Recognition II

General recipe

- Fix hypothesis class

$$h(x; \mathbf{w}, b) = \sigma(\mathbf{w}^T \phi(x) + b)$$

- Define loss function

$$L(h(x; \mathbf{w}, b), y) = -y \log h(x; \mathbf{w}, b) + (1 - y) \log(1 - h(x; \mathbf{w}, b))$$

- Minimize total loss on the training set

$$\min_{\mathbf{w}, b} \sum_{i=1}^{N} L(h(x_i; \mathbf{w}, b), y_i)$$

- *Why should this work?*
- *How do we do the minimization in practice?*

Training = Optimization

- Need to minimize an objective

$$\min_{\mathbf{w},b} \sum_{i=1}^{N} L(h(x_i; \mathbf{w}, b), y_i)$$

- More generally, objective takes the form

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} f(x_i, y_i, \boldsymbol{\theta}) \equiv \min_{\boldsymbol{\theta}} F(\boldsymbol{\theta})$$

Training = optimization

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} f(x_i, y_i, \boldsymbol{\theta}) \equiv \min_{\boldsymbol{\theta}} F(\boldsymbol{\theta})$$

- How do we minimize this?
- Start from an initial estimate
- Iteratively reduce F. How?

Optimization and function gradients

- Suppose current estimate is $\boldsymbol{\theta}^{(t)}$
- Consider changing this to $\boldsymbol{\theta}^{(t)} + \Delta\boldsymbol{\theta}$
- How does the objective value change?
- For small $\Delta\boldsymbol{\theta}$, can approximate F using Taylor expansion
  - F is *locally linear*

$$F(\boldsymbol{\theta}^{(t)} + \Delta\boldsymbol{\theta}) \approx F(\boldsymbol{\theta}^{(t)}) + \nabla F(\boldsymbol{\theta}^{(t)})^T \Delta\boldsymbol{\theta}$$

$$\Rightarrow F(\boldsymbol{\theta}^{(t)} + \Delta\boldsymbol{\theta}) - F(\boldsymbol{\theta}^{(t)}) \approx \nabla F(\boldsymbol{\theta}^{(t)})^T \Delta\boldsymbol{\theta}$$

Optimization and function gradients

$$\Rightarrow F(\boldsymbol{\theta}^{(t)} + \Delta\boldsymbol{\theta}) - F(\boldsymbol{\theta}^{(t)}) \approx \nabla F(\boldsymbol{\theta}^{(t)})^T \Delta\boldsymbol{\theta}$$

- We want $F(\boldsymbol{\theta}^{(t)} + \Delta\boldsymbol{\theta}) - F(\boldsymbol{\theta}^{(t)})$ to be negative
  - As highly negative as possible
- So we want $\nabla F(\boldsymbol{\theta}^{(t)})^T \Delta\boldsymbol{\theta}$ to be as negative as possible

$$\Delta\boldsymbol{\theta} = -\lambda \nabla F(\boldsymbol{\theta}^{(t)})$$

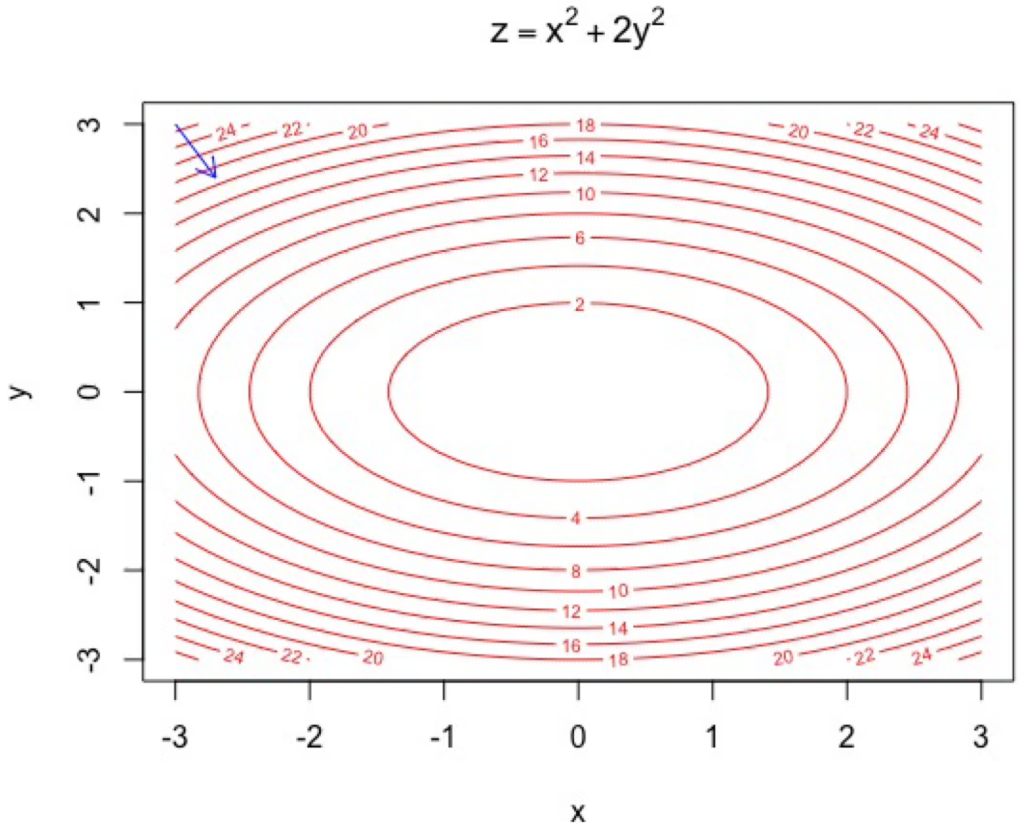
$$\Rightarrow \nabla F(\boldsymbol{\theta}^{(t)})^T \Delta\boldsymbol{\theta} = -\lambda \|\nabla F(\boldsymbol{\theta}^{(t)})\|^2$$

- λ is step size

Optimization using gradient descent

- Randomly initialize $\boldsymbol{\theta}^{(0)}$
- For i = 1 to max_iterations:
  - Compute gradient of F at $\boldsymbol{\theta}^{(t)}$
  - $\boldsymbol{\theta}^{(t+1)} \leftarrow \boldsymbol{\theta}^{(t)} - \lambda \nabla F(\boldsymbol{\theta}^{(t)})$
    - Function value will decrease by $\lambda ||\nabla F(\boldsymbol{\theta}^{(t)})||^2$
  - Repeat until $||\nabla F(\boldsymbol{\theta}^{(t)})||^2$ drops below a threshold

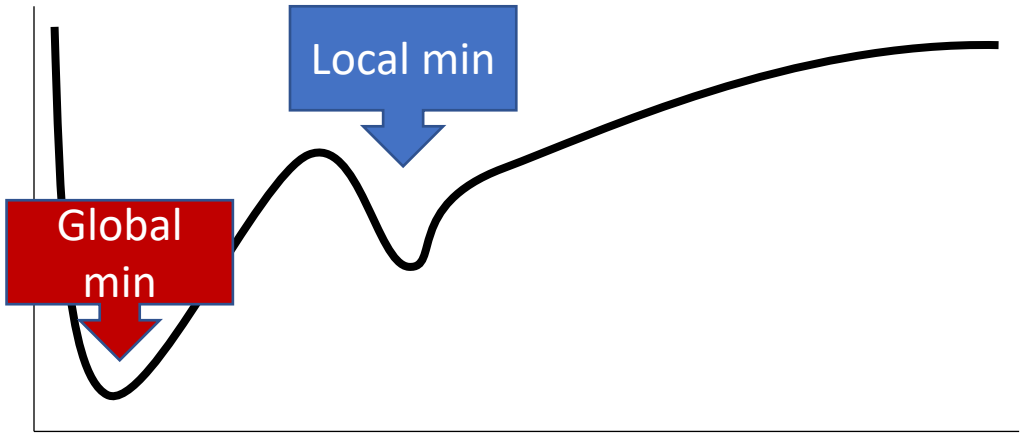
Gradient descent



https://yihui.name/animation/example/grad-desc/

Gradient descent - convergence

- Every step leads to a reduction in the function value
- If function is bounded below, we will eventually stop
- But will we stop at the right “global minimum”?
  - Not necessarily: local optimum!



Gradient descent in machine learning

$$\min_{\boldsymbol{\theta}} \sum_{i=1}^{N} f(x_i, y_i, \boldsymbol{\theta}) \equiv \min_{\boldsymbol{\theta}} F(\boldsymbol{\theta})$$

$$\nabla F(\boldsymbol{\theta}) = \sum_{i=1}^{N} \nabla f(x_i, y_i, \boldsymbol{\theta})$$

- Computing the gradient requires a *loop over all training examples*
- Very expensive for large datasets

Stochastic gradient descent

$$\nabla F(\boldsymbol{\theta}) = \sum_{i=1}^{N} \nabla f(x_i, y_i, \boldsymbol{\theta})$$

$$\nabla F(\boldsymbol{\theta}) \approx \sum_{j=1}^{K} \nabla f(x_{i_j}, y_{i_j}, \boldsymbol{\theta})$$

- Randomly sample small subset of examples
- Compute gradient on small subset
  - *Unbiased estimate of true gradient*
- Take step along estimated gradient

General recipe

Logistic Regression!

- Fix hypothesis class

$$h(x; \mathbf{w}, b) = \sigma(\mathbf{w}^T \phi(x) + b)$$

- Define loss function

$$L(h(x; \mathbf{w}, b), y) = -y \log h(x; \mathbf{w}, b) + (1 - y) \log(1 - h(x; \mathbf{w}, b))$$

- Minimize total loss on the training set using SGD

$$\min_{\mathbf{w}, b} \sum_{i=1}^{N} L(h(x_i; \mathbf{w}, b), y_i)$$

General recipe

- Fix hypothesis class

$$h(x; \mathbf{w}, b) = \sigma(\mathbf{w}^T \phi(x) + b)$$

- Define loss function

$$L(h(x; \mathbf{w}, b), y) = -y \log h(x; \mathbf{w}, b) + (1 - y) \log(1 - h(x; \mathbf{w}, b))$$

- Minimize total loss on the training set using SGD

$$\min_{\mathbf{w}, b} \sum_{i=1}^{N} L(h(x_i; \mathbf{w}, b), y_i)$$

- *Why should this work?*

Why should this work?

- Let us look at the objective more carefully

$$\min_{\mathbf{w},b} \sum_{i=1}^{N} L(h(x_i; \mathbf{w}, b), y_i)$$

$$\equiv \min_{\mathbf{w},b} \frac{1}{N} \sum_{i=1}^{N} L(h(x_i; \mathbf{w}, b), y_i)$$

- We are basically minimizing average loss on the training set
- Is this what we actually care about?

Risk

- Given:
  - Distribution $\mathcal{D}$ over (x,y) pairs
  - A hypothesis $h \in H$ from hypothesis class H
  - Loss function L
- We are interested in Expected Risk:

$$R(h) = \mathbb{E}_{(x,y)\sim\mathcal{D}} L(h(x), y)$$

- Given training set S, and a particular hypothesis h, Empirical Risk:

$$\hat{R}(S,h) = \frac{1}{|S|} \sum_{(x,y)\in S} L(h(x), y)$$

Risk

$$R(h) = \mathbb{E}_{(x,y)\sim\mathcal{D}} L(h(x), y) \qquad \hat{R}(S,h) = \frac{1}{|S|} \sum_{(x,y)\in S} L(h(x), y)$$

- Left: true quantity of interest, right: estimate
- How good is this estimate?
- If h is *randomly chosen,* actually a pretty good estimate!
  - In statistics-speak, it is an *unbiased estimator* : correct in expectation

$$\mathbb{E}_{S\sim\mathcal{D}^n} \hat{R}(S,h) = R(h)$$

Risk

- Empirical risk unbiased estimate of expected risk
- Want to minimize expected risk
- Idea: Minimize *empirical risk* instead
- This is the **Empirical Risk Minimization Principle**

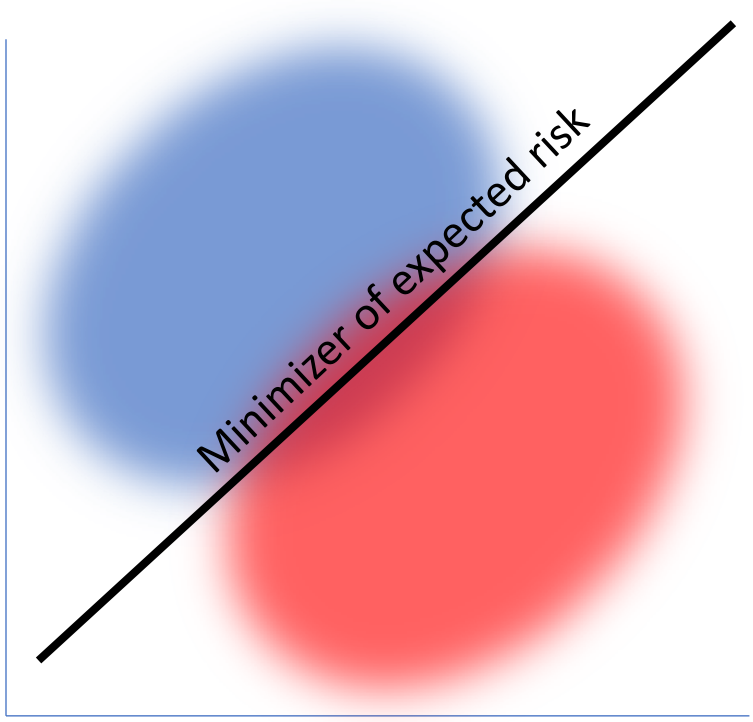
$$R(h) = \mathbb{E}_{(x,y)\sim\mathcal{D}} L(h(x), y) \qquad \hat{R}(S,h) = \frac{1}{|S|} \sum_{(x,y)\in S} L(h(x), y)$$

$$\boxed{h^* = \arg\min_{h\in H} \hat{R}(S,h)}$$

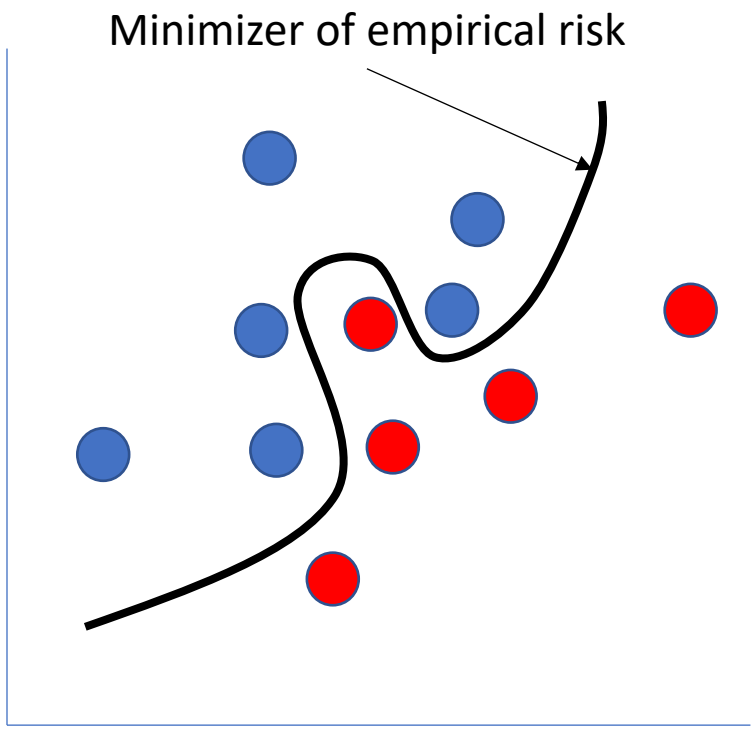
Overfitting

- For *randomly chosen* h, empirical risk (training error) good estimate of expected risk
- But we are *choosing* h by minimizing training error
- Empirical risk of chosen hypothesis *no longer* unbiased estimate:
  - We chose hypothesis based on S
  - Might have chosen h for which S is a special case
- Overfitting:
  - Minimize training error, but generalization error *increases*

Overfitting = fitting the noise



True distribution



Sampled training set

Generalization

$$R(h) = \mathbb{E}_{(x,y)\sim\mathcal{D}} L(h(x), y) \qquad \hat{R}(S,h) = \frac{1}{|S|} \sum_{(x,y)\in S} L(h(x), y)$$

$$R(h) = \hat{R}(S,h) + (R(h) - \hat{R}(S,h))$$

Training error

Generalization error

Controlling generalization error

- Variance of empirical risk inversely proportional to size of S
  - Choose very large S!
- *Larger* the hypothesis class H, *Higher* the chance of hitting bad hypotheses with low training error and high generalization error
  - Choose small H!
- For many models, can *bound* generalization error using some property of parameters
  - Regularize during optimization!
  - Eg. L2 regularization

Controlling the size of the hypothesis class

$$h(x; \mathbf{w}, b) = \sigma(\mathbf{w}^T \phi(x) + b)$$

- How many parameters (**w,** b) are there to find?
- Depends on dimensionality of ϕ
- Large dimensionality = large number of parameters = more chance of overfitting
- Rule of thumb: size of training set should be at least 10x number of parameters
- Often training sets are much smaller

Regularization

- Old objective

$$\min_{\mathbf{w},b} \sum_{i=1}^{N} L(h(x_i; \mathbf{w}, b), y_i)$$

- New objective

$$\min_{\mathbf{w},b} \sum_{i=1}^{N} L(h(x_i; \mathbf{w}, b), y_i) + \lambda \|\mathbf{w}\|^2$$

- Why does this help?

Regularization

$$\min_{\mathbf{w},b} \sum_{i=1}^{N} L(h(x_i; \mathbf{w}, b), y_i) + \lambda \|\mathbf{w}\|^2$$

- Ensures classifier does not weigh any one feature too highly
- Makes sure classifier scores *vary slowly* when image changes

$$|\mathbf{w}^T \phi(x_1) - \mathbf{w}^T \phi(x_2)| \leq \|\mathbf{w}\| \|\phi(x_1) - \phi(x_2)\|$$

Controlling generalization error

- How do we know we are overfitting?
  - Use a *held-out* “validation set”
  - To be an unbiased sample, must be completely *unseen*

Putting it all together

- Want model with least expected risk = expected loss
- But expected risk hard to evaluate
- Empirical Risk Minimization: minimize empirical risk in training set
- Might end up picking special case: overfitting
- Avoid overfitting by:
  - Constructing large training sets
  - Reducing size of model class
  - Regularization

Putting it all together

- Collect training set and validation set
- Pick hypothesis class
- Pick loss function
- Minimize empirical risk (+ regularization)
- Measure performance on held-out validation set
- Profit!

Loss functions and hypothesis classes

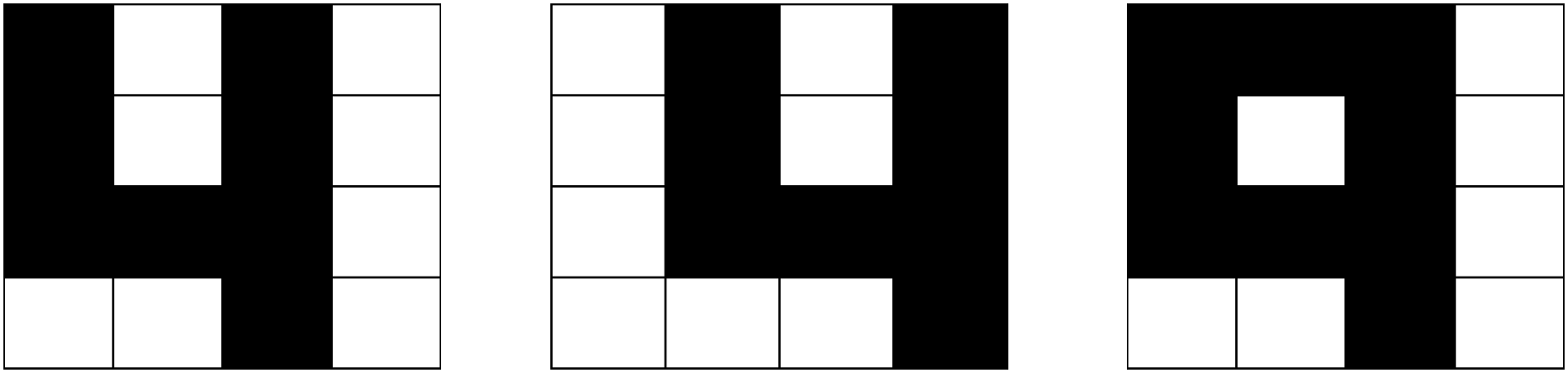
Loss function	Problem	Range of h	$\mathcal{Y}$	Formula
Log loss	Binary Classification	$\mathbb{R}$	$\{0, 1\}$	$\log(1 + e^{-yh(x)})$
Negative log likelihood	Multiclass classification	$[0, 1]^k$	$\{1, \ldots, k\}$	$-\log h_y(x)$
Hinge loss	Binary Classification	$\mathbb{R}$	$\{0, 1\}$	$\max(0, 1 - yh(x))$
MSE	Regression	$\mathbb{R}$	$\mathbb{R}$	$(y - h(x))^2$

Back to images

$$h(x; \mathbf{w}, b) = \sigma(\mathbf{w}^T \phi(x) + b)$$

- What should ϕ be?
- Simplest solution: string 2D image intensity values into vector

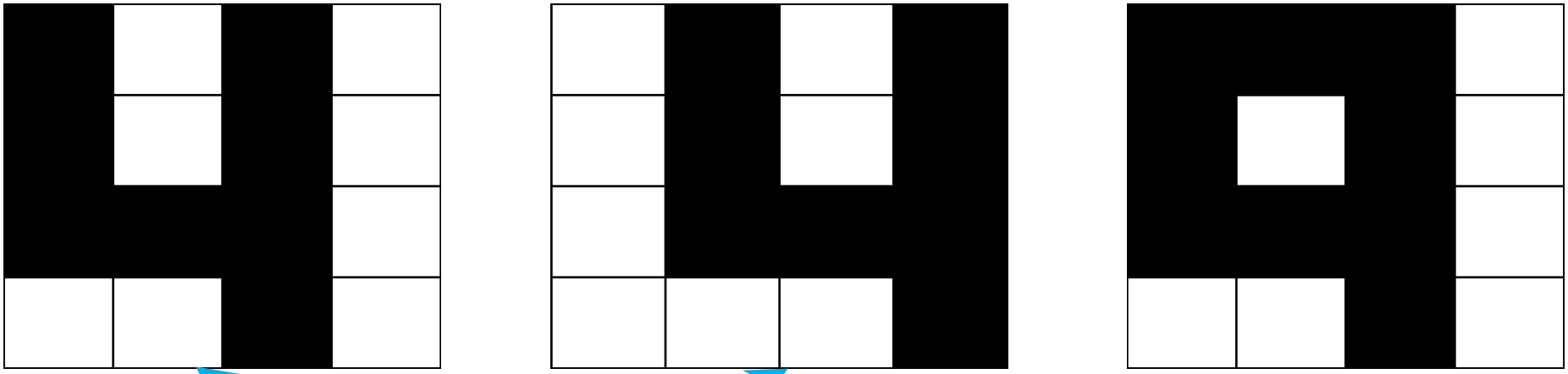
Linear classifiers on pixels are bad



- Solution 1: Better feature vectors
- Solution 2: Non-linear classifiers

Better feature vectors

These must have different feature vectors: *discriminability*



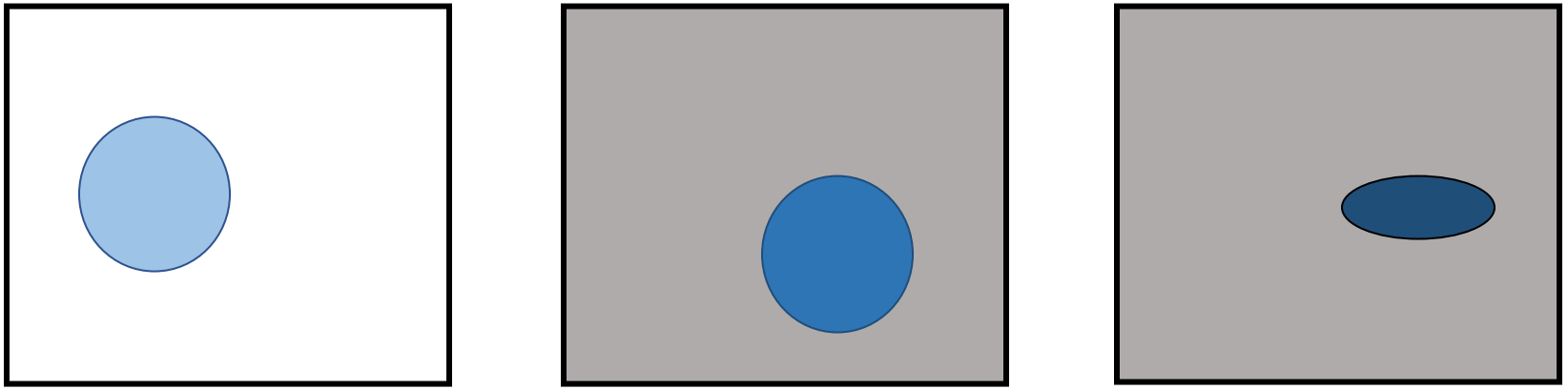
These must have similar feature vectors: *invariance*

Better feature vectors

- Invariance to
  - Illumination
  - Deformation
  - Translations/ rotations

Color and Lighting

Out-of-plane rotation

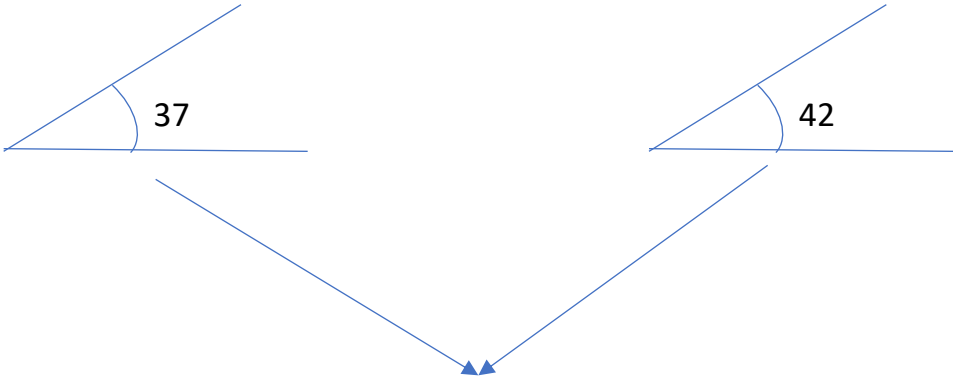


Out-of-plane rotation

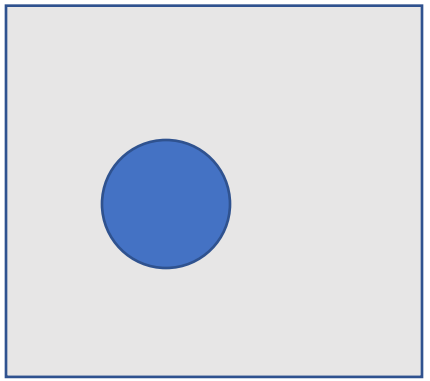
SIFT

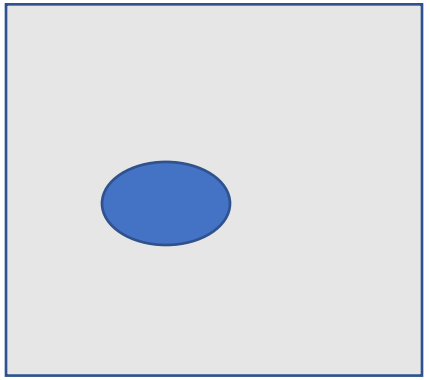
- Match *pattern of edges*
  - Edge orientation – clue to shape
- Be resilient to *small deformations*
  - Deformations might move pixels around, but slightly
  - Deformations might change edge orientations, but slightly

Invariance to deformation by quantization



Between 30 and 45

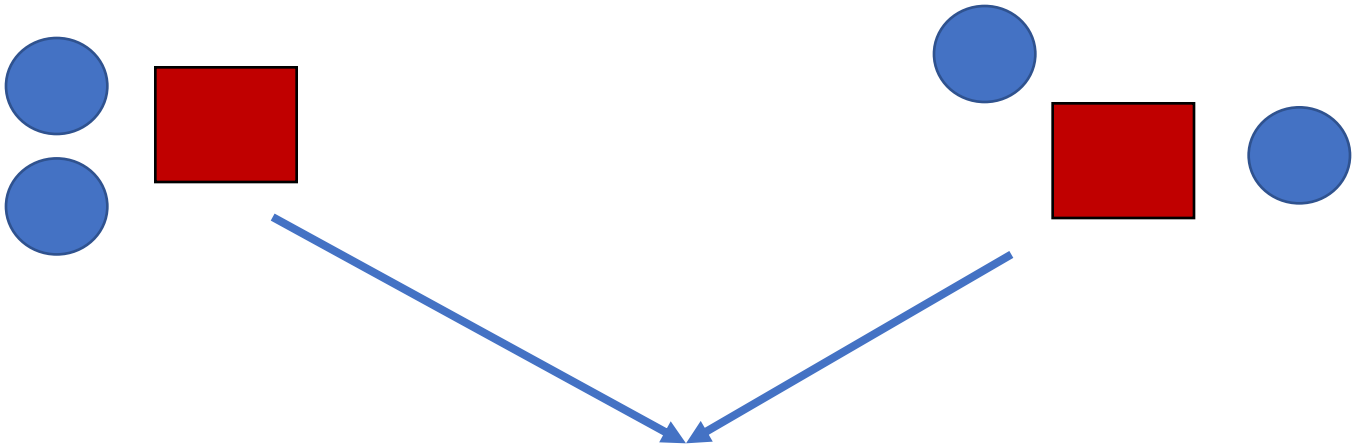




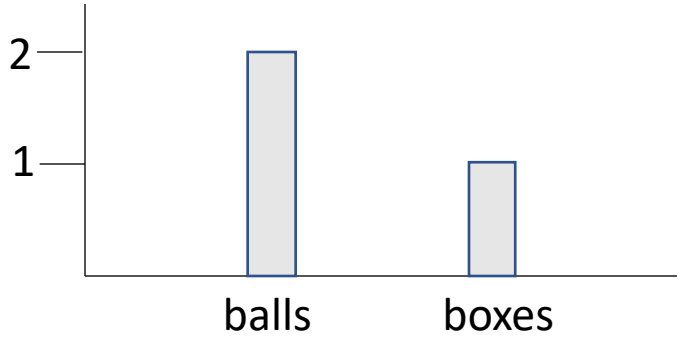
Invariance to deformation by quantization

$$g(\theta) = \begin{cases} 0 & \text{if } 0 < \theta < 2\pi/N \\ 1 & \text{if } 2\pi/N < \theta < 4\pi/N \\ 2 & \text{if } 4\pi/N < \theta < 6\pi/N \\ & \dots \\ N-1 & \text{if } 2(N-1)\pi/N \end{cases}$$

Spatial invariance by histograms



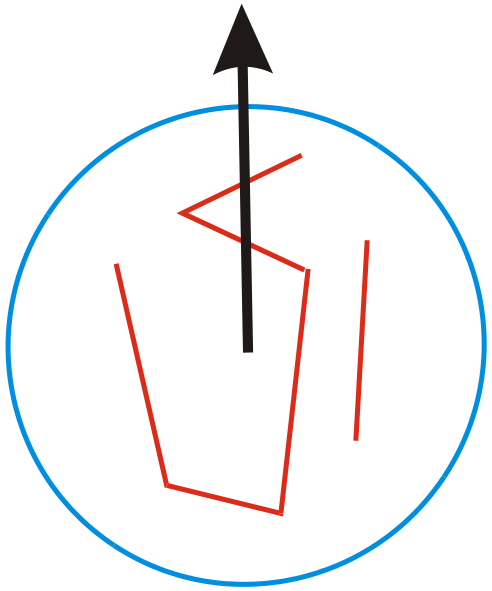
2 blue balls, one red box

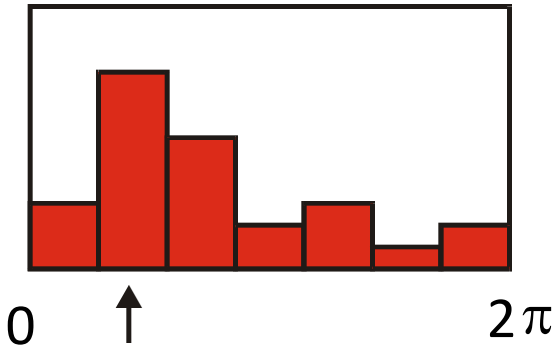


Rotation Invariance by Orientation Normalization

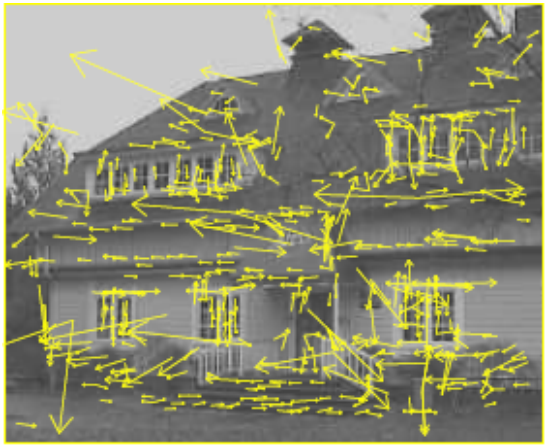
[Lowe, SIFT, 1999]

- Compute orientation histogram
- Select dominant orientation
- Normalize: rotate to fixed orientation





The SIFT descriptor



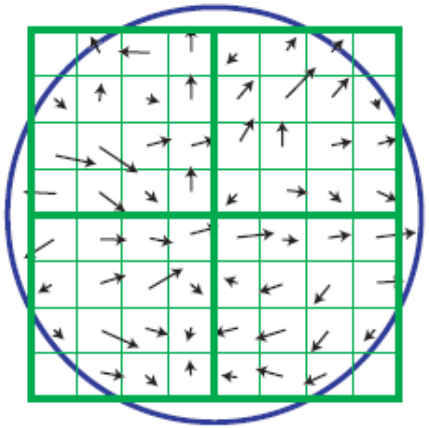
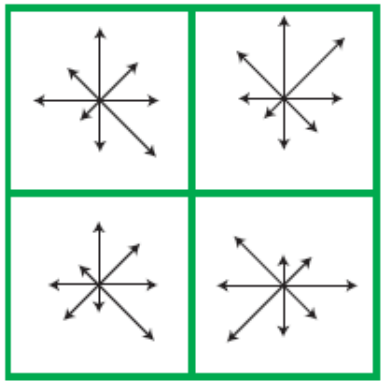


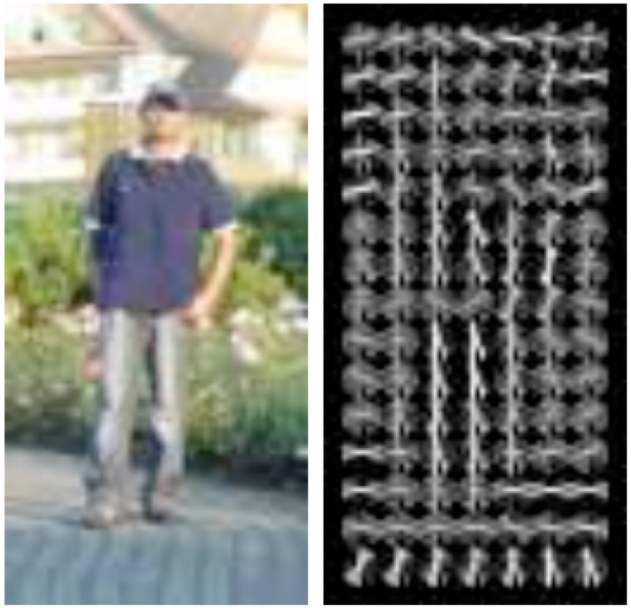
Image gradients



Keypoint descriptor

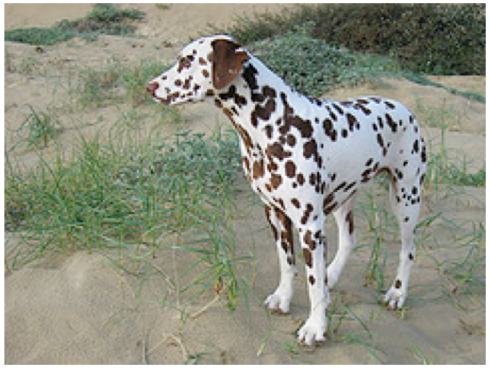
SIFT – Lowe IJCV 2004

Same but different: HOG



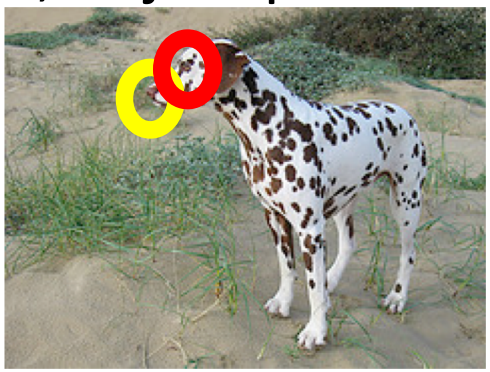
Histogram of oriented gradients
Same as SIFT but without orientation normalization. Why?

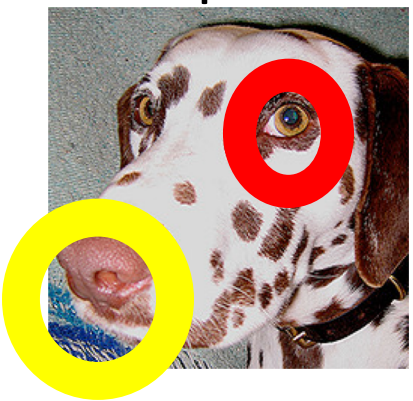
Invariance to large deformations



Invariance to large deformations

- Large deformations can cause objects / object parts to move a lot (much more than single grid cell)
- Yet, object parts themselves have precise appearance





- Idea: want to represent the image as a “bag of object parts”