Camera calibration
Triangulation

Perspective projection in homogenous coordinates

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} I & \mathbf{0} \end{bmatrix} \begin{bmatrix} R & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img} \equiv \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Matrix transformations in 2D

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

$$K = \begin{bmatrix} 1 & 0 & t_u \\ 0 & 1 & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Translation

$$K = \begin{bmatrix} s_x & 0 & t_u \\ 0 & s_y & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Scaling of Image x and y (conversion from “meters” to “pixels”)

$$K = \begin{bmatrix} s_x & \alpha & t_u \\ 0 & s_y & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Added skew if image x and y axes are not perpendicular

Final perspective projection

Camera extrinsics: where your camera is relative to the world. Changes if you move the camera

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Camera intrinsics: how your camera handles pixel. Changes if you change your camera

$$\vec{\mathbf{x}}_{img} \equiv P \vec{\mathbf{x}}_w$$

Final perspective projection

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

Camera parameters

$$\vec{\mathbf{x}}_{img} \equiv P \vec{\mathbf{x}}_w$$

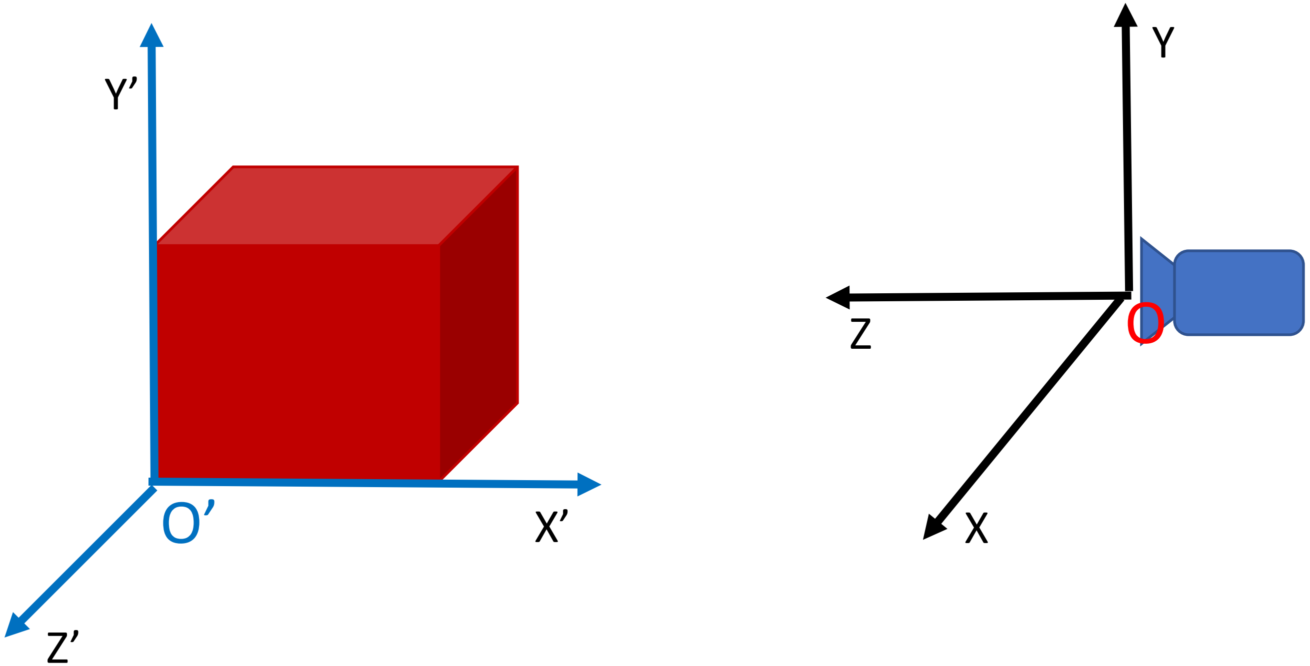
Camera calibration

- Goal: find the parameters of the camera

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

- Why?
  - Tells you where the camera is relative to the world/particular objects
  - Equivalently, tells you where objects are relative to the camera
  - Can allow you to ”render” new objects into the scene

Camera calibration



Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \equiv P \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Need to convert equivalence into equality.

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

Note: λ is unknown

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = P \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

$$\begin{bmatrix} \lambda x \\ \lambda y \\ \lambda \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & P_{13} & P_{14} \\ P_{21} & P_{22} & P_{23} & P_{24} \\ P_{31} & P_{32} & P_{33} & P_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

$$\lambda x = P_{11}X + P_{12}Y + P_{13}Z + P_{14}$$

$$\lambda y = P_{21}X + P_{22}Y + P_{23}Z + P_{24}$$

$$\lambda = P_{31}X + P_{32}Y + P_{33}Z + P_{34}$$

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

- Suppose we know that (X,Y,Z) in the world projects to (x,y) in the image.
- How many equations does this provide?

$$(P_{31}X + P_{32}Y + P_{33}Z + P_{34})x = P_{11}X + P_{12}Y + P_{13}Z + P_{14}$$

$$(P_{31}X + P_{32}Y + P_{33}Z + P_{34})y = P_{21}X + P_{22}Y + P_{23}Z + P_{24}$$

- 2 equations!
- Are the equations linear in the parameters?
- How many equations do we need?

Camera calibration

$$(P_{31}X + P_{32}Y + P_{33}Z + P_{34})x = P_{11}X + P_{12}Y + P_{13}Z + P_{14}$$

$$XxP_{31} + YxP_{32} + ZxP_{33} + xP_{34} - XP_{11} - YP_{12} - ZP_{13} - P_{14} = 0$$

- In matrix vector form: Ap = 0
- 6 points give 12 equations, 12 variables to solve for
- But can only solve upto scale

Camera calibration

- In matrix vector form: Ap = 0
- We want non-trivial solutions
- If p is a solution, αp is a solution too
- Let's just search for a solution with unit norm

$$A\mathbf{p} = 0$$

s.t

$$||\mathbf{p}|| = 1$$

- How do you solve this?

Camera calibration

- In matrix vector form: Ap = 0
- We want non-trivial solutions
- If p is a solution, αp is a solution too
- Let's just search for a solution with unit norm

$$A\mathbf{p} = 0$$

s.t

$$||\mathbf{p}|| = 1$$

Camera calibration

- What happens if there are more than 6 points?
- What if there is noise in the point locations?

$$\min_{\mathbf{p}} \|A\mathbf{p}\|^2$$

$$\sout{A\mathbf{p} = 0}$$

s.t

$$\|\mathbf{p}\| = 1$$

Camera calibration

- What happens if there are more than 6 points?
- What if there is noise in the point locations?

$$\min_{\mathbf{p}} \mathbf{p}^T A^T A \mathbf{p}$$

$$\cancel{A\mathbf{p} = 0}$$

s.t

$$\|\mathbf{p}\| = 1$$

- Look at eigenvector of A^TA with the smallest eigenvalue!

Camera calibration

- >=6 points with known 3D coordinates + known image coordinates

$$XxP_{31} + YxP_{32} + ZxP_{33} + xP_{34} - XP_{11} - YP_{12} - ZP_{13} - P_{14} = 0$$

- In matrix vector form: want Ap = 0
- Resilience to noise:

$$\min_{\mathbf{p}} \mathbf{p}^T A^T A \mathbf{p}$$

s.t

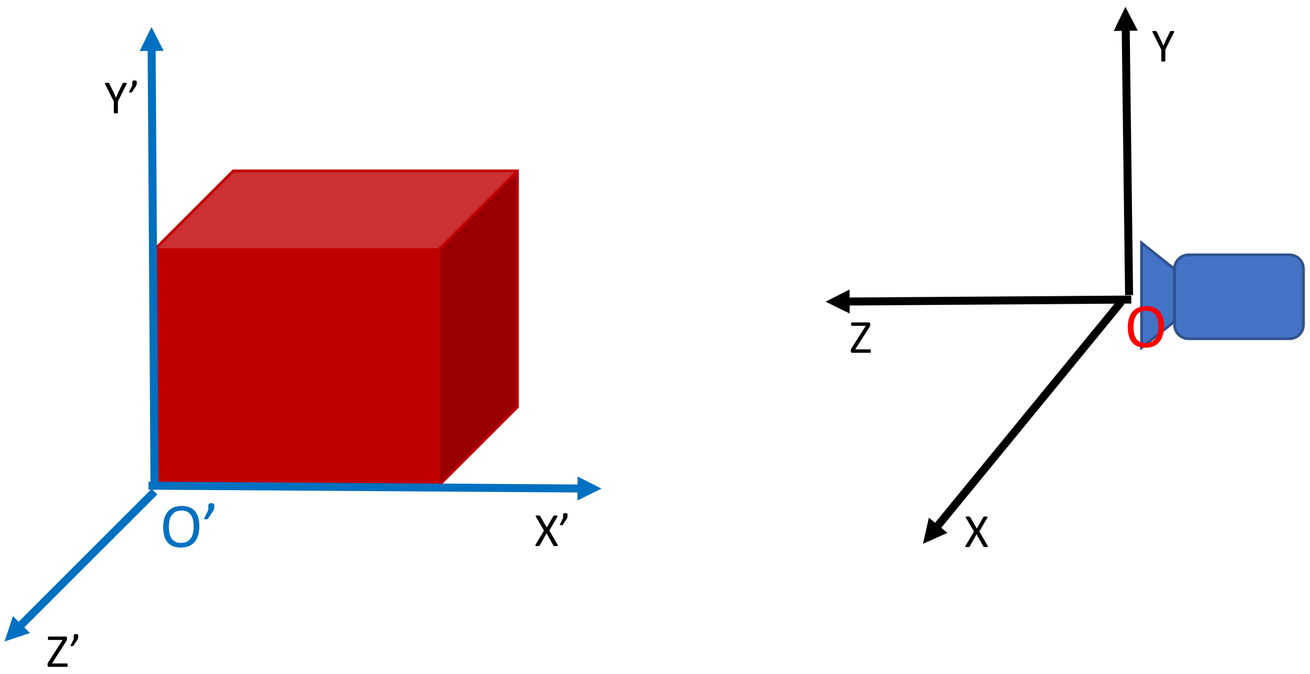
$$||\mathbf{p}|| = 1$$

- Look at eigenvector of A^TA with the smallest eigenvalue!

Camera calibration

- We need 6 world points for which we know image locations
- Would any 6 points work?
  - What if all 6 points are the same?
- Need at least 6 non-coplanar points!

Camera calibration



Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

- How do we get K, R and t from P?
- Need to make some assumptions about K
- What if K is identity?

Camera calibration

$$\vec{\mathbf{x}}_{img} \equiv P\vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_{img} \equiv K \begin{bmatrix} R & \mathbf{t} \end{bmatrix} \vec{\mathbf{x}}_w$$

- How do we get K, R and t from P?
- Need to make some assumptions about K
- What if K is upper triangular?

$$K = \begin{bmatrix} s_x & \alpha & t_u \\ 0 & s_y & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

Added skew if image x and y axes are not perpendicular

Camera calibration

- How do we get K, R and t from P?
- Need to make some assumptions about K
- What if K is upper triangular?

$$K = \begin{bmatrix} s_x & \alpha & t_u \\ 0 & s_y & t_v \\ 0 & 0 & 1 \end{bmatrix}$$

- P = K [R t]
- First 3 x 3 matrix of P is KR
- “RQ” decomposition: decomposes an n x n matrix into product of upper triangular and rotation matrix

Camera calibration

- How do we get K, R and t from P?
- Need to make some assumptions about K
- What if K is upper triangular?
- P = K [R t]
- First 3 x 3 matrix of P is KR
- "RQ" decomposition: decomposes an n x n matrix into product of upper triangular and rotation matrix
- $t = K^{-1}P[:,2]$ ← last column of P

Camera calibration and pose estimation

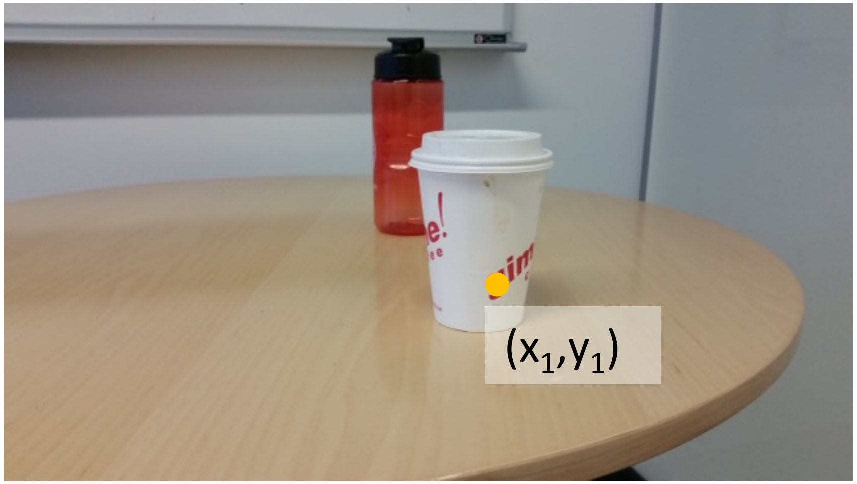
Triangulation

- Suppose we have two cameras
  - Calibrated: parameters known
- And a pair of corresponding pixels
- Find 3D location of point!

Triangulation

- Suppose we have two cameras
  - Calibrated: parameters known
- And a pair of corresponding pixels
- Find 3D location of point!

$P^{(1)}$



$P^{(2)}$

Triangulation

$$\begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \leftarrow \vec{\mathbf{x}}_{img}^{(1)} \equiv P^{(1)} \vec{\mathbf{x}}_w$$

$$\begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix} \leftarrow \vec{\mathbf{x}}_{img}^{(2)} \equiv P^{(2)} \vec{\mathbf{x}}_w$$

$$\vec{\mathbf{x}}_w \rightarrow \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Triangulation

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv P^{(1)}\vec{\mathbf{x}}_w$$

$$\lambda x_1 = P_{11}^{(1)}X + P_{12}^{(1)}Y + P_{13}^{(1)}Z + P_{14}^{(1)}$$

$$\lambda y_1 = P_{21}^{(1)}X + P_{22}^{(1)}Y + P_{23}^{(1)}Z + P_{24}^{(1)}$$

$$\lambda = P_{31}^{(1)}X + P_{32}^{(1)}Y + P_{33}^{(1)}Z + P_{34}^{(1)}$$

$$(P_{31}^{(1)}X + P_{32}^{(1)}Y + P_{33}^{(1)}Z + P_{34}^{(1)})x_1 = P_{11}^{(1)}X + P_{12}^{(1)}Y + P_{13}^{(1)}Z + P_{14}^{(1)}$$

$$X(P_{31}^{(1)}x_1 - P_{11}^{(1)}) + Y(P_{32}^{(1)}x_1 - P_{12}^{(1)}) + Z(P_{33}^{(1)}x_1 - P_{13}^{(1)}) + (P_{34}^{(1)}x_1 - P_{14}^{(1)}) = 0$$

Triangulation

$$\vec{\mathbf{x}}_{img}^{(1)} \equiv P^{(1)}\vec{\mathbf{x}}_w$$

$$X(P_{31}^{(1)}x_1 - P_{11}^{(1)}) + Y(P_{32}^{(1)}x_1 - P_{12}^{(1)}) + Z(P_{33}^{(1)}x_1 - P_{13}^{(1)}) + (P_{34}^{(1)}x_1 - P_{14}^{(1)}) = 0$$

$$X(P_{31}^{(1)}y_1 - P_{21}^{(1)}) + Y(P_{32}^{(1)}y_1 - P_{22}^{(1)}) + Z(P_{33}^{(1)}y_1 - P_{23}^{(1)}) + (P_{34}^{(1)}y_1 - P_{24}^{(1)}) = 0$$

- 1 image gives 2 equations
- Need 2 images!
- Solve linear equations to get 3D point location

Linear vs non-linear optimization

$$\lambda x_1 = P^{(1)}_{11} X + P^{(1)}_{12} Y + P^{(1)}_{13} Z + P^{(1)}_{14}$$
$$\lambda y_1 = P^{(1)}_{21} X + P^{(1)}_{22} Y + P^{(1)}_{23} Z + P^{(1)}_{24}$$
$$\lambda = P^{(1)}_{31} X + P^{(1)}_{32} Y + P^{(1)}_{33} Z + P^{(1)}_{34}$$

$$x_1 = \frac{P^{(1)}_{11} X + P^{(1)}_{12} Y + P^{(1)}_{13} Z + P^{(1)}_{14}}{P^{(1)}_{31} X + P^{(1)}_{32} Y + P^{(1)}_{33} Z + P^{(1)}_{34}}$$

$$y_1 = \frac{P^{(1)}_{21} X + P^{(1)}_{22} Y + P^{(1)}_{23} Z + P^{(1)}_{24}}{P^{(1)}_{31} X + P^{(1)}_{32} Y + P^{(1)}_{33} Z + P^{(1)}_{34}}$$

Linear vs non-linear optimization

$$x_1 = \frac{P_{11}^{(1)}X + P_{12}^{(1)}Y + P_{13}^{(1)}Z + P_{14}^{(1)}}{P_{31}^{(1)}X + P_{32}^{(1)}Y + P_{33}^{(1)}Z + P_{34}^{(1)}}$$

$$y_1 = \frac{P_{21}^{(1)}X + P_{22}^{(1)}Y + P_{23}^{(1)}Z + P_{24}^{(1)}}{P_{31}^{(1)}X + P_{32}^{(1)}Y + P_{33}^{(1)}Z + P_{34}^{(1)}}$$

$$(x_1 - \frac{P_{11}^{(1)}X + P_{12}^{(1)}Y + P_{13}^{(1)}Z + P_{14}^{(1)}}{P_{31}^{(1)}X + P_{32}^{(1)}Y + P_{33}^{(1)}Z + P_{34}^{(1)}})^2$$
$$+(y_1 - \frac{P_{21}^{(1)}X + P_{22}^{(1)}Y + P_{23}^{(1)}Z + P_{24}^{(1)}}{P_{31}^{(1)}X + P_{32}^{(1)}Y + P_{33}^{(1)}Z + P_{34}^{(1)}})^2$$

Reprojection error

Linear vs non-linear optimization

$$(x_1 - \frac{P^{(1)}_{11}X + P^{(1)}_{12}Y + P^{(1)}_{13}Z + P^{(1)}_{14}}{P^{(1)}_{31}X + P^{(1)}_{32}Y + P^{(1)}_{33}Z + P^{(1)}_{34}})^2$$

$$+(y_1 - \frac{P^{(1)}_{21}X + P^{(1)}_{22}Y + P^{(1)}_{23}Z + P^{(1)}_{24}}{P^{(1)}_{31}X + P^{(1)}_{32}Y + P^{(1)}_{33}Z + P^{(1)}_{34}})^2$$

Reprojection error

- Reprojection error is the squared error between the true image coordinates of a point and the projected coordinates of hypothesized 3D point
- Actual error we care about
- Minimize total sum of reprojection error across all images
- *Non-linear optimization*