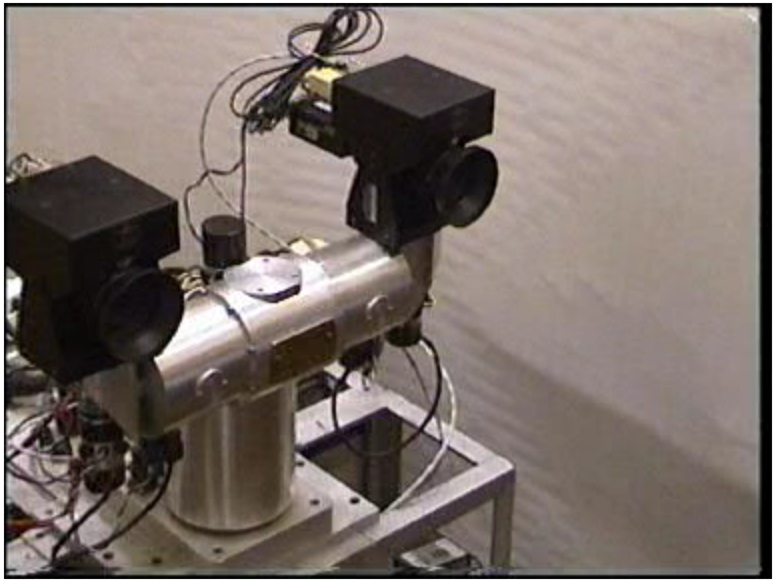
Two-view geometry

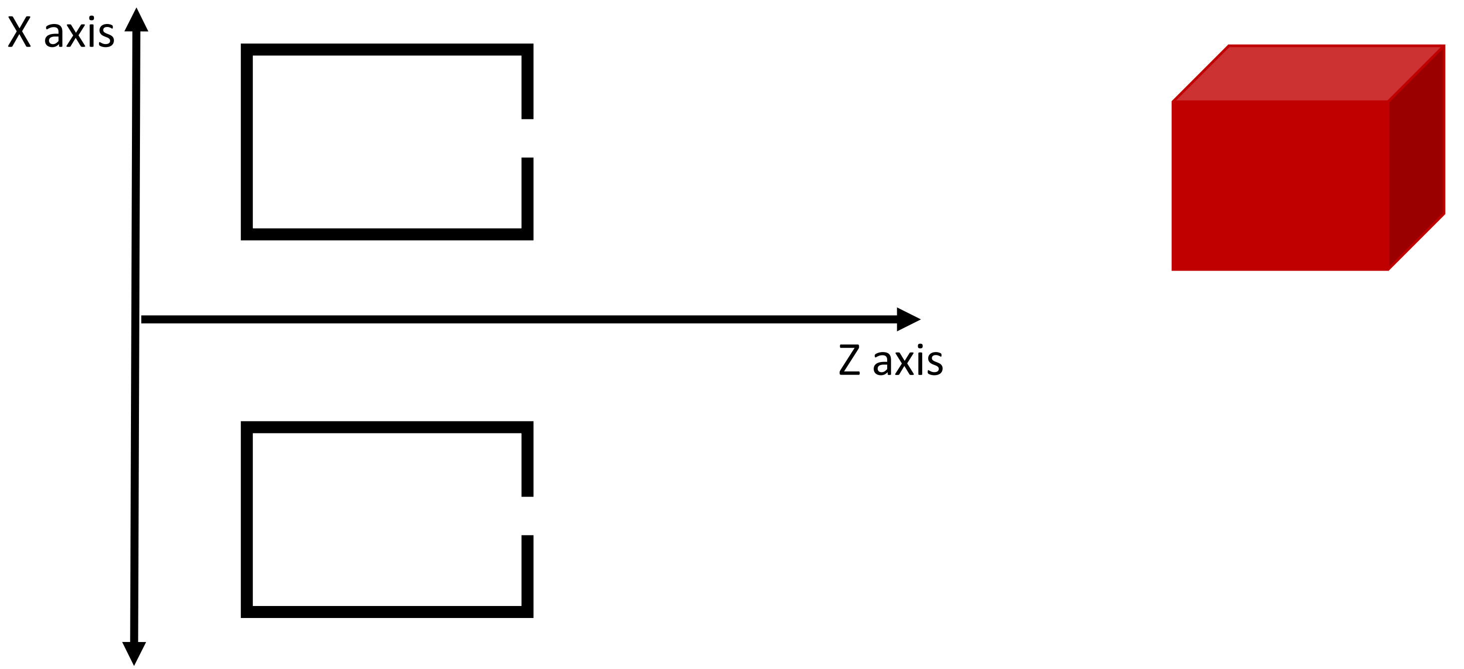
Stereo head



Kinect / depth cameras

Stereo with *rectified cameras*

- Special case: cameras are parallel to each other and translated along X axis



Perspective projection in rectified cameras

- Without loss of generality, assume origin is at pinhole of 1st camera

$$\vec{\mathbf{x}}^{(1)}_{img} \equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}^{(2)}_{img} \equiv \begin{bmatrix} I & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\mathbf{t} = \begin{bmatrix} t_x \\ 0 \\ 0 \end{bmatrix}$$

Perspective projection in rectified cameras

- Without loss of generality, assume origin is at pinhole of 1st camera

X coordinate differs by t_x/Z

$$x_1 = \frac{X}{Z} \qquad x_2 = \frac{X + t_x}{Z}$$

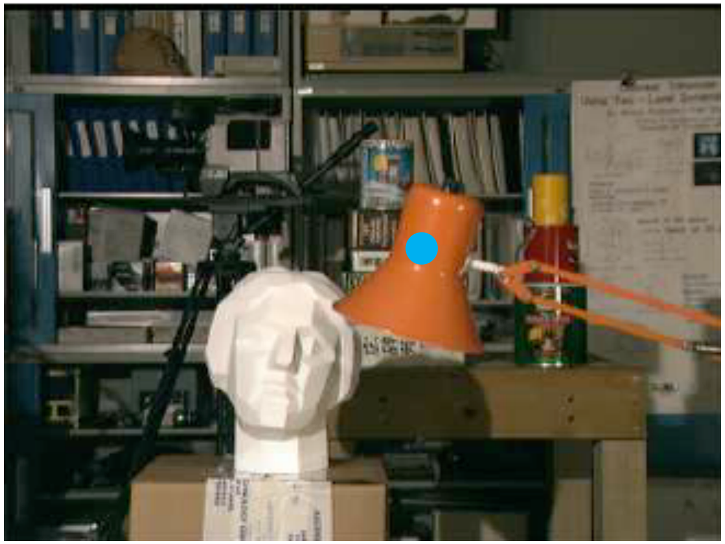
$$y_1 = \frac{Y}{Z} \qquad y_2 = \frac{Y}{Z}$$

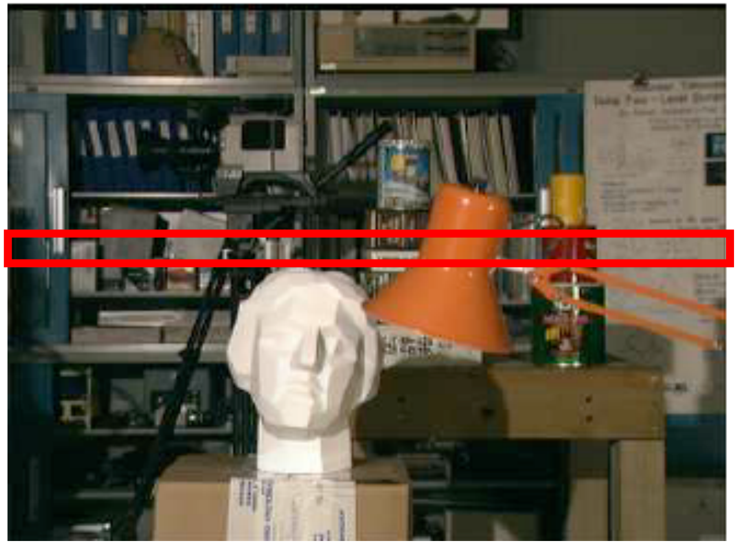
Y coordinate is the same!

Perspective projection in rectified cameras

- disparity = t_x/Z
- If t_x is known, disparity gives Z
- Otherwise, disparity gives Z in units of t_x
  - Small-baseline, near depth = large-baseline, far depth

Perspective projection in rectified cameras

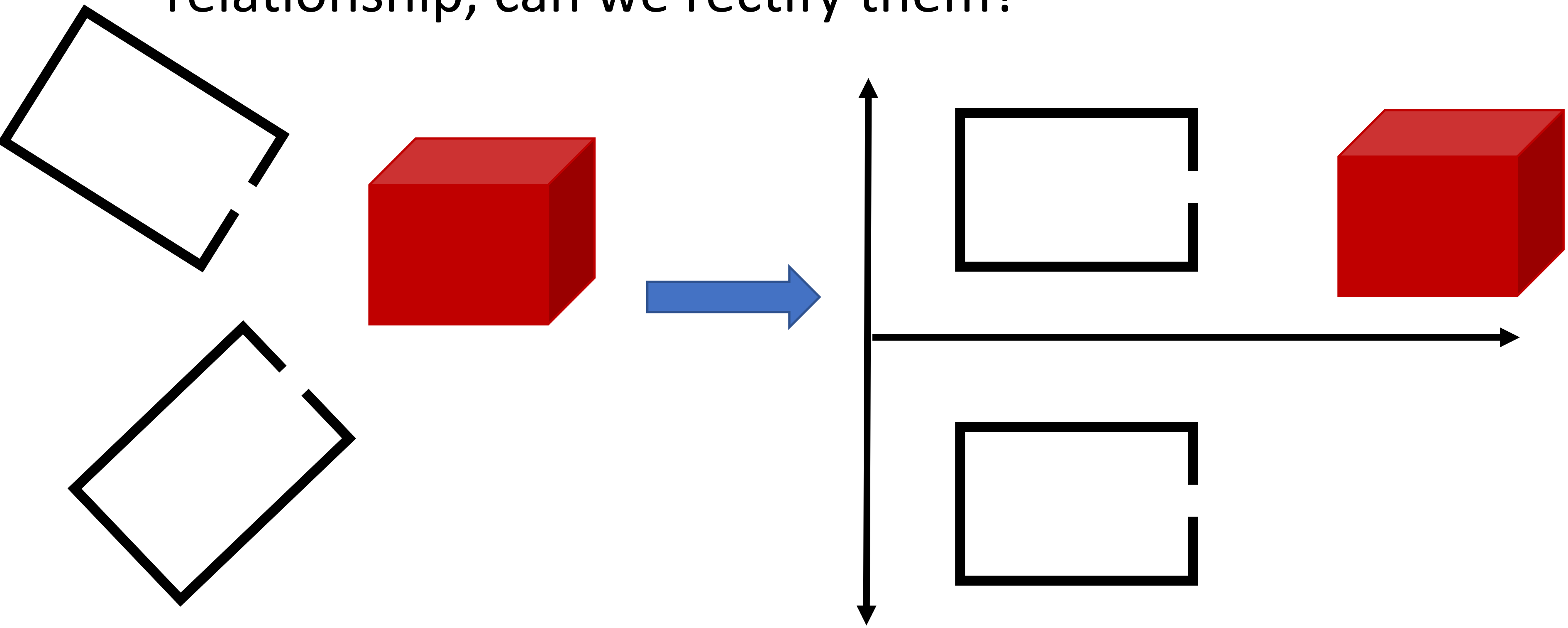




- For rectified cameras, correspondence problem is easier
- Only requires searching along a particular *row.*

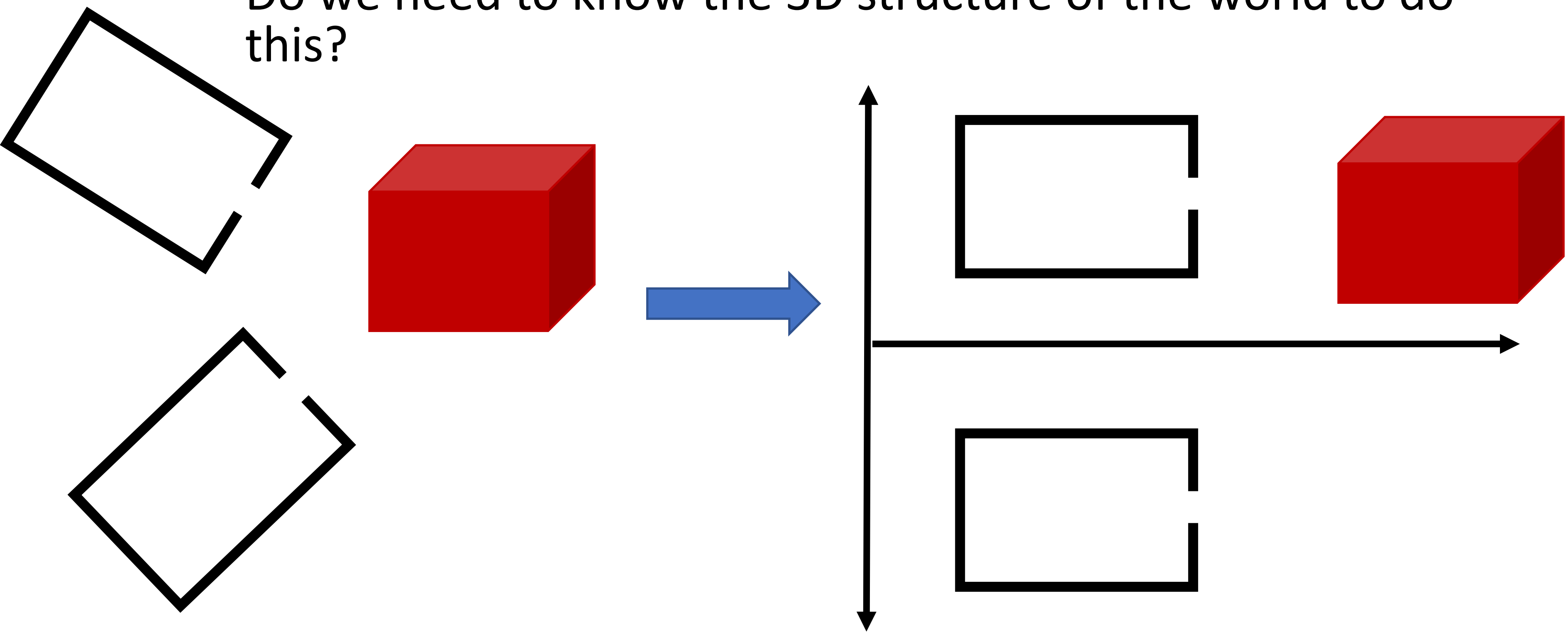
Rectifying cameras

- Given two images from two cameras with known relationship, can we rectify them?



Rectifying cameras

- Can we rotate / translate cameras?
  - Do we need to know the 3D structure of the world to do this?



Rotating cameras

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

- Assume K is identity
- Assume coordinate system at camera pinhole

$$\begin{aligned} \vec{\mathbf{x}}_{img} &\equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \vec{\mathbf{x}}_w \\ &\equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} \\ &\equiv \mathbf{x}_w \end{aligned}$$

Rotating cameras

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

- Assume K is identity
- Assume coordinate system at camera pinhole

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix}$$

$$\equiv \mathbf{x}_w$$

Rotating cameras

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix}$$

$$\vec{\mathbf{x}}_{img} \equiv \mathbf{x}_w$$

- What happens if the camera is rotated?

$$\vec{\mathbf{x}}'_{img} \equiv \begin{bmatrix} R & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix}$$

$$\equiv R\mathbf{x}_w$$

$$\equiv R\vec{\mathbf{x}}_{img}$$

Rotating cameras

- What happens if the camera is rotated?

Rotation matrix

$$\vec{\mathbf{x}}'_{img} \equiv R\vec{\mathbf{x}}_{img}$$

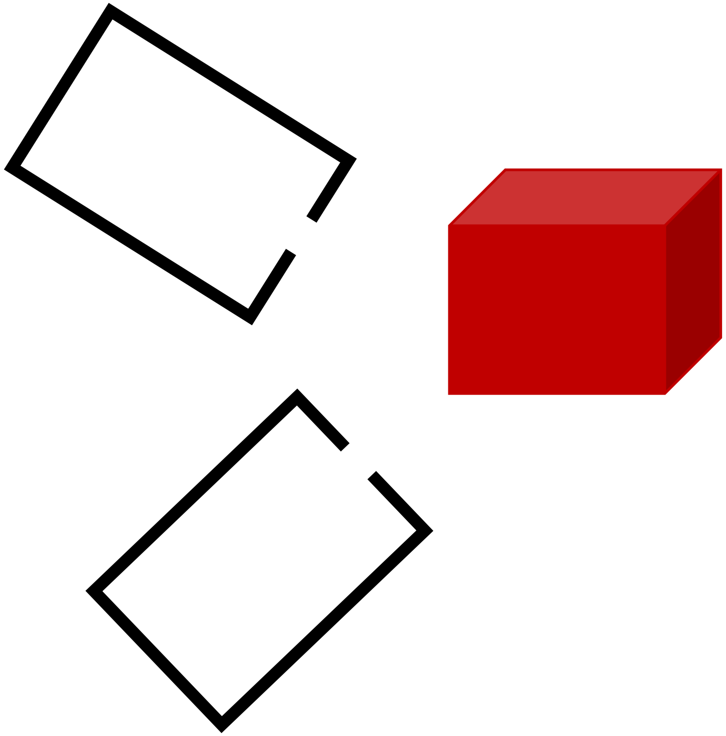
Homogenous coordinates of mapped pixel

Homogenous coordinates of original pixel

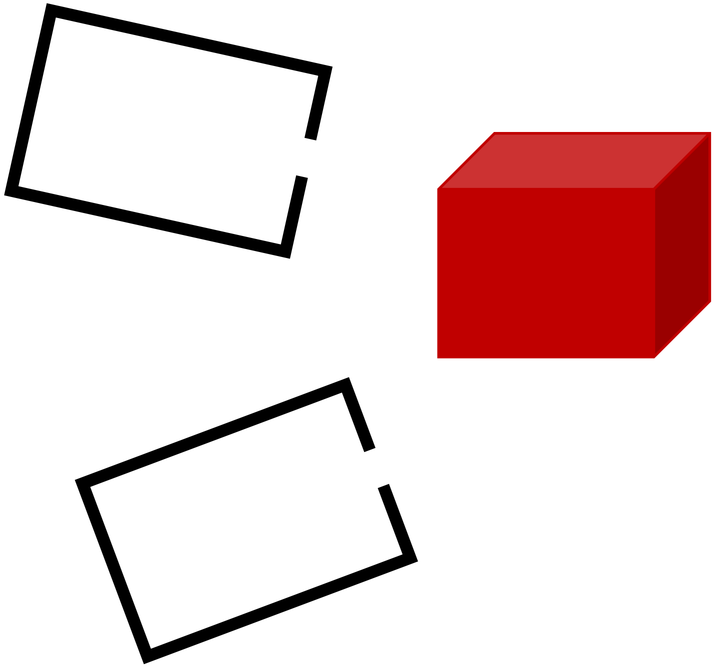
- No need to know the 3D structure

Rotating cameras

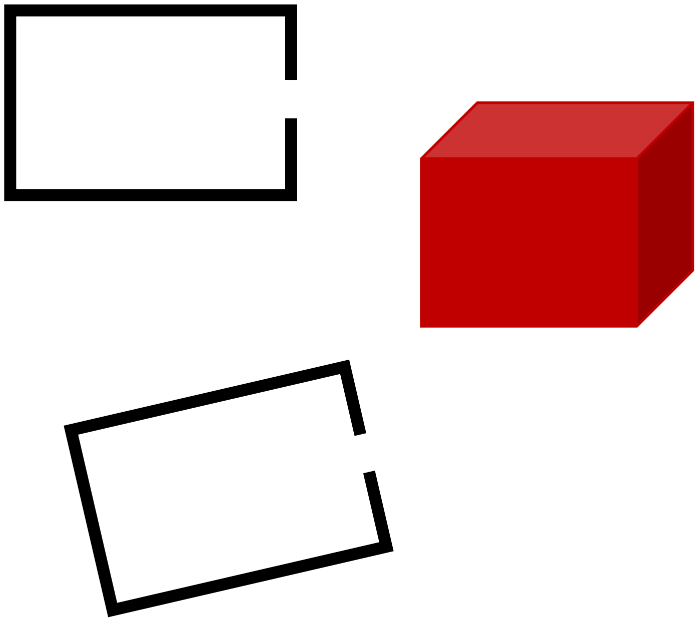
Rectifying cameras



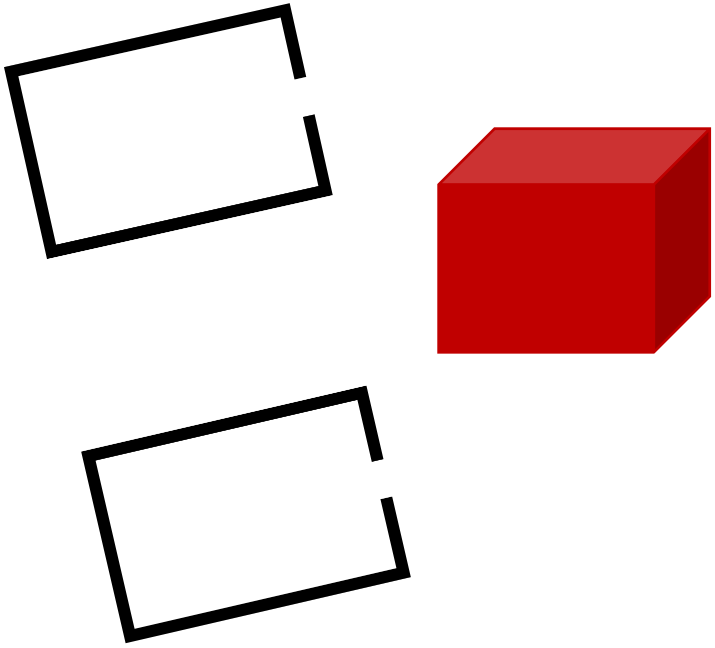
Rectifying cameras



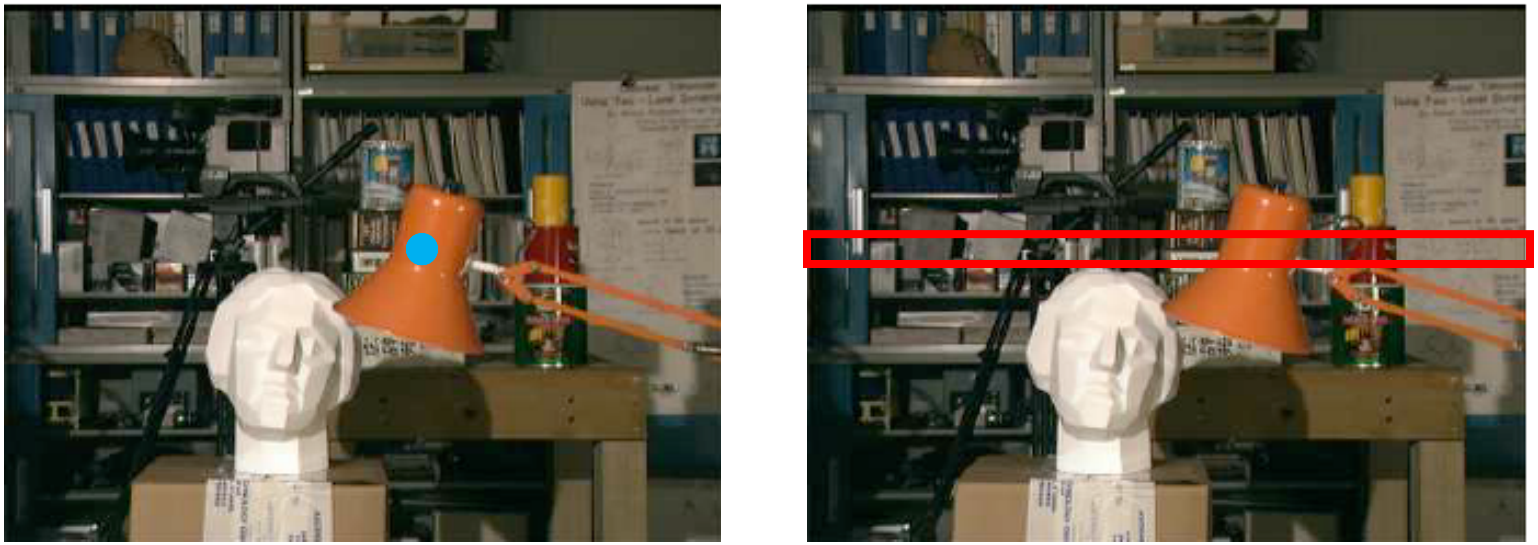
Rectifying cameras



Rectifying cameras

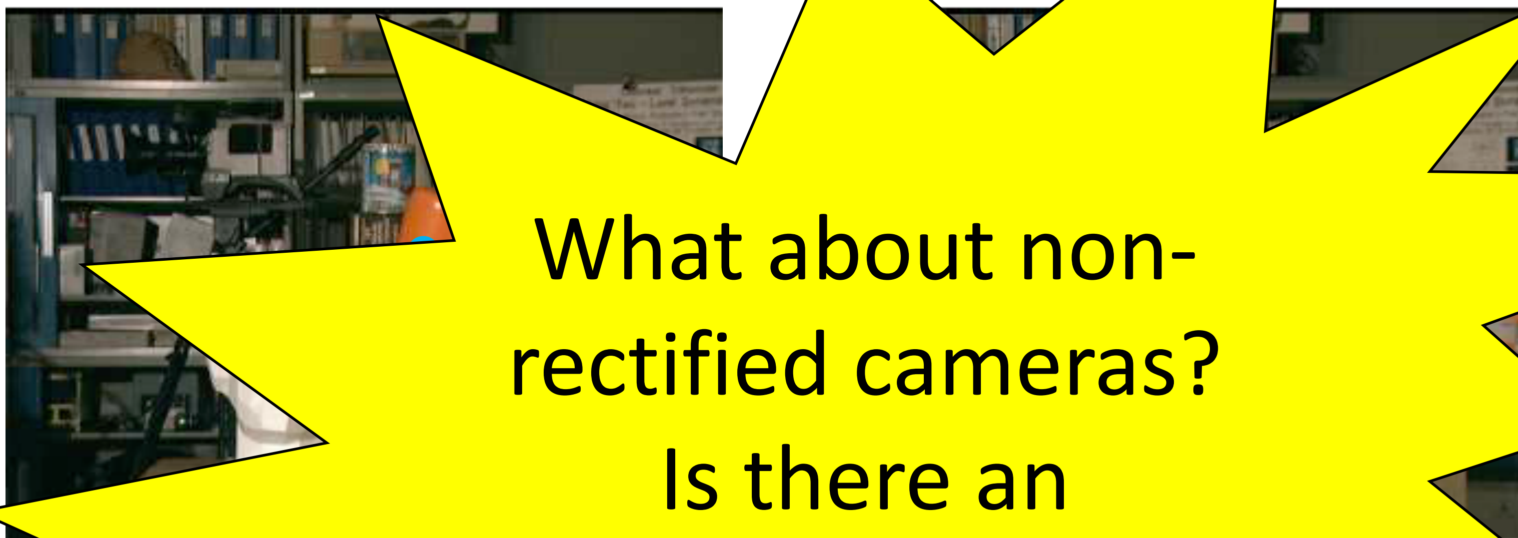


Perspective projection in rectified cameras



- For rectified cameras, correspondence problem is easier
- Only requires searching along a particular *row.*

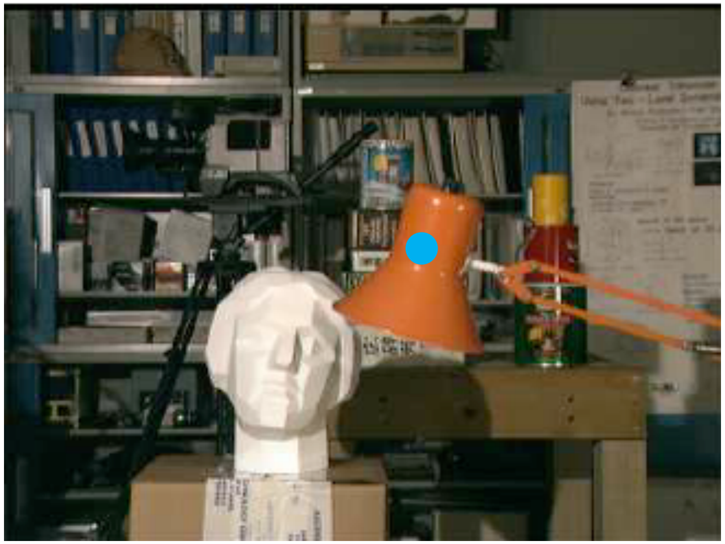
Perspective projection in rectified cameras

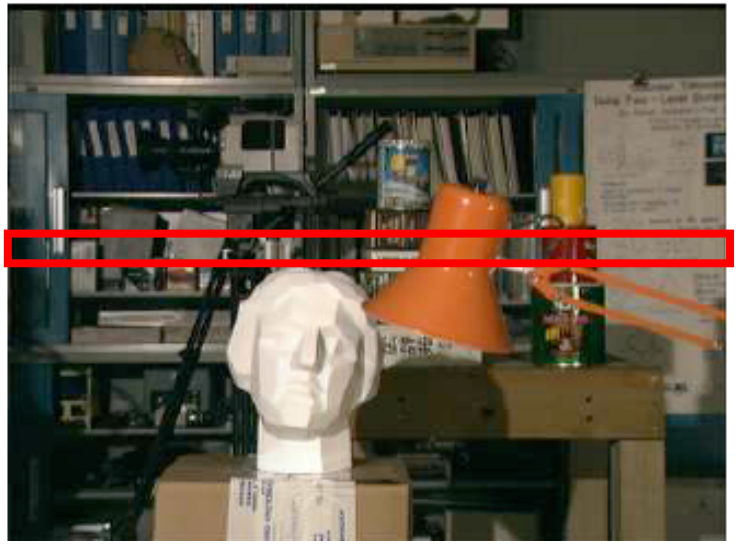


What about non-rectified cameras? Is there an equivalent?

- For re… problem… easi…
- Only requi… c… along … particular *row.*

Epipolar constraint





- *Reduces 2D search problem to search along a particular line: epipolar line*

Epipolar constraint

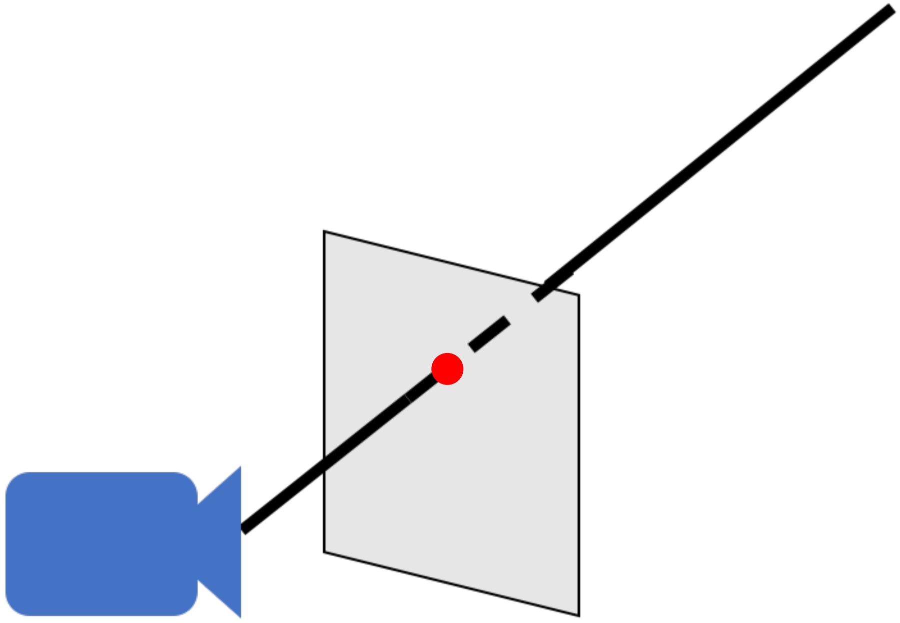
True in general!

- Given pixel (x,y) in one image, corresponding pixel in the other image must lie on a line
- Line function of (x,y) and parameters of camera
- These lines are called *epipolar line*

Epipolar geometry

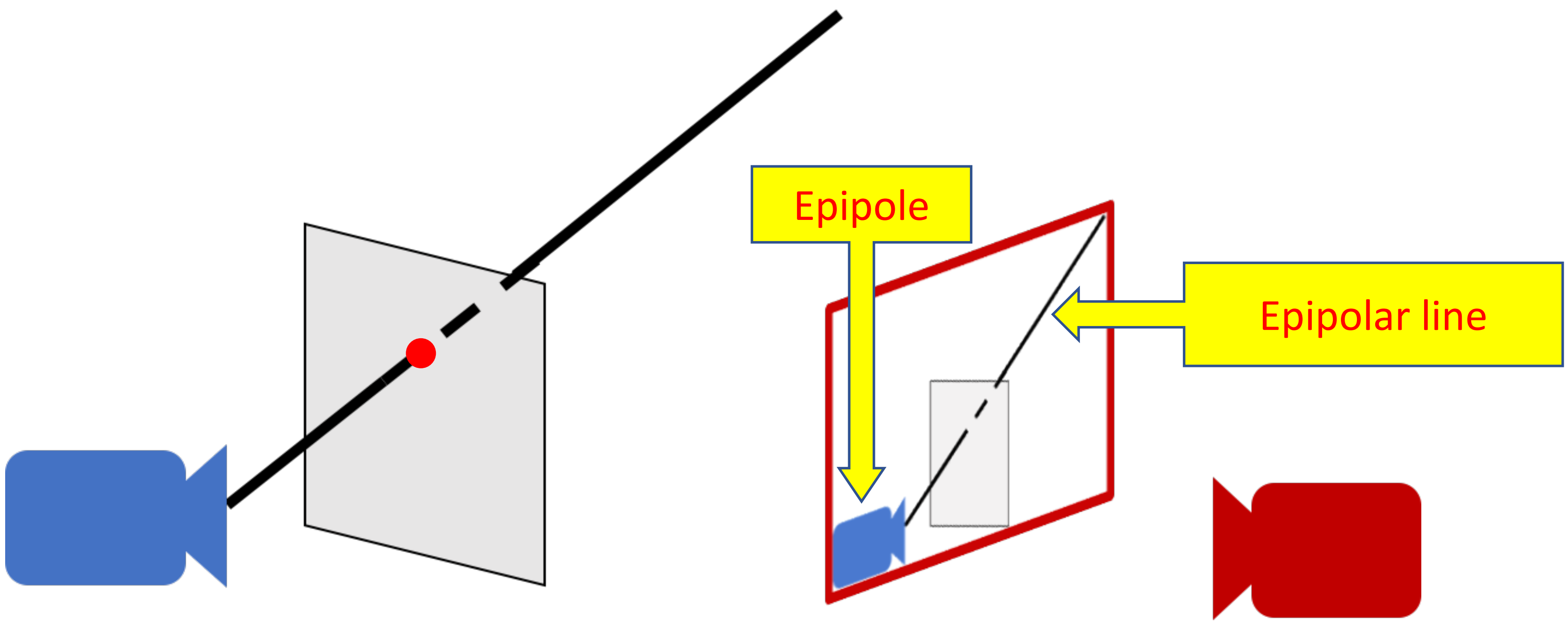
Epipolar geometry - why?

- For a single camera, pixel in image plane must correspond to point somewhere along a ray



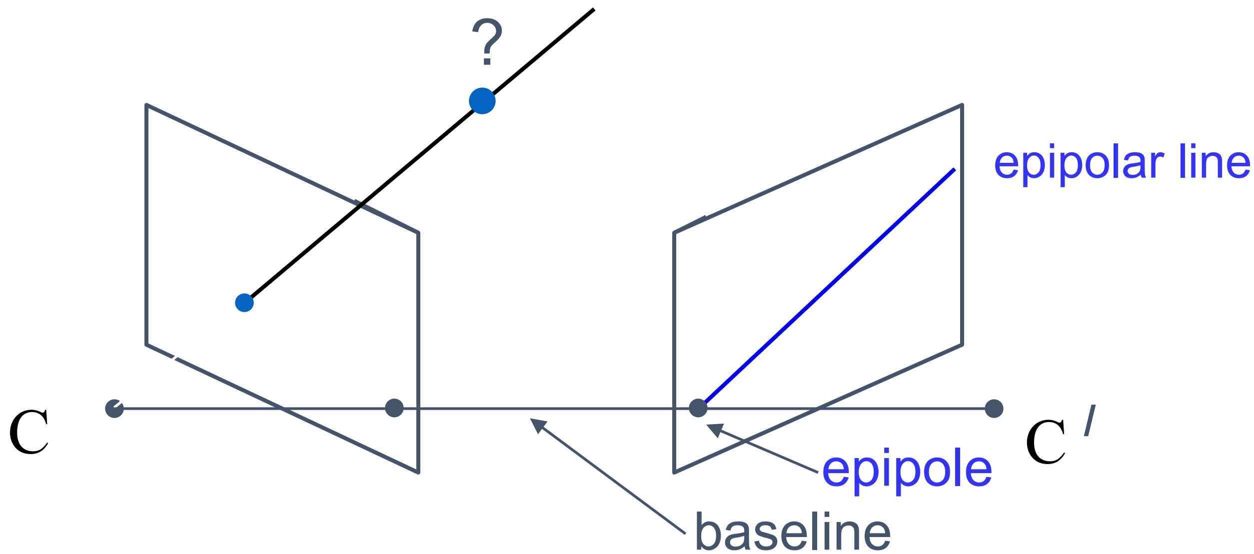
Epipolar geometry

- When viewed in second image, this ray looks like a line: *epipolar line*
- The epipolar line must pass through image of the first camera in the second image - *epipole*



Epipolar geometry

Given an image point in one view, where is the corresponding point in the other view?



- A point in one view "generates" an epipolar line in the other view
- The corresponding point lies on this line

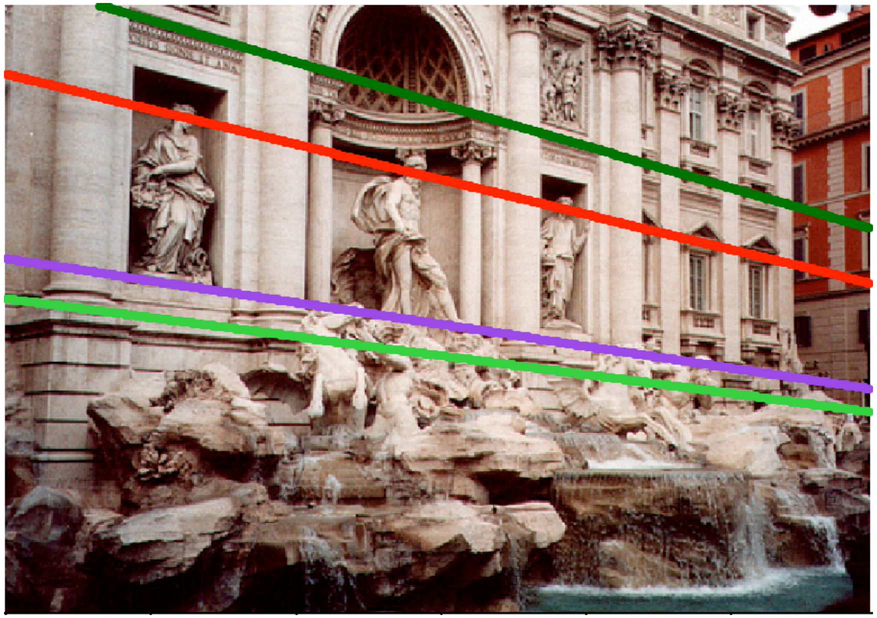
Epipolar line

Epipolar constraint

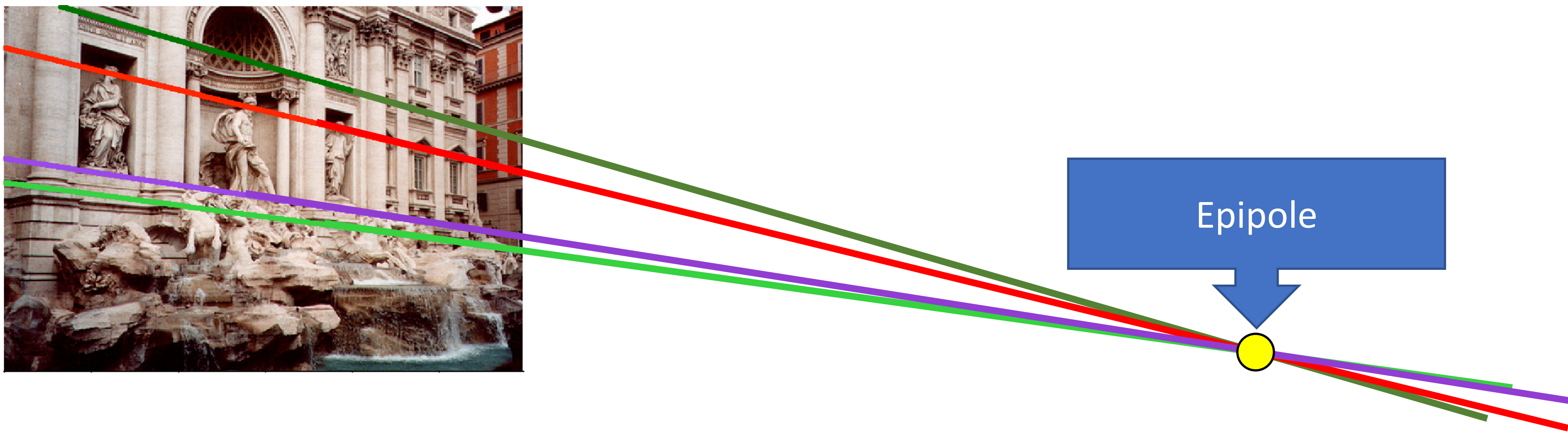
- Reduces correspondence problem to 1D search along an epipolar line

Epipolar lines

Epipolar lines

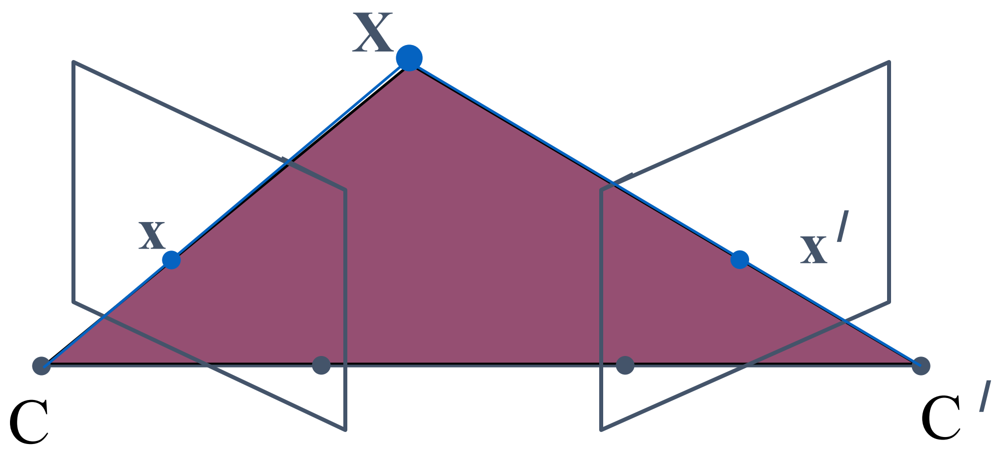


Epipolar lines



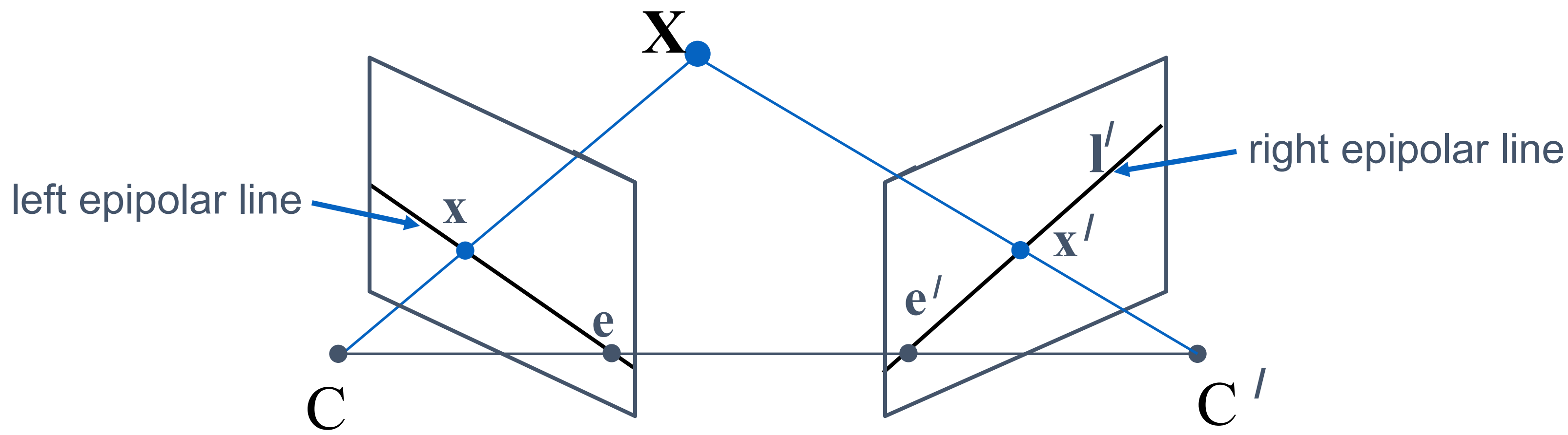
Epipolar geometry continued

Epipolar geometry is a consequence of the coplanarity of the camera centres and scene point



The camera centres, corresponding points and scene point lie in a single plane, known as the epipolar plane

Nomenclature



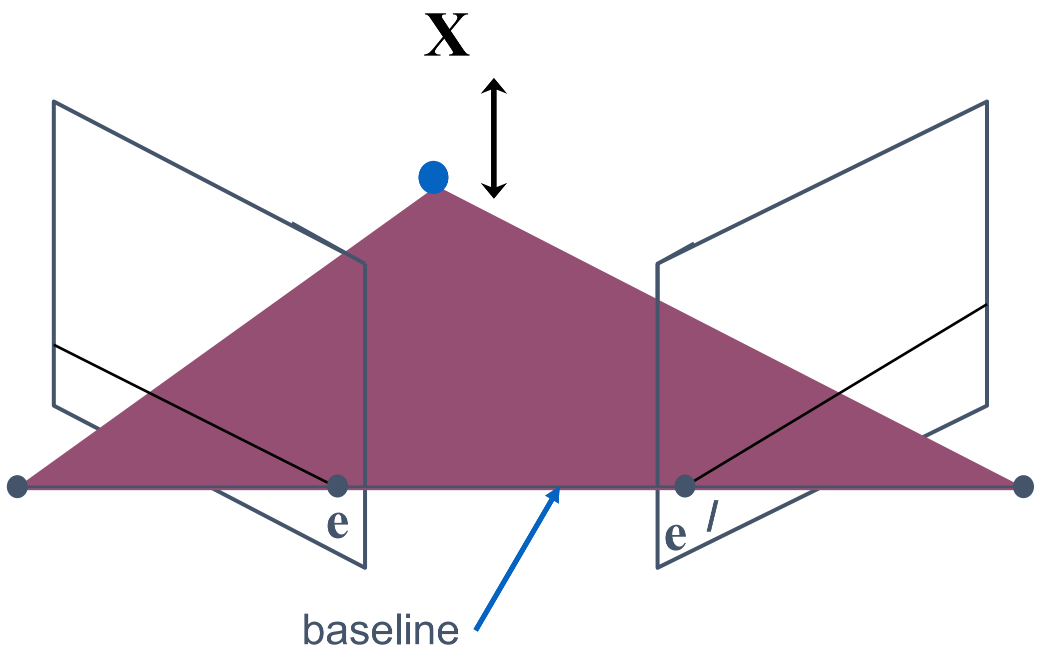
• The epipolar line $\mathbf{l}'$ is the image of the ray through $\mathbf{x}$

• The epipole $\mathbf{e}$ is the point of intersection of the line joining the camera centres with the image plane

- this line is the baseline for a stereo rig, and
- the translation vector for a moving camera

• The epipole is the image of the centre of the other camera: $\mathbf{e} = \mathrm{P}\mathbf{C}'$, $\mathbf{e}' = \mathrm{P}'\mathbf{C}$

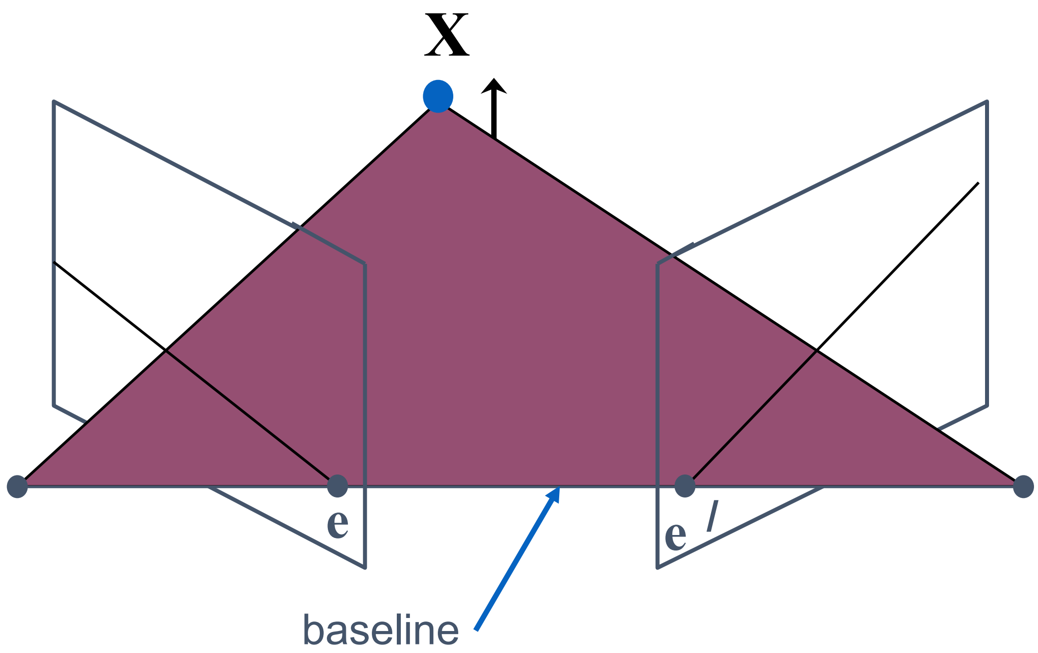
The epipolar pencil



As the position of the 3D point $\mathbf{X}$ varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil (a pencil is a one parameter family).

All epipolar lines intersect at the epipole.

The epipolar pencil



As the position of the 3D point $\mathbf{X}$ varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil (a pencil is a one parameter family).

All epipolar lines intersect at the epipole.

Epipolar geometry - the math

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv K_1 \begin{bmatrix} R_1 & \mathbf{t}_1 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv K_2 \begin{bmatrix} R_2 & \mathbf{t}_2 \end{bmatrix} \vec{\mathbf{x}}_w$$

Epipolar geometry - the math

- Assume intrinsic parameters K are identity
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Epipolar geometry - the math

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv \begin{bmatrix} I & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} = \mathbf{x}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \begin{bmatrix} \mathbf{x}_w \\ 1 \end{bmatrix} = R\mathbf{x}_w + \mathbf{t}$$

Epipolar geometry - the math

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv \mathbf{x}_w$$

$$\vec{\mathbf{x}}_{img}^{(2)} \equiv R\mathbf{x}_w + \mathbf{t}$$

Epipolar geometry - the math

- Assume intrinsic parameters K are identity
- Assume world coordinate system is centered at 1st camera pinhole with Z along viewing direction

$$\lambda_1 \vec{\mathbf{x}}_{img}^{(1)} = \mathbf{x}_w$$

$$\lambda_2 \vec{\mathbf{x}}_{img}^{(2)} = R\mathbf{x}_w + \mathbf{t}$$

Epipolar geometry - the math

$$\lambda_2 \vec{\mathbf{x}}^{(2)}_{img} = \lambda_1 R \vec{\mathbf{x}}^{(1)}_{img} + \mathbf{t}$$

$$\lambda_2 \mathbf{t} \times \vec{\mathbf{x}}^{(2)}_{img} = \lambda_1 \mathbf{t} \times R \vec{\mathbf{x}}^{(1)}_{img} + \mathbf{t} \times \mathbf{t}$$

0

$$\lambda_2 \mathbf{t} \times \vec{\mathbf{x}}^{(2)}_{img} = \lambda_1 \mathbf{t} \times R \vec{\mathbf{x}}^{(1)}_{img}$$

$$\lambda_2 \vec{\mathbf{x}}^{(2)}_{img} \cdot \mathbf{t} \times \vec{\mathbf{x}}^{(2)}_{img} = \lambda_1 \vec{\mathbf{x}}^{(2)}_{img} \cdot \mathbf{t} \times R \vec{\mathbf{x}}^{(1)}_{img}$$

0

$$0 = \lambda_1 \vec{\mathbf{x}}^{(2)}_{img} \cdot \mathbf{t} \times R \vec{\mathbf{x}}^{(1)}_{img}$$

Epipolar geometry - the math

$$\vec{\mathbf{x}}_{img}^{(2)} \cdot \mathbf{t} \times R\vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Can we write this as matrix vector operations?
- Cross product can be written as a matrix

$$[\mathbf{t}]_\times = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

$$[\mathbf{t}]_\times \mathbf{a} = \mathbf{t} \times \mathbf{a}$$

Epipolar geometry - the math

$$\vec{\mathbf{x}}_{img}^{(2)} \cdot [\mathbf{t}]_{\times} R \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Can we write this as matrix vector operations?
- Dot product can be written as a vector-vector times

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$$

Epipolar geometry - the math

$$\vec{\mathbf{x}}_{img}^{(2)} \cdot [\mathbf{t}]_{\times} R \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Can we write this as matrix vector operations?
- Dot product can be written as a vector-vector times

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^T \mathbf{b}$$

Epipolar geometry - the math

$$\vec{\mathbf{x}}_{img}^{(2)T}[\mathbf{t}]_\times R\vec{\mathbf{x}}_{img}^{(1)} = 0$$

$$\vec{\mathbf{x}}_{img}^{(2)T} E \vec{\mathbf{x}}_{img}^{(1)} = 0$$

Epipolar geometry - the math

Homogenous coordinates of point in image 2

Homogenous coordinates of point in image 1

$$\vec{\mathbf{x}}_{img}^{(2)T} E \vec{\mathbf{x}}_{img}^{(1)} = 0$$

Essential matrix

Epipolar constraint and epipolar lines

$$\vec{\mathbf{x}}_{img}^{(2)T} E \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Consider a known, fixed pixel in the first image
- What constraint does this place on the corresponding pixel?
- $\vec{\mathbf{x}}_{img}^{(2)T} \mathbf{l} = 0$ where $\mathbf{l} = E \vec{\mathbf{x}}_{img}^{(1)}$
- What kind of equation is this?

Epipolar constraint and epipolar lines

$$\vec{\mathbf{x}}_{img}^{(2)T} E \vec{\mathbf{x}}_{img}^{(1)} = 0$$

- Consider a known, fixed pixel in the first image
- $\vec{\mathbf{x}}_{img}^{(2)T} \mathbf{l} = 0$ where $\mathbf{l} = E \vec{\mathbf{x}}_{img}^{(1)}$

$$\vec{\mathbf{x}}_{img}^{(2)T} \mathbf{l} = 0$$

$$\Rightarrow \begin{bmatrix} x_2 & y_2 & 1 \end{bmatrix} \begin{bmatrix} l_x \\ l_y \\ l_z \end{bmatrix} = 0$$

$$\Rightarrow l_x x_2 + l_y y_2 + l_z = 0$$

Line!

Epipolar constraint: putting it all together

- If **p** is a pixel in first image and **q** is the corresponding pixel in the second image, then: $\mathbf{q}^T E \mathbf{p} = 0$
- $E = [\mathbf{t}]_X R$
- For fixed **p**, **q** must satisfy: $\mathbf{q}^T \mathbf{l} = 0$, where $\mathbf{l} = E\mathbf{p}$

  Epipolar line in 2nd image
- For fixed **q**, **p** must satisfy: $\mathbf{l}^T \mathbf{p} = 0$ where $\mathbf{l}^T = \mathbf{q}^T E$, or $\mathbf{l} = E^t \mathbf{q}$

  Epipolar line in 1st image
- These are epipolar lines!

Essential matrix and epipoles

- $E = [\mathbf{t}]_X R$

$$\vec{\mathbf{c}_2} = \mathbf{t}$$
$$\vec{\mathbf{c}_2}^T E = \mathbf{t}^T E = \mathbf{t}^T [\mathbf{t}]_\times R = 0$$
$$\vec{\mathbf{c}_2}^T E \mathbf{p} = 0 \ \ \forall \mathbf{p}$$

- E**p** is an epipolar line in 2^{nd} image
- All epipolar lines in second image pass through c_2
- c_2 is epipole in 2^{nd} image

Essential matrix and epipoles

- $E = [\mathbf{t}]_X R$

$$\vec{\mathbf{c}_1} = \mathbf{R}^T \mathbf{t}$$

$$E\vec{\mathbf{c}_1} = [\mathbf{t}]_\times R R^T \mathbf{t} = [\mathbf{t}]_\times \mathbf{t} = 0$$

$$\mathbf{q}^T E \vec{\mathbf{c}_1} = 0 \;\; \forall \mathbf{q}$$

- $E^T\mathbf{q}$ is an epipolar line in 1st image
- All epipolar lines in first image pass through c_1
- c_1 is the epipole in 1st image