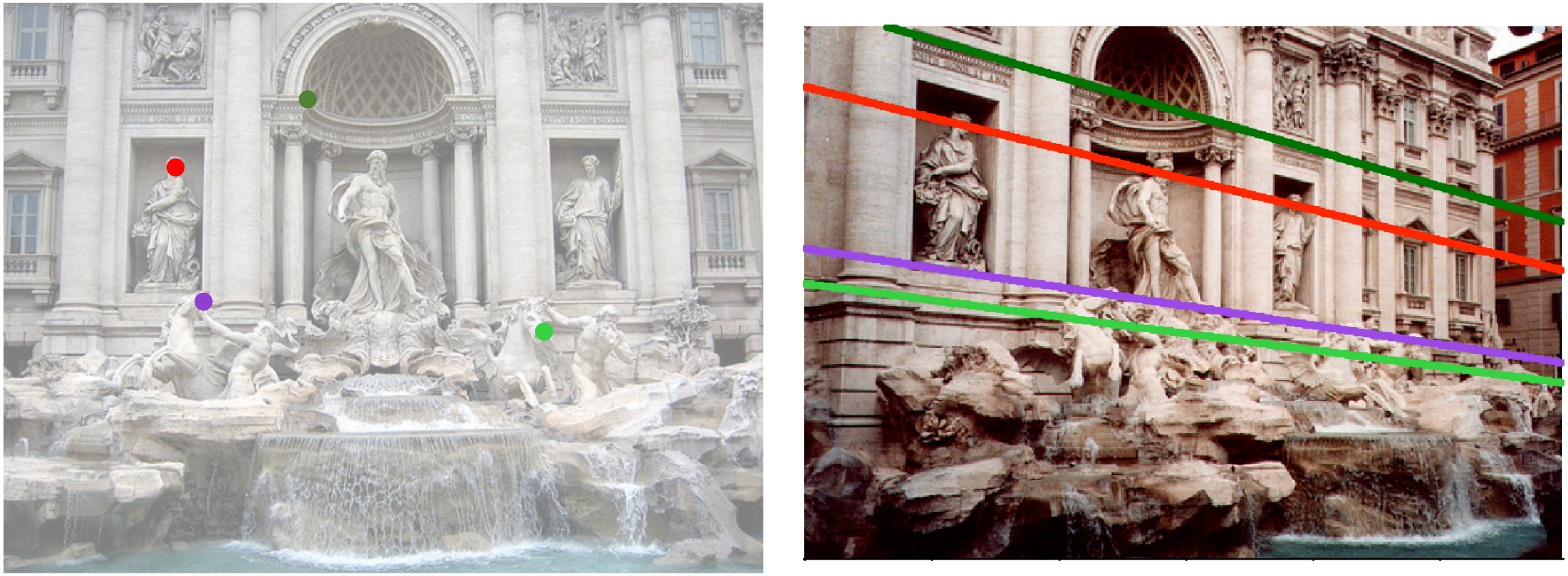
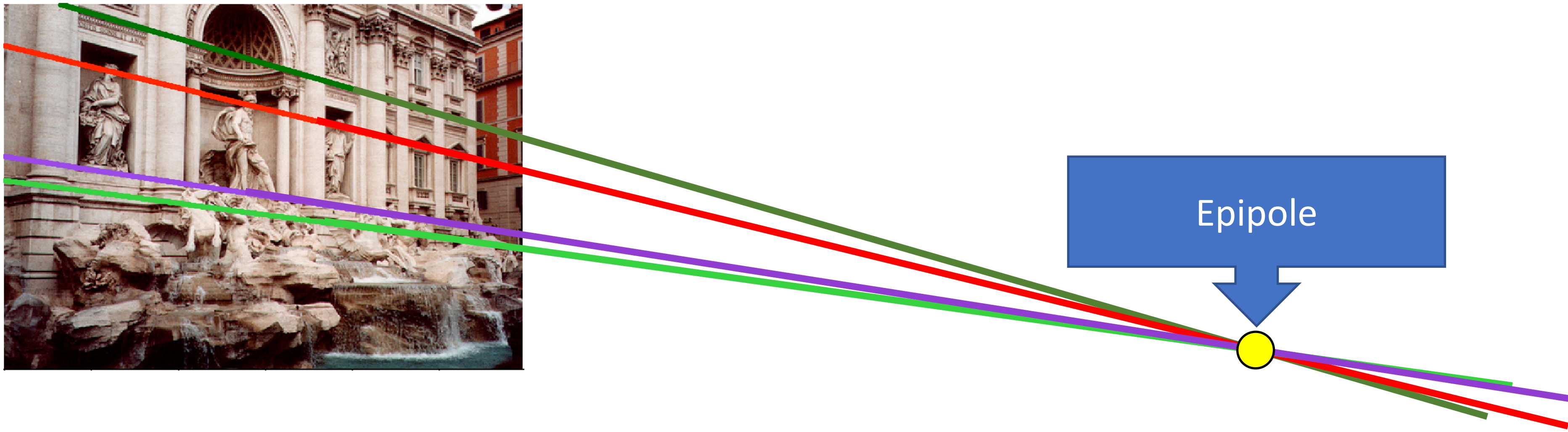
Epipolar geometry continued

Epipolar lines

Epipolar lines

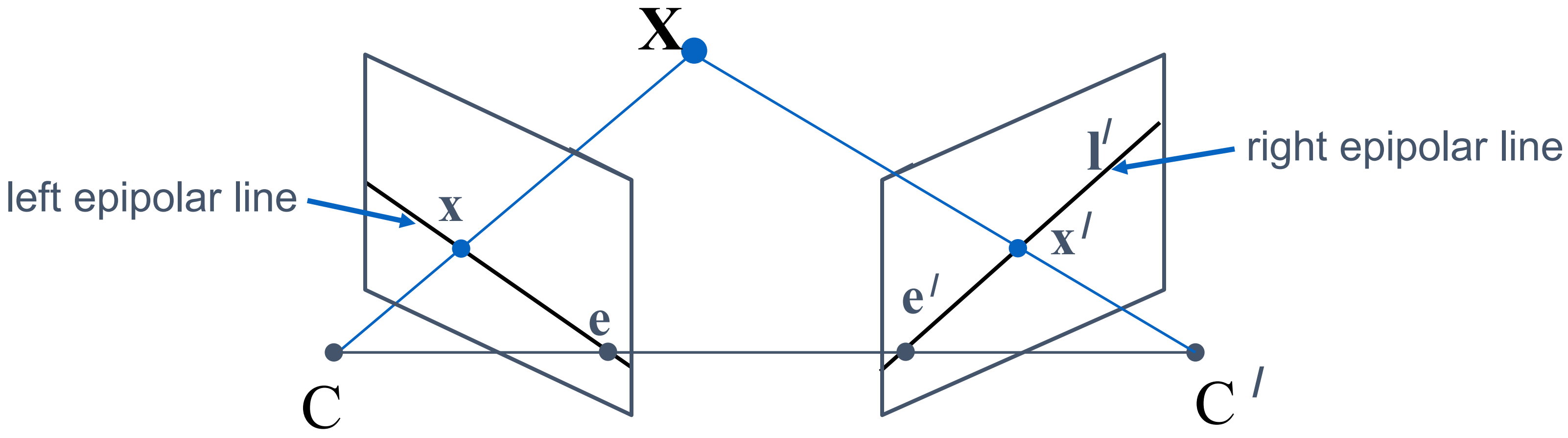


Epipolar lines



Epipole

Nomenclature



- The epipolar line $\mathbf{l}'$ is the image of the ray through $\mathbf{x}$
- The epipole $\mathbf{e}$ is the point of intersection of the line joining the camera centres with the image plane
  - this line is the baseline for a stereo rig, and
  - the translation vector for a moving camera
- The epipole is the image of the centre of the other camera: $\mathbf{e} = P\mathbf{C}'$, $\mathbf{e}' = P'\mathbf{C}$

Epipolar geometry - the math

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv K_1 \begin{bmatrix} R_1 & \mathbf{t}_1 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv K_2 \begin{bmatrix} R_2 & \mathbf{t}_2 \end{bmatrix} \vec{\mathbf{x}}_w$$

Epipolar geometry - the math

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv \begin{bmatrix} I & 0 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Epipolar geometry - the math

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} = \mathbf{x}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} = R\mathbf{x}_w + \mathbf{t}$$

Epipolar geometry - the math

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv \mathbf{x}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv R\mathbf{x}_w + \mathbf{t}$$

Epipolar geometry - the math

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\lambda_1 \vec{\mathbf{x}}_{img}^{(1)} = \mathbf{x}_w$$

$$\lambda_2 \vec{\mathbf{x}}_{img}^{(2)} = R\mathbf{x}_w + \mathbf{t}$$

Epipolar geometry - the math

$$\lambda_2 \vec{\mathbf{x}}^{(2)}_{img} = \lambda_1 R \vec{\mathbf{x}}^{(1)}_{img} + \mathbf{t}$$

$$\lambda_2 \mathbf{t} \times \vec{\mathbf{x}}^{(2)}_{img} = \lambda_1 \mathbf{t} \times R \vec{\mathbf{x}}^{(1)}_{img} + \cancelto{0}{\mathbf{t} \times \mathbf{t}}$$

$$\lambda_2 \mathbf{t} \times \vec{\mathbf{x}}^{(2)}_{img} = \lambda_1 \mathbf{t} \times R \vec{\mathbf{x}}^{(1)}_{img}$$

$$\cancelto{0}{\lambda_2 \vec{\mathbf{x}}^{(2)}_{img} \cdot \mathbf{t} \times \vec{\mathbf{x}}^{(2)}_{img}} = \lambda_1 \vec{\mathbf{x}}^{(2)}_{img} \cdot \mathbf{t} \times R \vec{\mathbf{x}}^{(1)}_{img}$$

$$0 = \lambda_1 \vec{\mathbf{x}}^{(2)}_{img} \cdot \mathbf{t} \times R \vec{\mathbf{x}}^{(1)}_{img}$$

Epipolar geometry - the math

$$\vec{\mathbf{x}}_{img}^{(2)} \cdot \mathbf{t} \times R\vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Can we write this as matrix vector operations?
- Cross product can be written as a matrix

$$[\mathbf{t}]_{\times} = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

$$[\mathbf{t}]_{\times}\mathbf{a} = \mathbf{t} \times \mathbf{a}$$

Epipolar geometry - the math

$$\vec{\mathbf{x}}_{img}^{(2)} \cdot [\mathbf{t}]_{\times} R \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Can we write this as matrix vector operations?
- Dot product can be written as a vector-vector times

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$$

Epipolar geometry - the math

$$\vec{\mathbf{x}}^{(2)}_{img} \cdot [\mathbf{t}]_{\times} R \vec{\mathbf{x}}^{(1)}_{img} = 0$$

- Can we write this as matrix vector operations?
- Dot product can be written as a vector-vector times

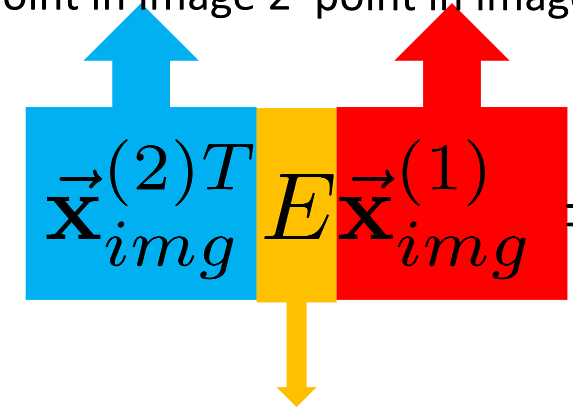
$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$$

Epipolar geometry - the math

$$\vec{\mathbf{x}}_{img}^{(2)T}[\mathbf{t}]_{\times}R\vec{\mathbf{x}}_{img}^{(1)} = 0$$

$$\vec{\mathbf{x}}_{img}^{(2)T}E\vec{\mathbf{x}}_{img}^{(1)} = 0$$

Epipolar geometry - the math



Homogenous coordinates of point in image 2

Homogenous coordinates of point in image 1

$$\vec{\mathbf{x}}_{img}^{(2)T} E \vec{\mathbf{x}}_{img}^{(1)} = 0$$

Essential matrix

Epipolar constraint and epipolar lines

$$\vec{\mathbf{x}}_{img}^{(2)T} E \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Consider a known, fixed pixel in the first image
- What constraint does this place on the corresponding pixel?
- $\vec{\mathbf{x}}_{img}^{(2)T} \mathbf{l} = 0$ where $\mathbf{l} = E \vec{\mathbf{x}}_{img}^{(1)}$
- What kind of equation is this?

Epipolar constraint and epipolar lines

$$\vec{\mathbf{x}}_{img}^{(2)T} E \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Consider a known, fixed pixel in the first image
- $\vec{\mathbf{x}}_{img}^{(2)T} \mathbf{l} = 0$ where $\mathbf{l} = E \vec{\mathbf{x}}_{img}^{(1)}$

$$\vec{\mathbf{x}}_{img}^{(2)T} \mathbf{l} = 0$$

$$\Rightarrow \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} = 0$$

$$\Rightarrow l_x x_2 + l_y y_2 + l_z = 0$$

Line!

Epipolar constraint: putting it all together

- If **p** is a pixel in first image and **q** is the corresponding pixel in the second image, then: $\mathbf{q}^T E \mathbf{p} = 0$
- $E = [\mathbf{t}]_X R$
- For fixed **p**, **q** must satisfy: $\mathbf{q}^T \mathbf{l} = 0$, where $\mathbf{l} = E\mathbf{p}$

Epipolar line in 2nd image

- For fixed **q, p** must satisfy: $\mathbf{l}^T \mathbf{p} = 0$ where $\mathbf{l}^T = \mathbf{q}^T E$, or $\mathbf{l} = E^t \mathbf{q}$

Epipolar line in 1st image

- These are epipolar lines!

Essential matrix and epipoles

- E = $[\mathbf{t}]_X R$

$$\vec{\mathbf{c}_2} = \mathbf{t}$$
$$\vec{\mathbf{c_2}}^T E = \mathbf{t}^T E = \mathbf{t}^T [\mathbf{t}]_\times R = 0$$
$$\vec{\mathbf{c}_2}^T E \mathbf{p} = 0 \ \forall \mathbf{p}$$

- E**p** is an epipolar line in 2nd image
- All epipolar lines in second image pass through c_2
- c_2 is epipole in 2nd image

Essential matrix and epipoles

- E = $[\mathbf{t}]_X R$

$$\vec{\mathbf{c}_1} = \mathbf{R}^T \mathbf{t}$$

$$E\vec{\mathbf{c}_1} = [\mathbf{t}]_\times R R^T \mathbf{t} = [\mathbf{t}]_\times \mathbf{t} = 0$$

$$\mathbf{q}^T E \vec{\mathbf{c}_1} = 0 \ \ \forall \mathbf{q}$$

- $E^T\mathbf{q}$ is an epipolar line in 1st image
- All epipolar lines in first image pass through c_1
- c_1 is the epipole in 1st image

Essential matrix

- Assume intrinsic parameters (K) are identity
- For corresponding pixels **p** and **q**
- $\mathbf{q}^T E \mathbf{p} = 0$
- $E = [\mathbf{t}]_X R$
- Given pixel **p** in first image $E\mathbf{p}$ is corresponding epipolar line in the second image
- Given pixel **q** in second image, $E^T\mathbf{q}$ is corresponding epipolar line in first image

Epipolar geometry - the math

- We assumed that intrinsic parameters K are identity
- What if they are not?

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv K_1 \begin{bmatrix} R_1 & \mathbf{t}_1 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv K_2 \begin{bmatrix} R_2 & \mathbf{t}_2 \end{bmatrix} \vec{\mathbf{x}}_w$$

Fundamental matrix

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv K_1 \begin{bmatrix} I & 0 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv K_2 \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Fundamental matrix

$$\lambda_1 \vec{\mathbf{x}}_{img}^{(1)} = K_1 \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\lambda_2 \vec{\mathbf{x}}_{img}^{(2)} = K_2 \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Fundamental matrix

$$\begin{aligned}\lambda_1 \vec{\mathbf{x}}^{(1)}_{img} =& K_1 \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \vec{\mathbf{x}}_w \\ =& K_1 \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} \\ =& K_1 \mathbf{x}_w \end{aligned}$$

$$\Rightarrow \lambda_1 K_1^{-1} \vec{\mathbf{x}}^{(1)}_{img} = \mathbf{x}_w$$

Fundamental matrix

$$\begin{aligned}
\lambda_2 \vec{\mathbf{x}}_{img}^{(2)} =& K_2 \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} \\
=& K_2 R \mathbf{x}_w + K_2 \mathbf{t} \\
=& \lambda_1 K_2 R K_1^{-1} \vec{\mathbf{x}}_{img}^{(1)} + K_2 \mathbf{t}
\end{aligned}$$

$$\Rightarrow \lambda_2 K_2^{-1} \vec{\mathbf{x}}_{img}^{(2)} = \lambda_1 R K_1^{-1} \vec{\mathbf{x}}_{img}^{(1)} + \mathbf{t}$$

$$\Rightarrow \lambda_2 [\mathbf{t}]_\times K_2^{-1} \vec{\mathbf{x}}_{img}^{(2)} = \lambda_1 [\mathbf{t}]_\times R K_1^{-1} \vec{\mathbf{x}}_{img}^{(1)}$$

$$\Rightarrow 0 = \vec{\mathbf{x}}_{img}^{(2)} K_2^{-T} [\mathbf{t}]_\times R K_1^{-1} \vec{\mathbf{x}}_{img}^{(1)}$$

Fundamental matrix

$$\Rightarrow 0 = \vec{\mathbf{x}}_{img}^{(2)} K_2^{-T} [\mathbf{t}]_{\times} R K_1^{-1} \vec{\mathbf{x}}_{img}^{(1)}$$

$$\Rightarrow 0 = \vec{\mathbf{x}}_{img}^{(2)} F \vec{\mathbf{x}}_{img}^{(1)}$$

Fundamental matrix

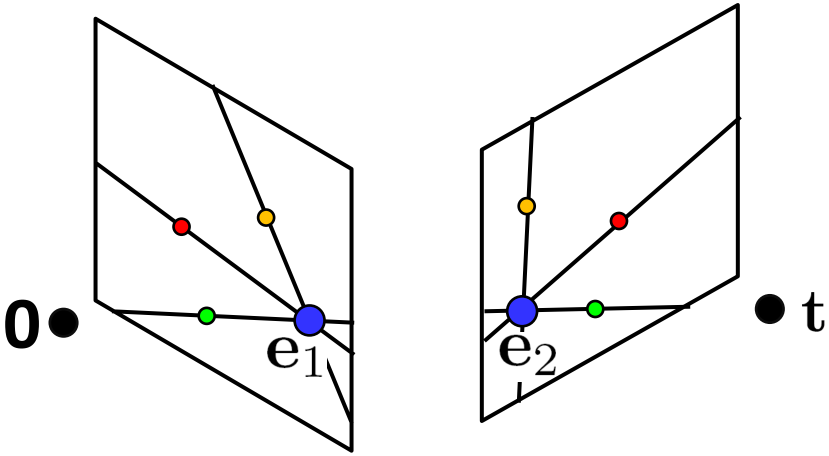
Fundamental matrix result

$$\mathbf{q}^T \mathbf{F} \mathbf{p} = 0$$

(Longuet-Higgins, 1981)

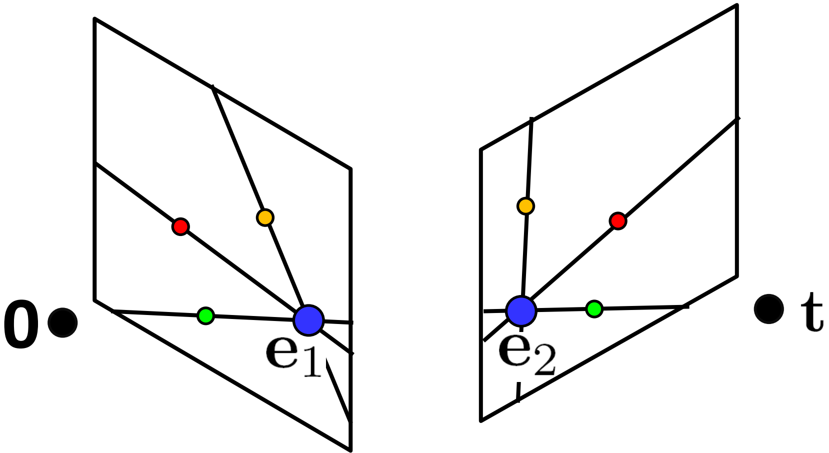
Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$



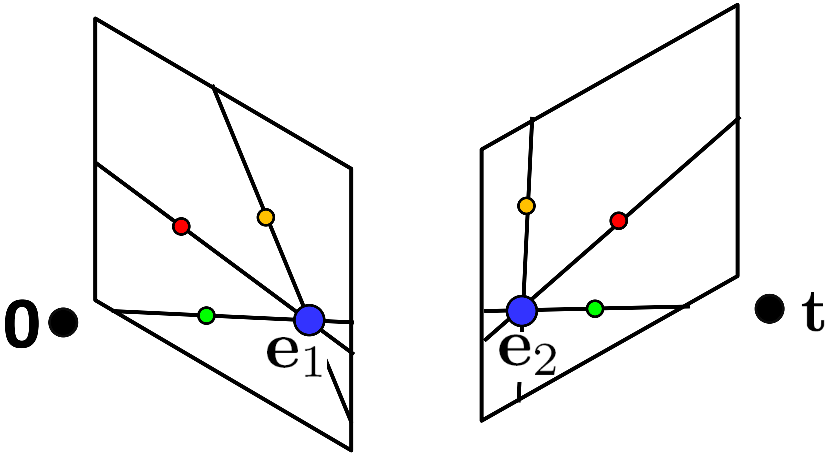
Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$
- $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{F}^T\mathbf{e}_2 = \mathbf{0}$
- All epipolar lines contain epipole



Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$
- $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{F}^T\mathbf{e}_2 = \mathbf{0}$
- $\mathbf{F}$ is rank 2

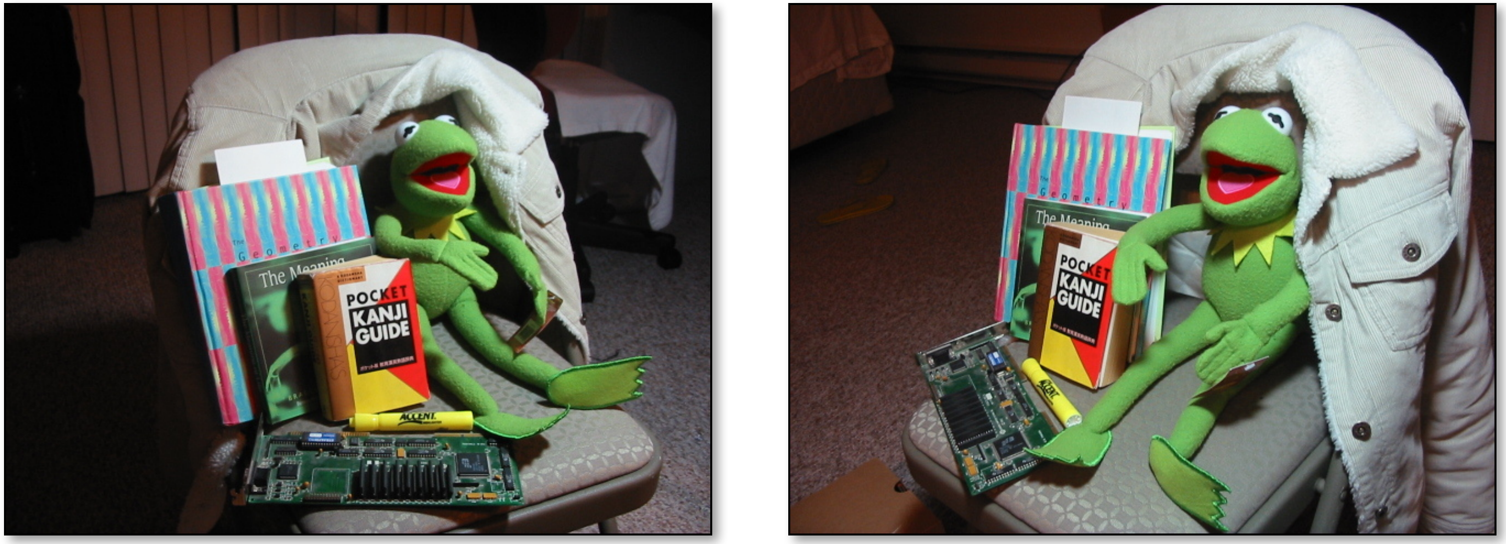


Why is F rank 2?

- F is a 3 x 3 matrix
- But there is a vector c_1 and c_2 such that $Fc_1 = 0$ and $F^Tc_2 = 0$

Fundamental matrix song

Estimating F



- If we don’t know $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{R}$, or $\mathbf{t}$, can we estimate $\mathbf{F}$ for two images?

- Yes, given enough correspondences

Estimating F – 8-point algorithm

- The fundamental matrix F is defined by

$$\mathbf{x}'^{\mathrm{T}}\mathbf{F}\mathbf{x} = 0$$

for any pair of matches x and x' in two images.

- Let x=$(u,v,1)^T$ and x'=$(u',v',1)^T$,

$$\mathbf{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

each match gives a linear equation

$$uu' f_{11} + vu' f_{12} + u' f_{13} + uv' f_{21} + vv' f_{22} + v' f_{23} + uf_{31} + vf_{32} + f_{33} = 0$$

8-point algorithm

$$\begin{bmatrix} u_1u_1' & v_1u_1' & u_1' & u_1v_1' & v_1v_1' & v_1' & u_1 & v_1 & 1 \\ u_2u_2' & v_2u_2' & u_2' & u_2v_2' & v_2v_2' & v_2' & u_2 & v_2 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_nu_n' & v_nu_n' & u_n' & u_nv_n' & v_nv_n' & v_n' & u_n & v_n & 1 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{12} \\ f_{13} \\ f_{21} \\ f_{22} \\ f_{23} \\ f_{31} \\ f_{32} \\ f_{33} \end{bmatrix} = 0$$

- In reality, instead of solving $\mathbf{Af} = 0$, we seek $\mathbf{f}$ to minimize $\|\mathbf{Af}\|$, least eigenvector of $\mathbf{A}^{\mathrm{T}}\mathbf{A}$.

8-point algorithm – Problem?

- **F** should have rank 2
- To enforce that **F** is of rank 2, F is replaced by F’ that minimizes $\|\mathbf{F}-\mathbf{F}'\|$ subject to the rank constraint.

- This is achieved by SVD. Let $\mathbf{F}=\mathbf{U}\Sigma\mathbf{V}^{\mathrm{T}}$, where

$$\Sigma=\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & \sigma_3\end{bmatrix}, \text{ let } \quad \Sigma'=\begin{bmatrix}\sigma_1 & 0 & 0\\ 0 & \sigma_2 & 0\\ 0 & 0 & 0\end{bmatrix}$$

then $\mathbf{F}'=\mathbf{U}\Sigma'\mathbf{V}^{\mathrm{T}}$ is the solution.

Recovering camera parameters from F / E

- Can we recover R and t between the cameras from F?

$$F = K_2^{-T}[\mathbf{t}]_\times R K_1^{-1}$$

- No: K_1 and K_2 are in principle arbitrary matrices
- What if we knew K_1 and K_2 to be identity?

$$E = [\mathbf{t}]_\times R$$

Recovering camera parameters from E

$$E = [\mathbf{t}]_{\times} R$$

$$\mathbf{t}^T E = \mathbf{t}^T [\mathbf{t}]_{\times} R = 0$$

$$E^T \mathbf{t} = 0$$

- **t** is a solution to $E^T\mathbf{x} = 0$
- Can’t distinguish between **t** and c**t** for constant scalar c
- How do we recover R?

Recovering camera parameters from E

$$E = [\mathbf{t}]_\times R$$

- We know E and **t**
- Consider taking SVD of E and $[\mathbf{t}]_\times$

$$[\mathbf{t}]_\times = U \Sigma V^T$$

$$E = U' \Sigma' V'^T$$

$$U' \Sigma' V'^T = E = [\mathbf{t}]_\times R = U \Sigma V^T R$$

$$U' \Sigma' V'^T = U \Sigma V^T R$$

$$V'^T = V^T R$$

Recovering camera parameters from E

$$E = [\mathbf{t}]_\times R$$

$$\mathbf{t}^T E = \mathbf{t}^T [\mathbf{t}]_\times R = 0$$

$$E^T \mathbf{t} = 0$$

- **t** is a solution to $E^T\mathbf{x} = 0$
- Can’t distinguish between **t** and c**t** for constant scalar c

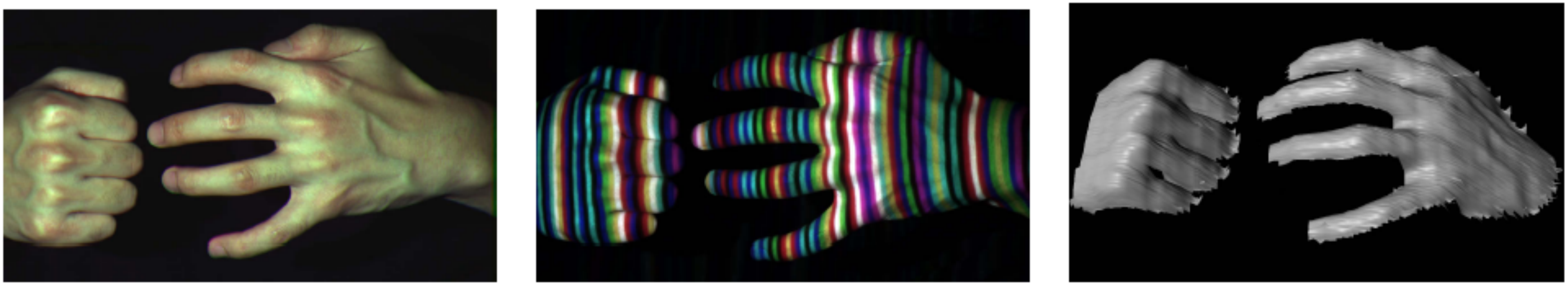
8-point algorithm

- Pros: it is linear, easy to implement and fast
- Cons: susceptible to noise
- Degenerate: if points are on same plane

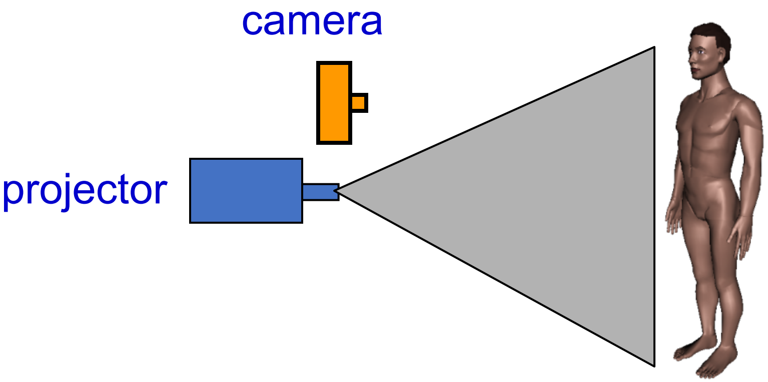
- Normalized 8-point algorithm: Hartley
  - Position origin at centroid of image points
  - Rescale coordinates so that center to farthest point is sqrt (2)

Other approaches to obtaining 3D structure

Active stereo with structured light

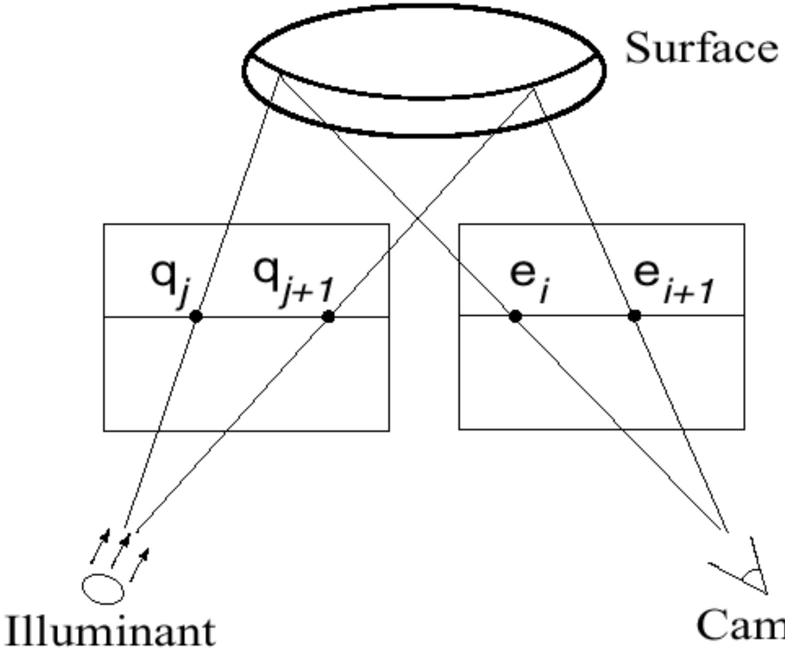


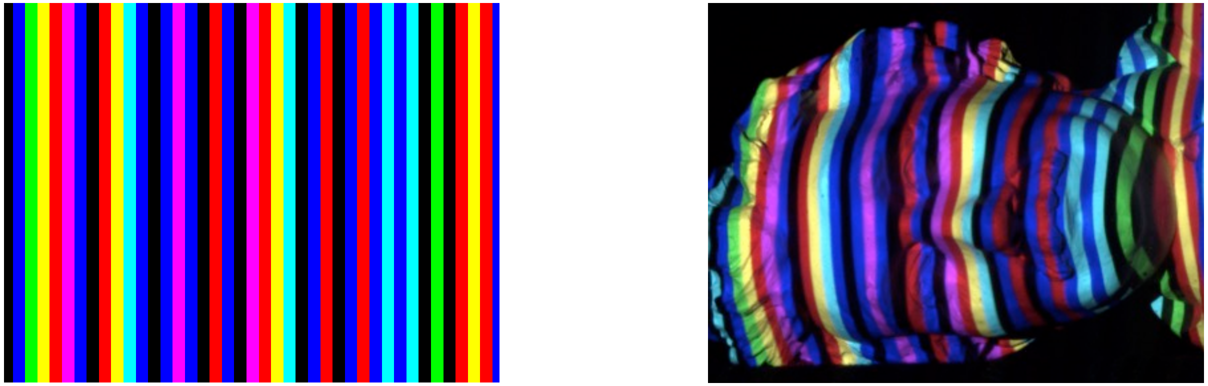
- Project “structured” light patterns onto the object
  - simplifies the correspondence problem
  - Allows us to use only one camera



L. Zhang, B. Curless, and S. M. Seitz. Rapid Shape Acquisition Using Color Structured Light and Multi-pass Dynamic Programming. *3DPVT* 2002

Active stereo with structured light





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Microsoft Kinect