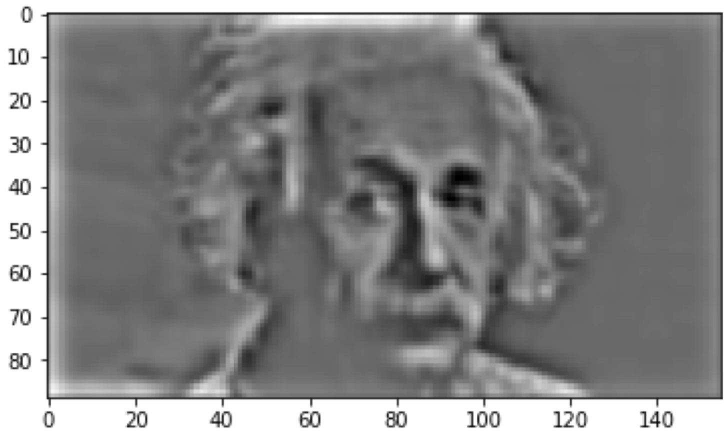
Images have structure at various scales

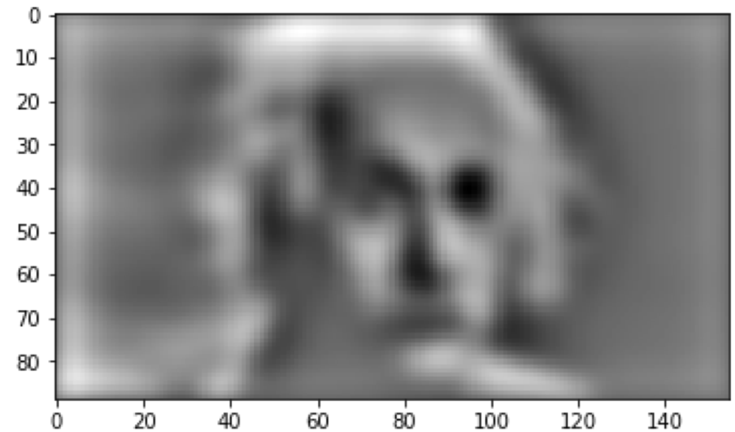


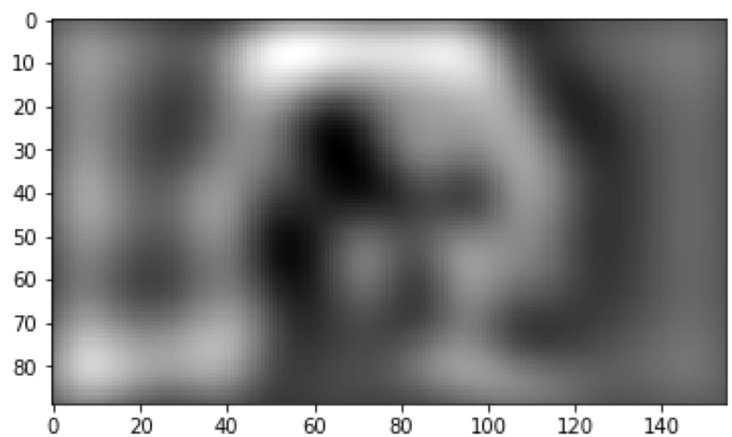


Images have structure at various scales

- Let’s formalize this!







A change of basis

- Want to write image I as:

$$I = a_1 B_1 + a_2 B_2 + \ldots + a_n B_n$$

- Basis: $B_1, \ldots, B_n$

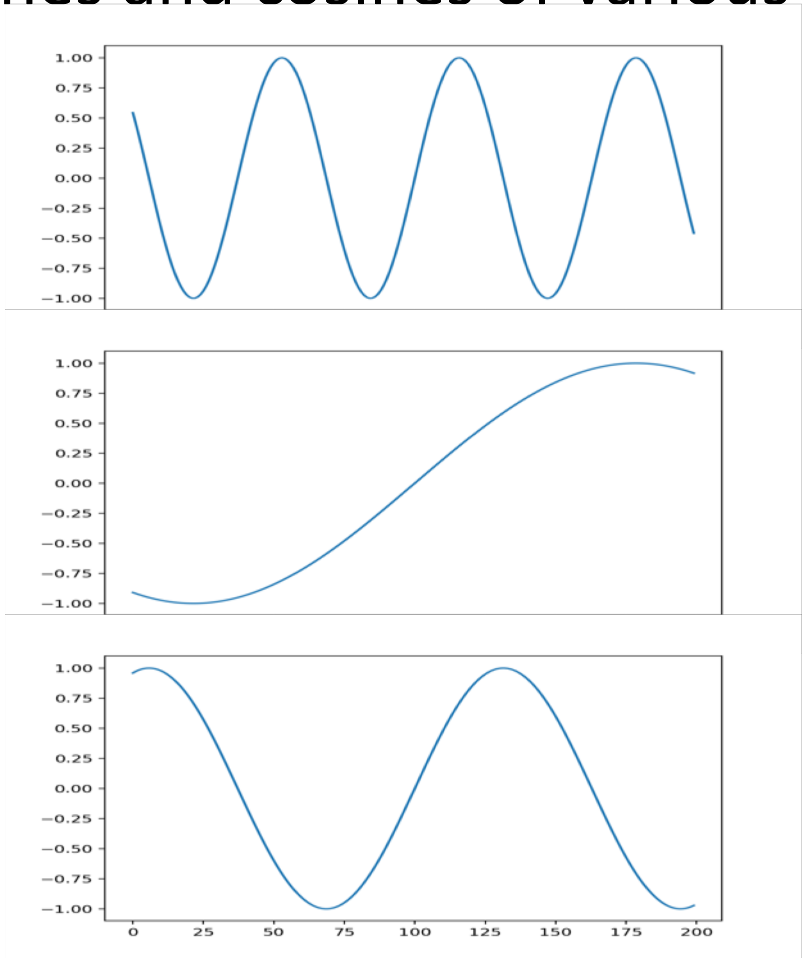
- What should basis be?

A change of basis

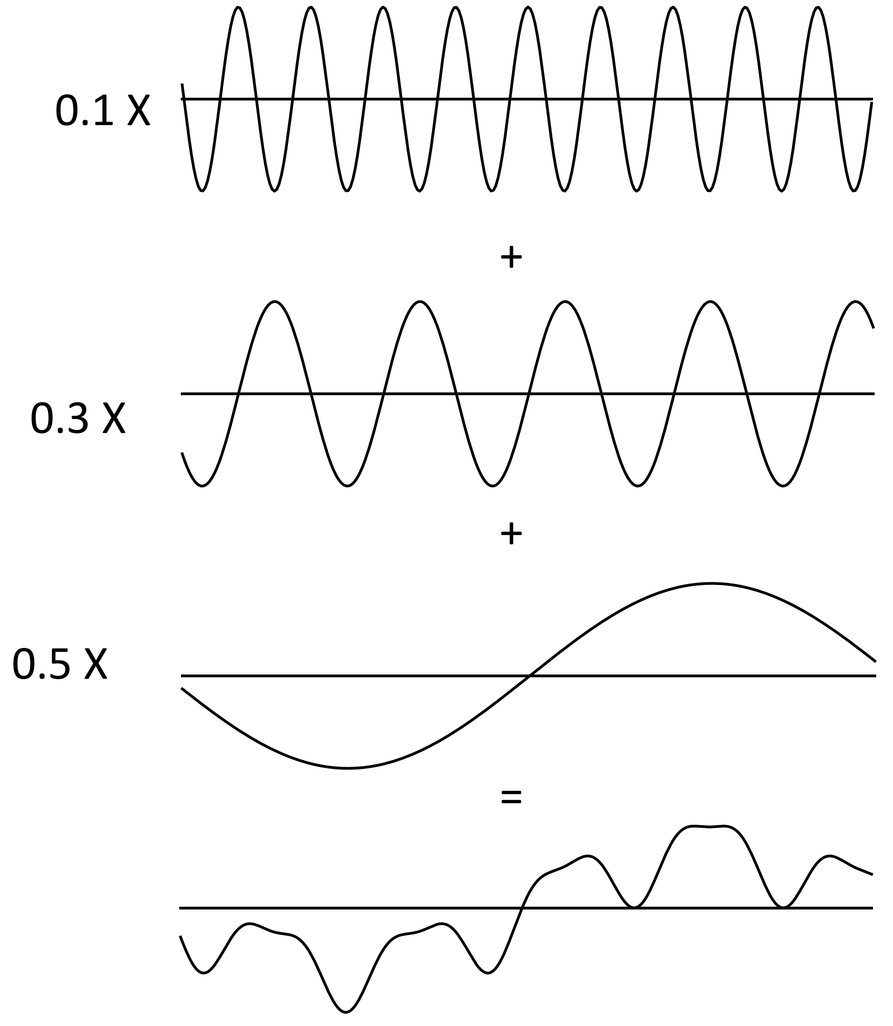
- Each basis should capture structure at a particular scale
- Coarse structure: Corresponding to large regions of the image
- Fine structure: corresponding to individual pixels as details
- One such basis: Fourier basis

Fourier bases for 1D signals

- Consider sines and cosines of various frequencies as basis



A combination of frequencies

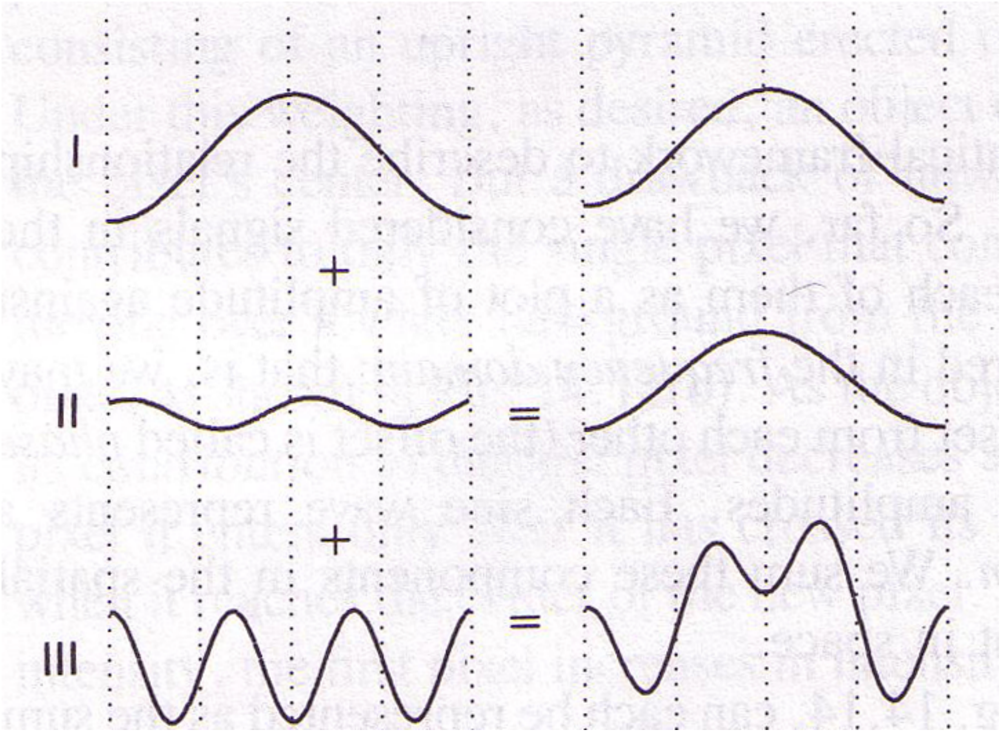


Fourier transform

- Can we figure out the canonical single-frequency signals that make up a complex signal?
  - *Yes!*
- Can *any* signal be decomposed in this way?
  - *Yes!*

Idea of Fourier Analysis

- Every signal (doesn't matter what it is)
  - Sum of sine/cosine waves



Idea of Fourier Analysis

- Every signal (doesn't matter what it is)
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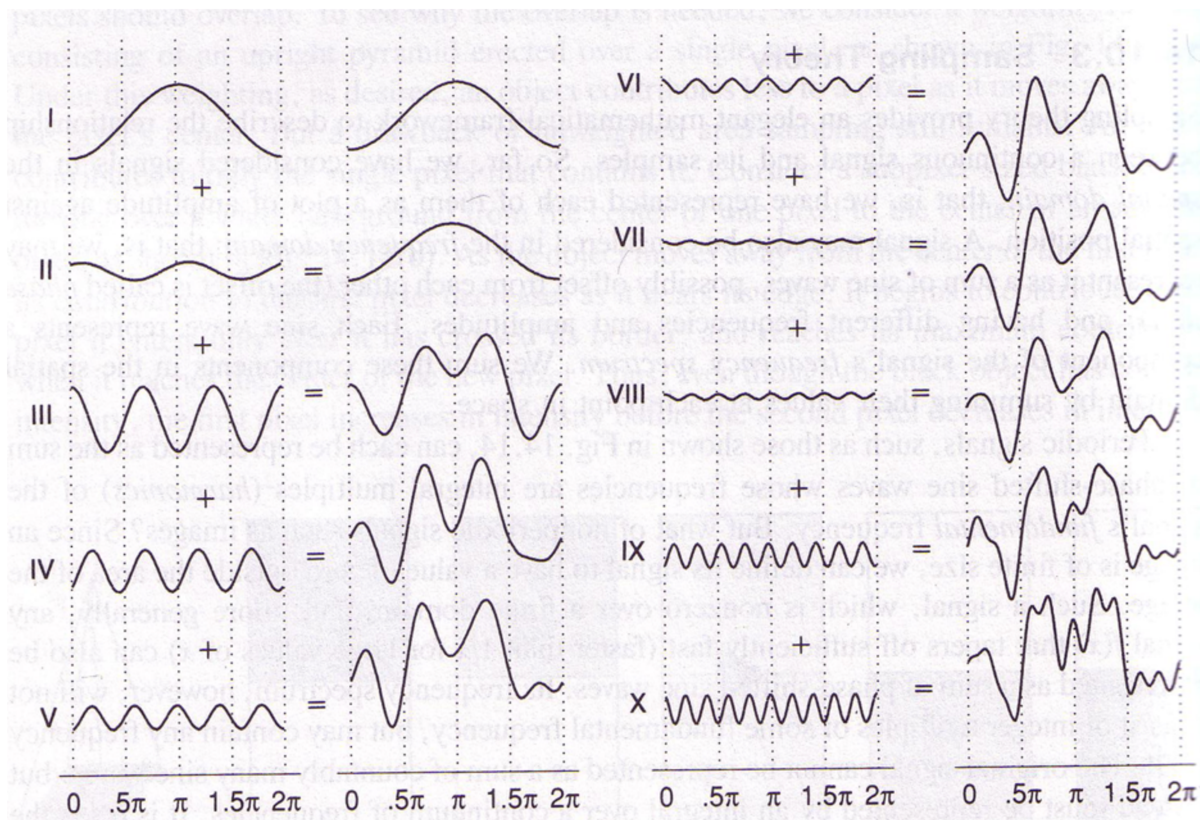
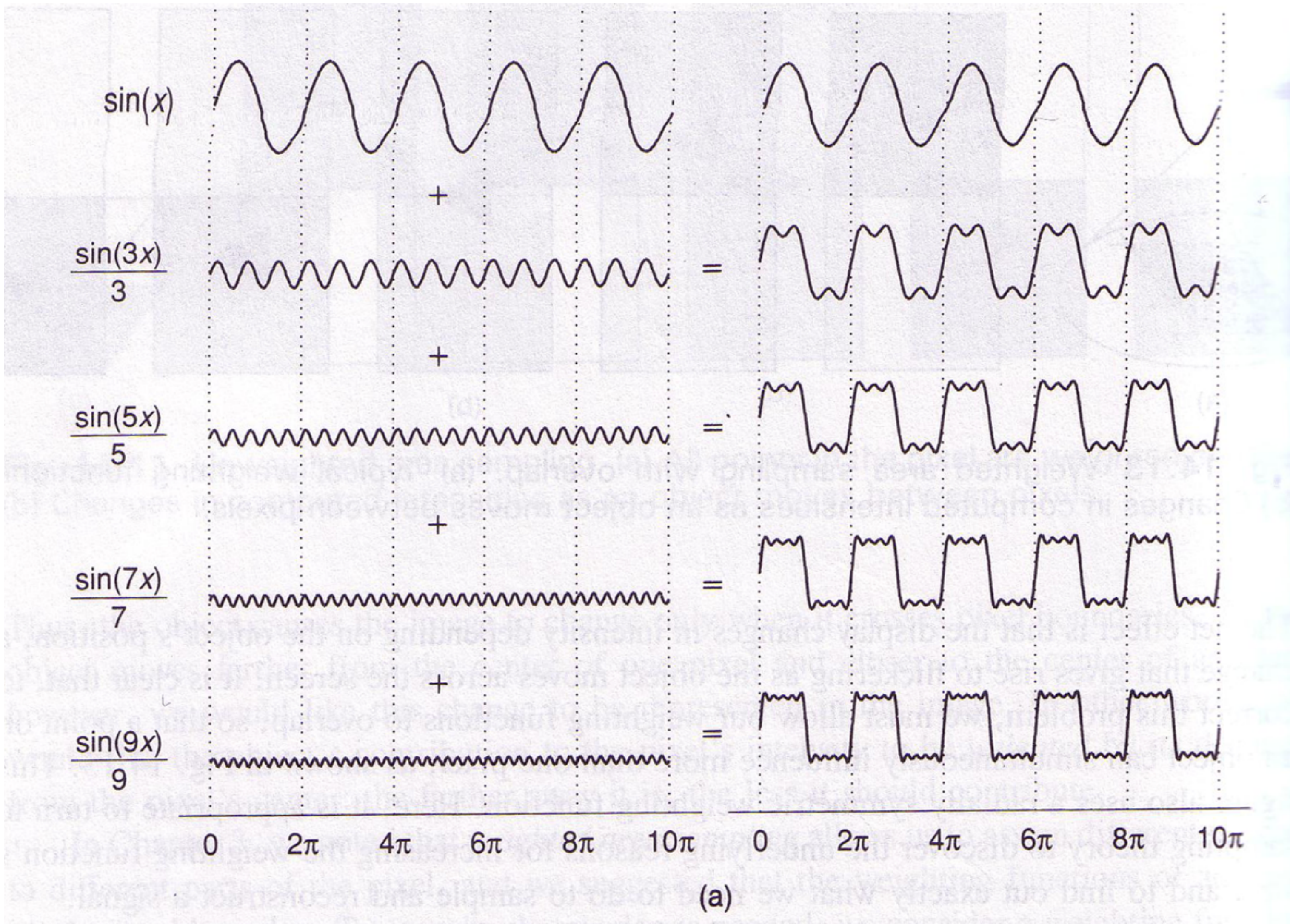


(b)

A box-like example

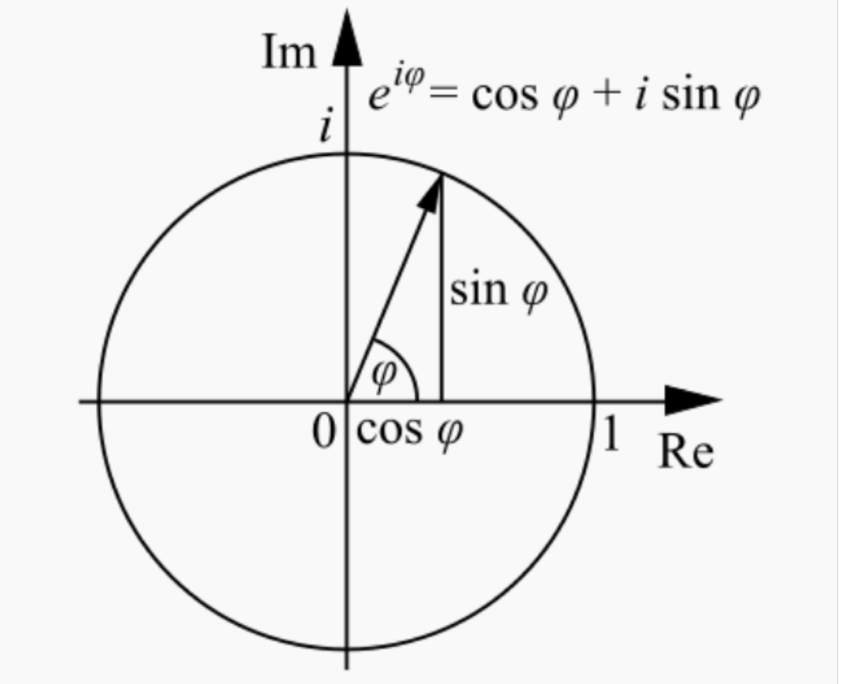


Fourier bases for 1D signals

- Not exactly sines and cosines, but *complex* variants

$$e^{ix} = \cos(x) + i\sin(x)$$
$$e^{i2\pi\nu t} = \cos(2\pi\nu t) + i\sin(2\pi\nu t)$$

- Euler's formula
- ν is the frequency
- Period is $\frac{1}{\nu}$



Fourier basis for 1D signals

- Consider a 1D array of size N
- k-th element of Fourier basis:
  - Has period $\frac{\mathrm{N}}{k}$
  - Varies between period of N (large scale structure ranging over entire array)
  - And period of 1 (fine scale structure ranging over individual elements)
- $B_k(n) = e^{\frac{2\pi i k n}{N}} = \cos\left(\frac{2\pi k n}{N}\right) + i \sin\left(\frac{2\pi k n}{N}\right)$

Fourier transform

- Consider any 1D signal x with N entries
- It can be expressed as a combination of Fourier basis elements:
- $x = a_0(x)B_0 + a_1(x)B_1 + \ \ldots + a_{N-1}(x)B_{N-1}$
- x can be represented using N **Fourier coefficients:**
  - X = $[a_0(x), a_1(x), \ldots, a_{N-1}(x)]$
- Fourier transform of x is X
- Inverse Fourier transform of X is x

Fourier transform

- Problem: basis is complex, but signal is real?
- Combine a pair of conjugate basis elements
- $B_{N-k}(\mathrm{n}) = e^{\frac{2\pi i(N-k)n}{N}} = e^{2\pi in - \frac{2\pi ikn}{N}} = e^{-\frac{2\pi ikn}{N}} = B_{-k}(n)$
- Consider $B_{-N/2}$ to $B_{N/2}$ as basis elements
- Real signals will have same coefficients for B_k and B_{-k}

Fourier transform

- What is B_0?
- $B_0 = e^{2\pi i 0 n/N} = 1$
- Coefficient at 0 acts as a "constant bias"
- All other basis elements average out to 0
- So average must come from 0 coefficient
- "DC component"