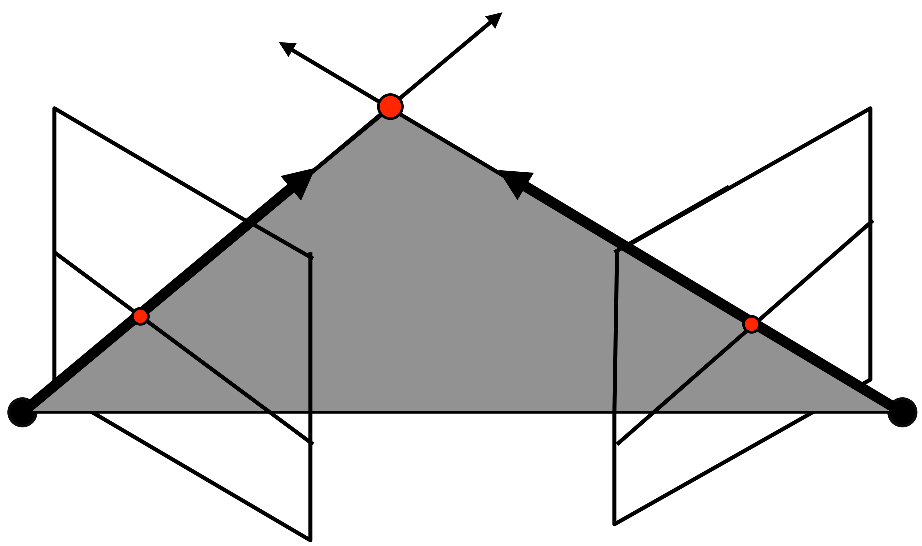
CS4670 / 5670: Computer Vision

Kavita Bala

Lec 26: Fundamental Matrix

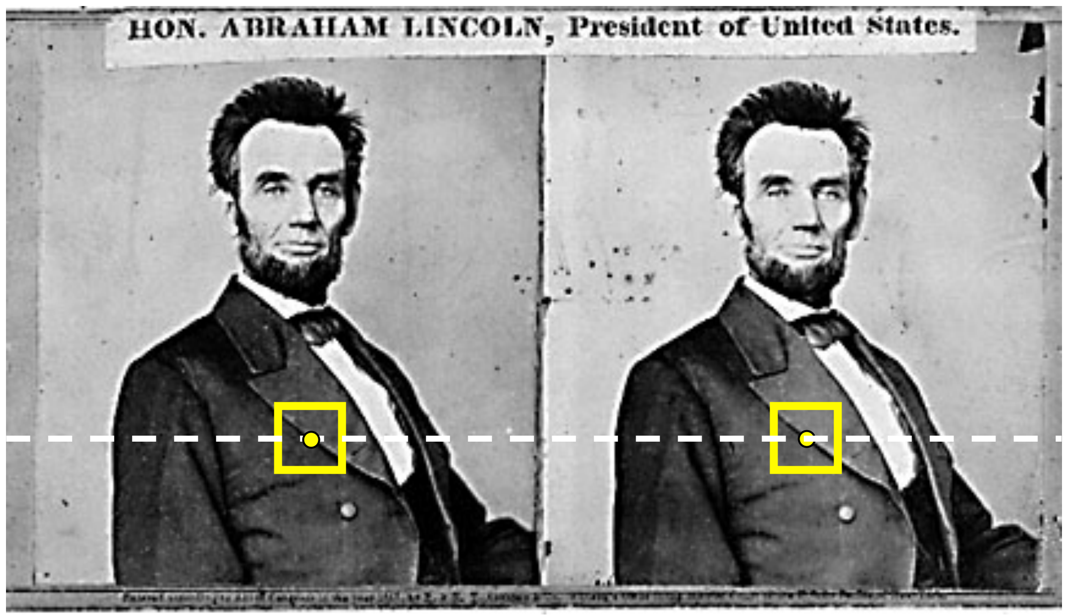


Readings

- Welcome back ….. to winter

- Szeliski, Chapter 7.2

Back to stereo



The objective

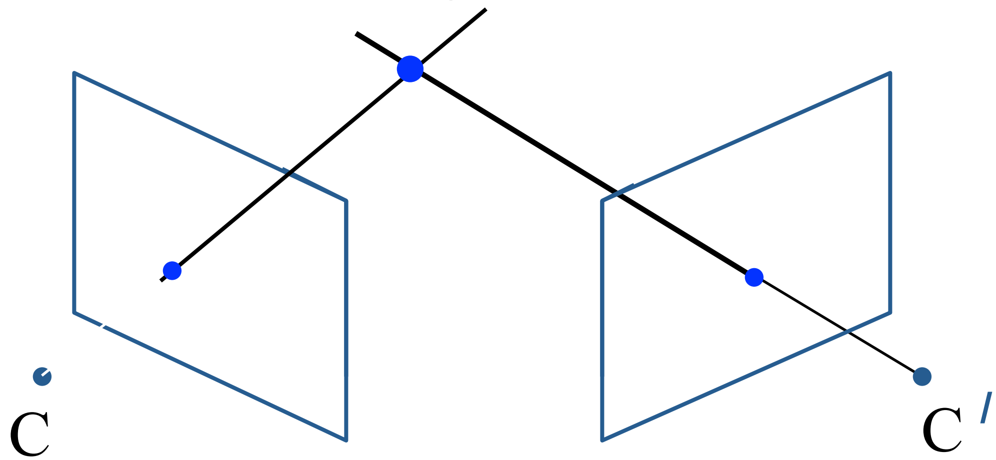
Given two images of a scene acquired by known cameras compute the 3D position of the scene (structure recovery)

Basic principle: triangulate from corresponding image points

Determine 3D point at intersection of two back-projected rays

Triangulation

Corresponding points are images of the same scene point



The back-projected points generate rays which intersect at the 3D scene point

An algorithm for stereo reconstruction

1. For each point in the first image determine the corresponding point in the second image

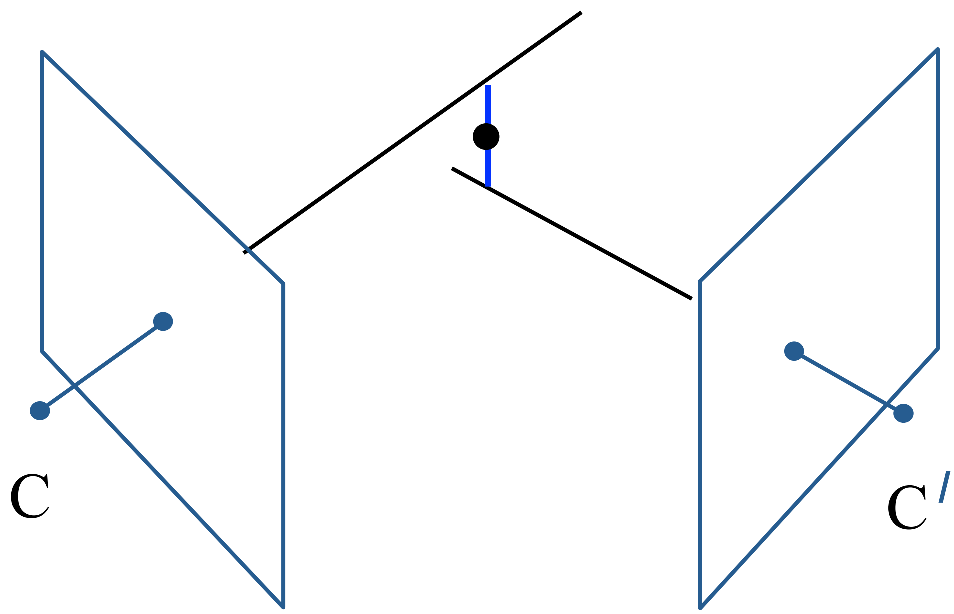
   (this is a search problem)

2. For each pair of matched points determine the 3D point by triangulation

   (this is an estimation problem)

Triangulation

Vector solution



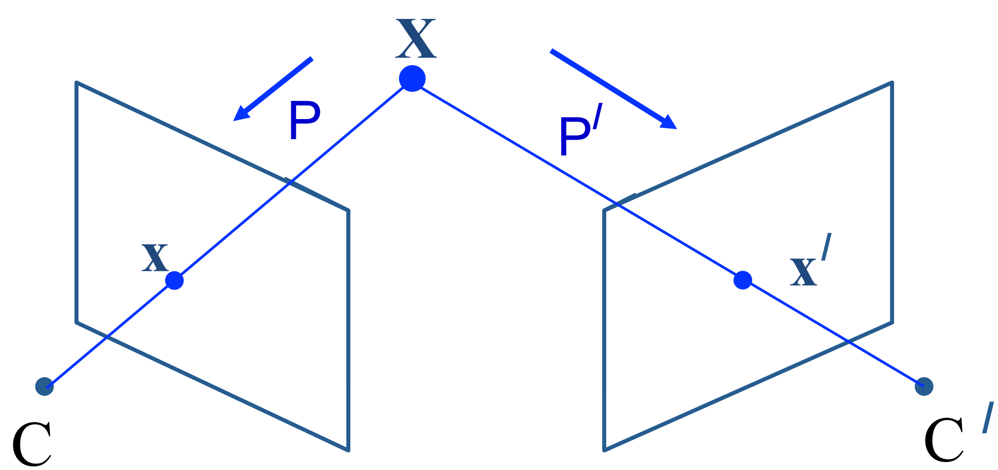
Compute the mid-point of the shortest line between the 2 rays

Problems: multiple cameras, one camera closer than other

Minimizing reprojection error

The two cameras are P and P', and a 3D point **X** is imaged as

$$\mathbf{x} = \mathtt{P}\mathbf{X} \qquad \mathbf{x}' = \mathtt{P}'\mathbf{X}$$



$\mathtt{P}$: 3×4 matrix

$\mathbf{X}$: 4-vector

$\mathbf{x}$: 3-vector

Remember

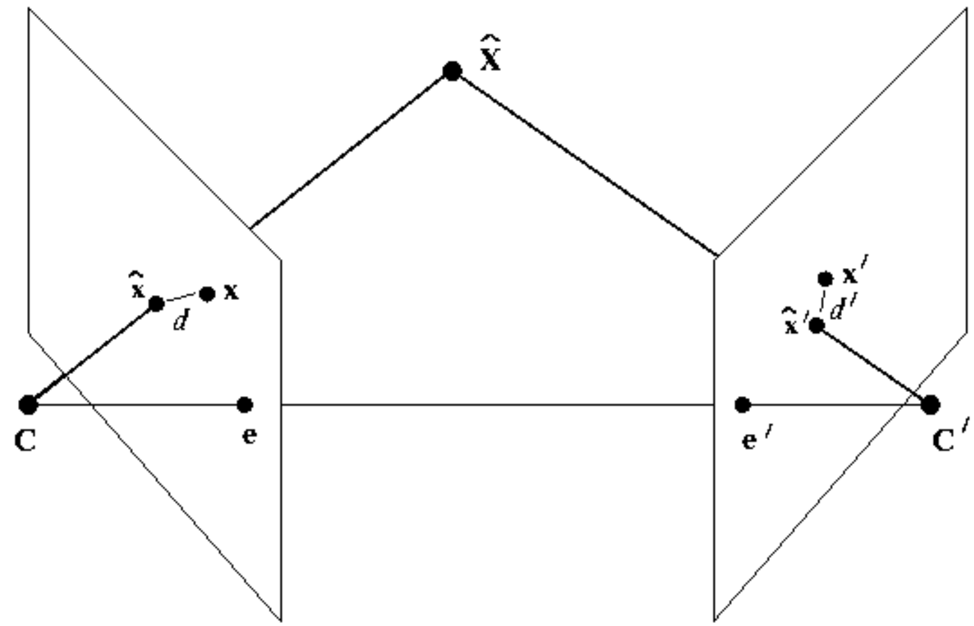
for equations involving homogeneous quantities '=' means 'equal up to scale'

Minimizing a geometric/statistical error

The idea is to estimate a 3D point $\widehat{\mathrm{X}}$ which exactly satisfies the supplied camera geometry, so it projects as

$$\hat{\mathrm{x}} = \mathrm{P}\widehat{\mathrm{X}} \qquad \hat{\mathrm{x}}' = \mathrm{P}'\widehat{\mathrm{X}}$$

and the aim is to estimate $\widehat{\mathrm{X}}$ from the image measurements $\mathbf{x}$ and $\mathbf{x}'$.



$$\min_{\widehat{\mathrm{X}}} \quad \mathcal{C}(\mathbf{x}, \mathbf{x}') = d(\mathbf{x}, \hat{\mathbf{x}})^2 + d(\mathbf{x}', \hat{\mathbf{x}}')^2$$

where $d(*, *)$ is the Euclidean distance between the points.

The correspondence problem

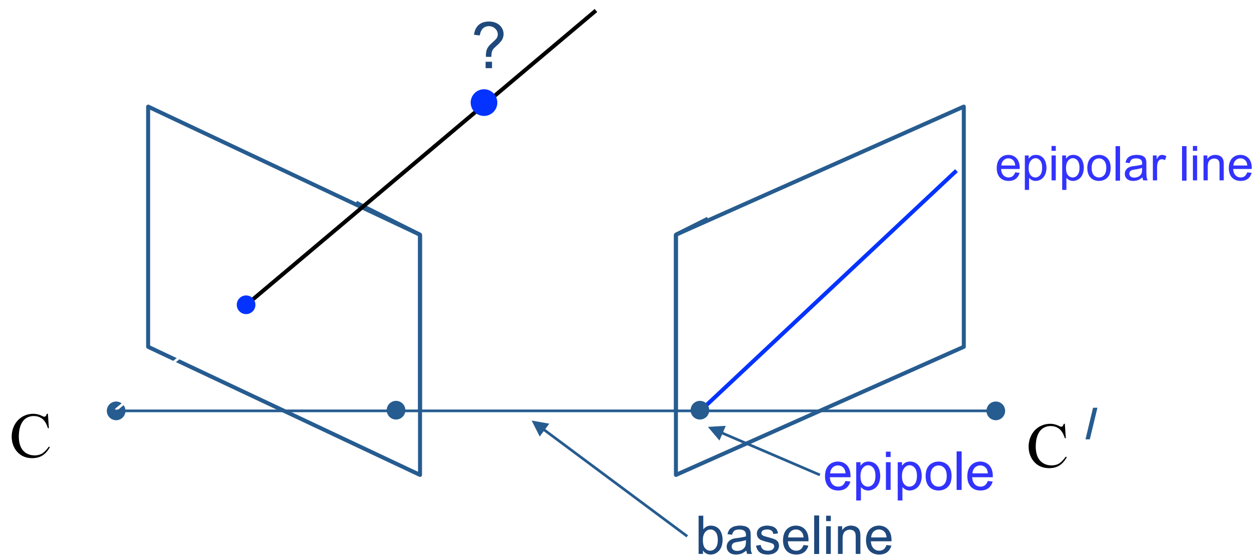
Given a point x in one image find the corresponding point in the other image

This appears to be a 2D search problem, but it is reduced to a 1D search by the epipolar constraint

Epipolar geometry

Epipolar geometry

Given an image point in one view, where is the corresponding point in the other view?



- A point in one view "generates" an epipolar line in the other view
- The corresponding point lies on this line

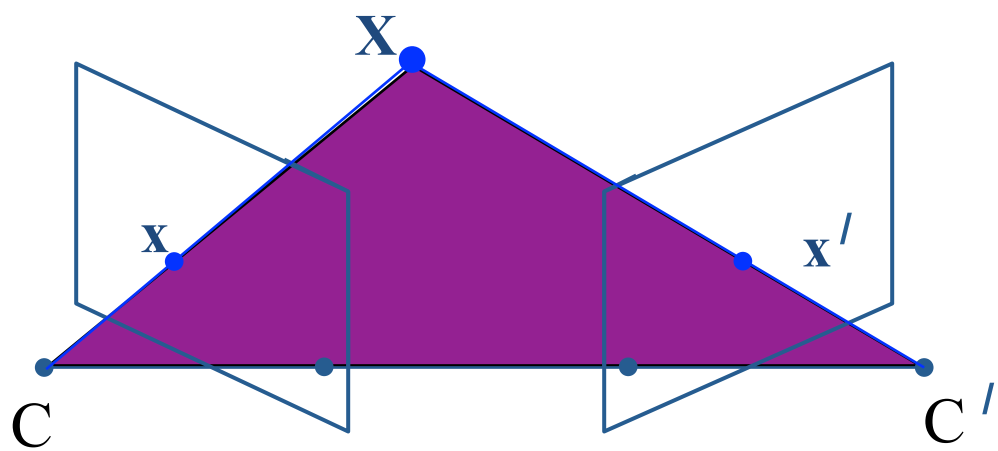
Epipolar line

Epipolar constraint

- Reduces correspondence problem to 1D search along an epipolar line

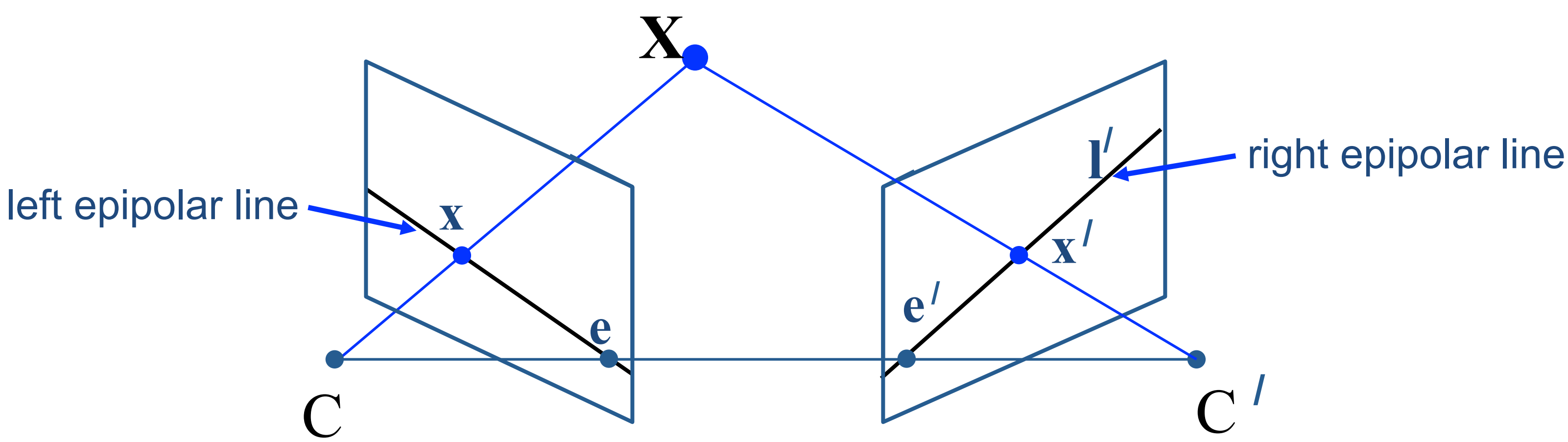
Epipolar geometry continued

Epipolar geometry is a consequence of the coplanarity of the camera centres and scene point



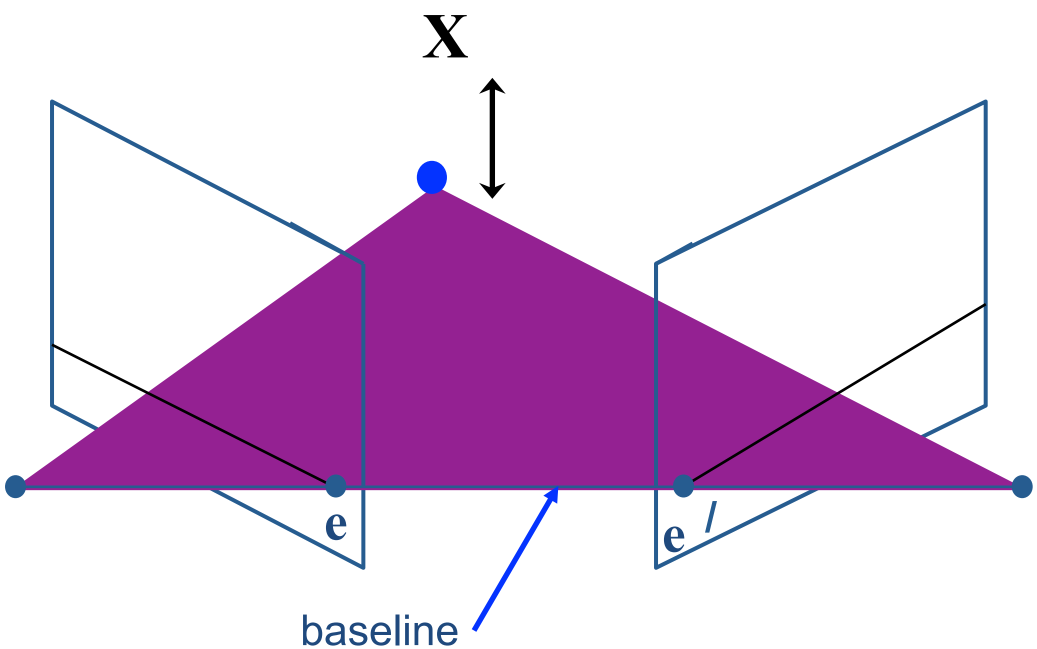
The camera centres, corresponding points and scene point lie in a single plane, known as the epipolar plane

Nomenclature



- The epipolar line $\mathbf{l}'$ is the image of the ray through $\mathbf{x}$
- The epipole $\mathbf{e}$ is the point of intersection of the line joining the camera centres with the image plane
  - this line is the baseline for a stereo rig, and
  - the translation vector for a moving camera
- The epipole is the image of the centre of the other camera: $\mathbf{e} = \mathrm{P}\mathbf{C}'$, $\mathbf{e}' = \mathrm{P}'\mathbf{C}$

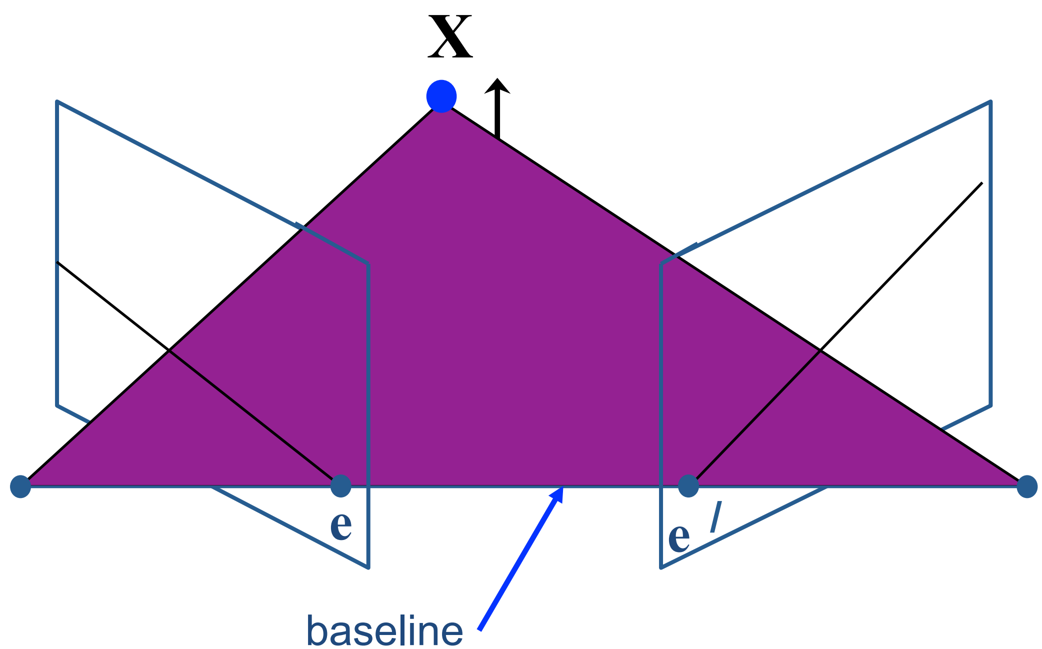
The epipolar pencil



As the position of the 3D point **X** varies, the epipolar planes "rotate" about the baseline. This family of planes is known as an epipolar pencil (a pencil is a one parameter family).

All epipolar lines intersect at the epipole.

The epipolar pencil

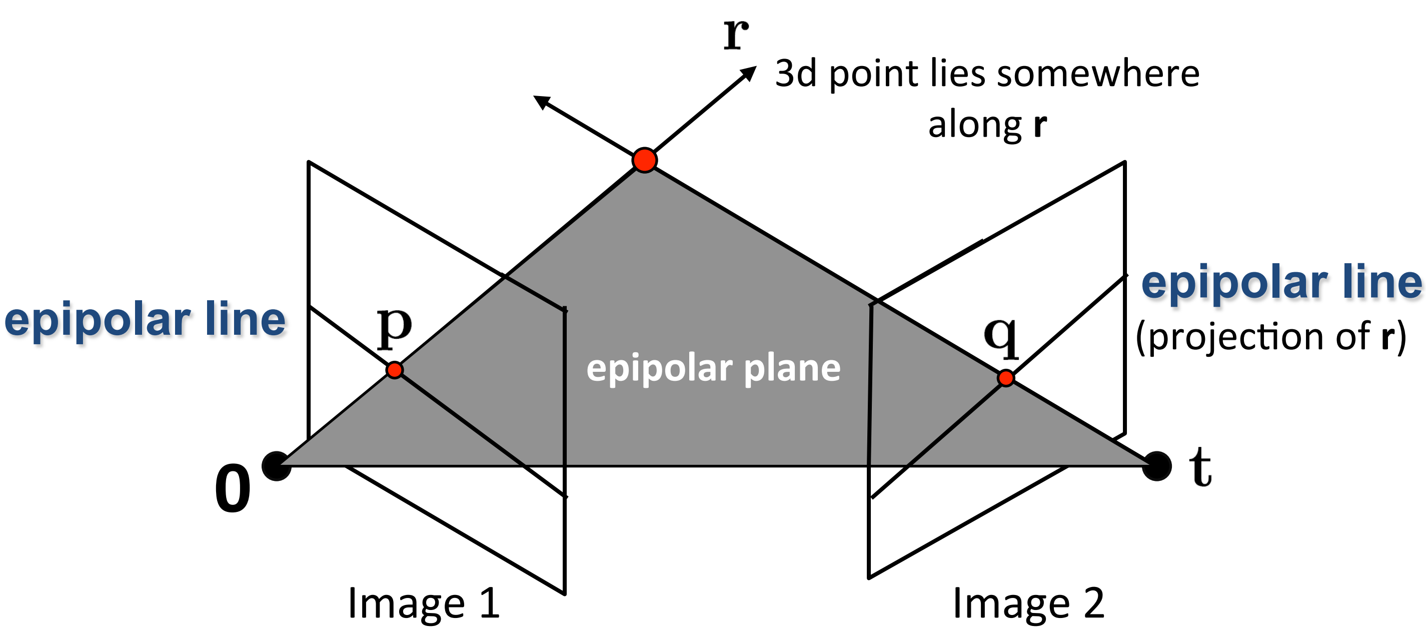


As the position of the 3D point $\mathbf{X}$ varies, the epipolar planes “rotate” about the baseline. This family of planes is known as an epipolar pencil (a pencil is a one parameter family).

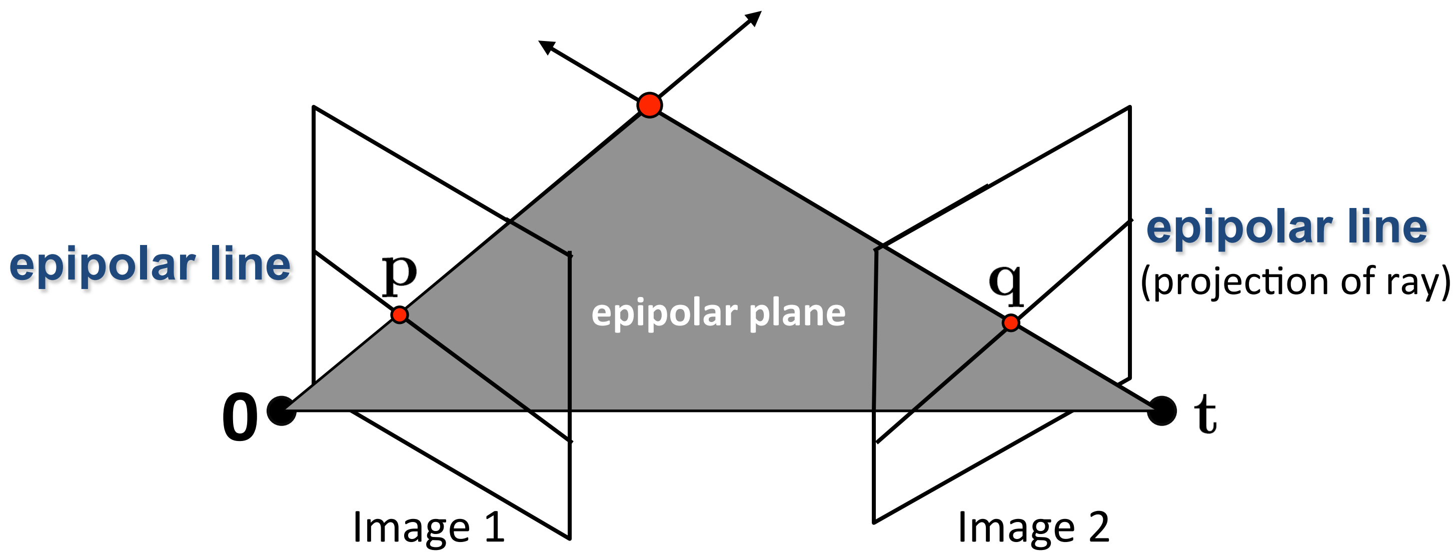
All epipolar lines intersect at the epipole.

Two-view geometry

- Two-view geometry represented by fundamental matrix

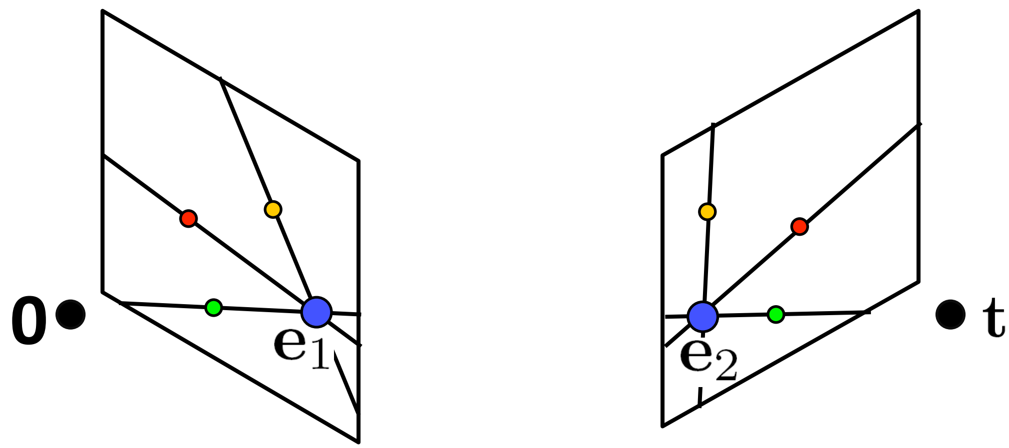


Fundamental matrix



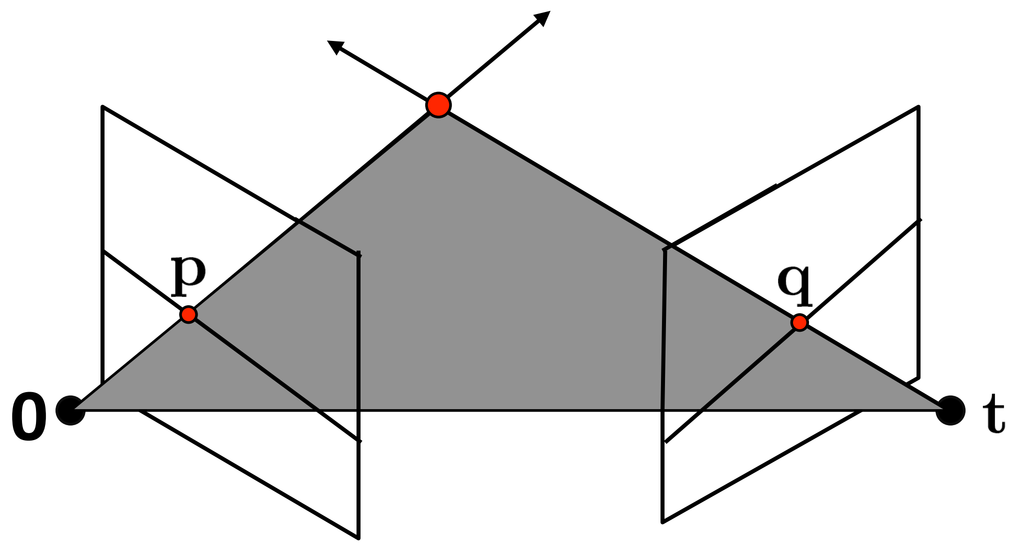
- *Epipolar geometry* of two views is described by a very special 3x3 matrix $\mathbf{F}$, called the *fundamental matrix*
- $\mathbf{F}$ maps (homogeneous) *points* in image 1 to *lines* in image 2!
- The epipolar line (in image 2) of point **p** is: $\mathbf{F}\mathbf{p}$
- *Epipolar constraint* on corresponding points: $\mathbf{q}^T\mathbf{F}\mathbf{p} = 0$

Fundamental matrix



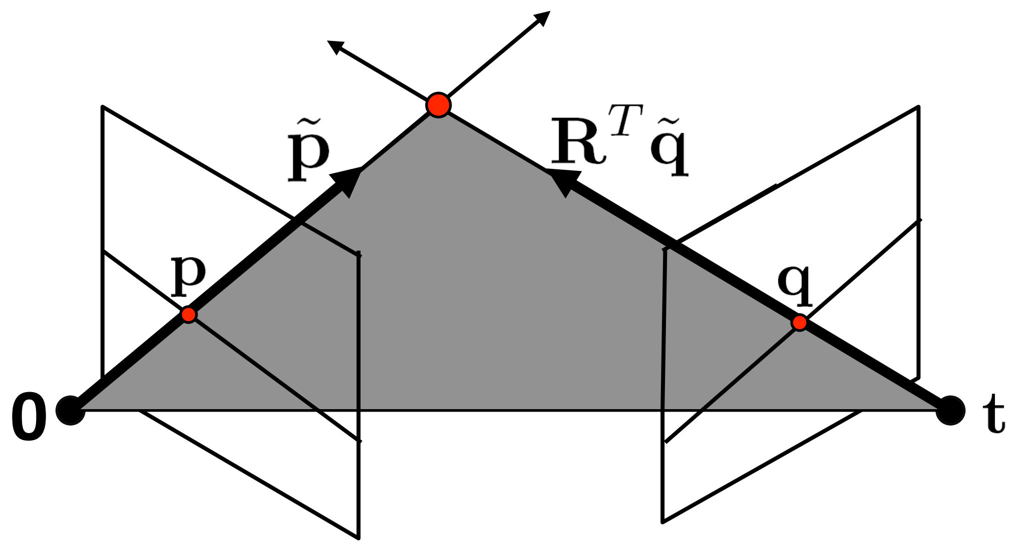
- Two special points: $\mathbf{e}_1$ and $\mathbf{e}_2$ (the *epipoles*): projection of one camera into the other
- All of the epipolar lines in an image pass through the epipole

Fundamental matrix



- Why does **F** exist?
- Let's derive it…

Fundamental matrix



$\mathbf{K}_1$: intrinsics of camera 1

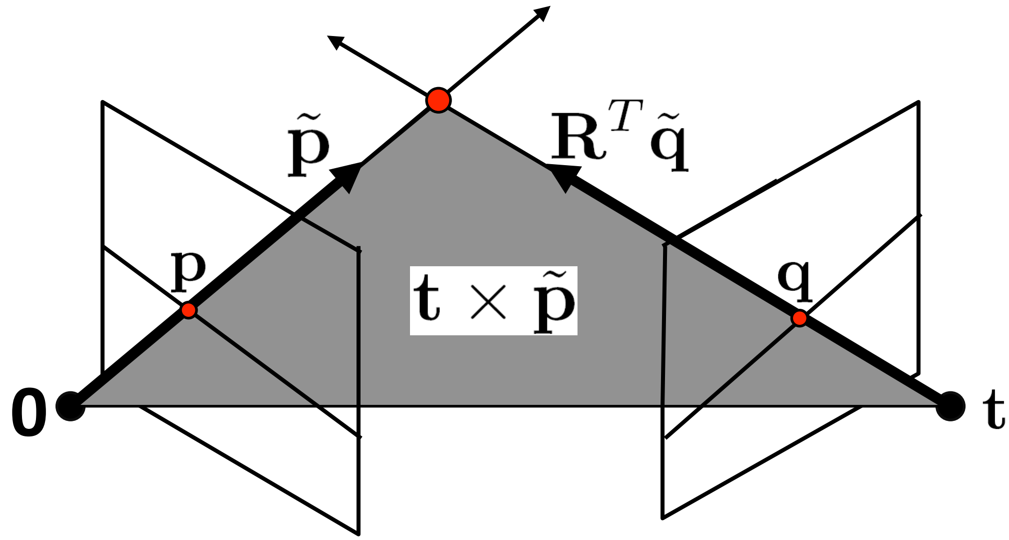
$\mathbf{K}_2$: intrinsics of camera 2

$\mathbf{R}$: rotation of image 2 w.r.t. camera 1

$\tilde{\mathbf{p}} = \mathbf{K}_1^{-1}\mathbf{p}$: ray through **p** in camera 1's (and world) coordinate system

$\tilde{\mathbf{q}} = \mathbf{K}_2^{-1}\mathbf{q}$: ray through **q** in camera 2's coordinate system

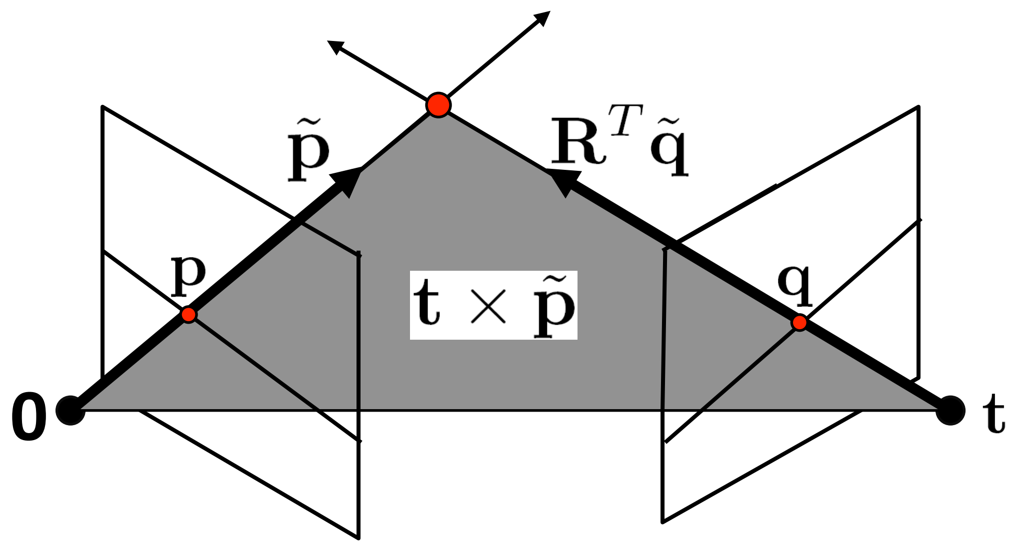
Fundamental matrix



- $\tilde{\mathbf{p}}$, $\mathbf{R}^T\tilde{\mathbf{q}}$, and $\mathbf{t}$ are coplanar
- epipolar plane can be represented as $\mathbf{t} \times \tilde{\mathbf{p}}$

$$(\mathbf{R}^T\tilde{\mathbf{q}})^T(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

Fundamental matrix



$$(\mathbf{R}^T\tilde{\mathbf{q}})^T(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

$$\tilde{\mathbf{q}}^T\mathbf{R}(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

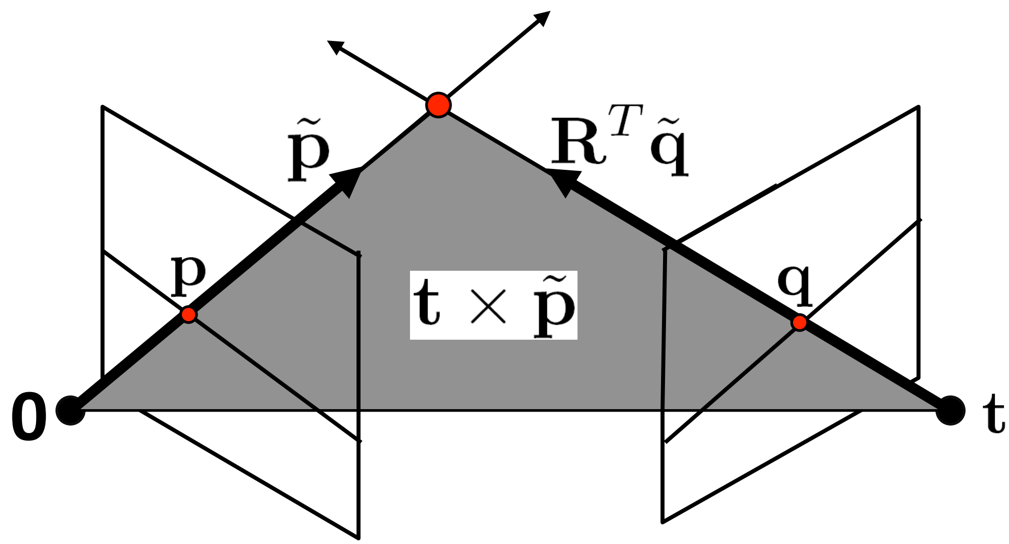
Cross-product as linear operator

Useful fact: Cross product with a vector **t** can be represented as multiplication with a (*skew-symmetric*) 3x3 matrix

$$[\mathbf{t}]_\times = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

$$\mathbf{t} \times \tilde{\mathbf{p}} = [\mathbf{t}]_\times \tilde{\mathbf{p}}$$

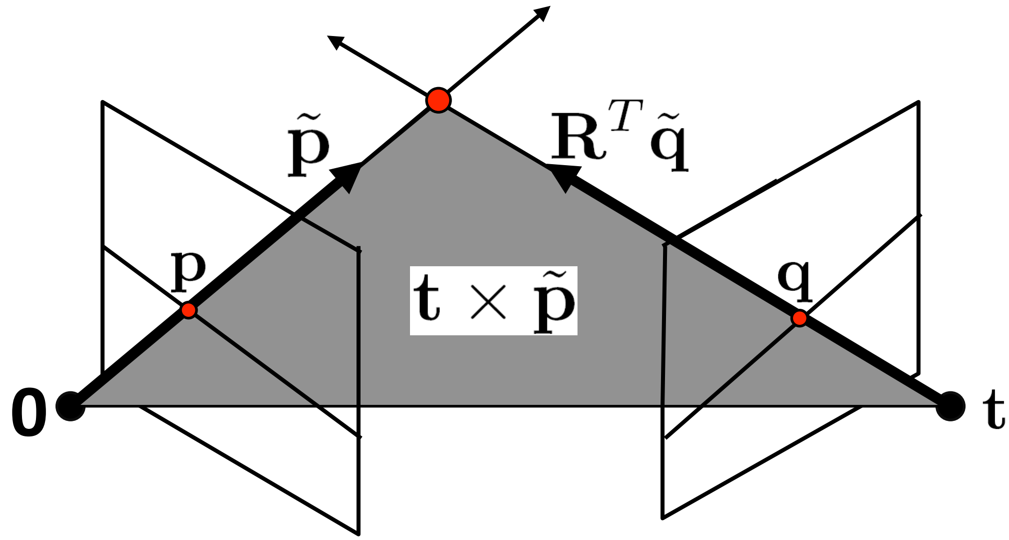
Fundamental matrix



$$\tilde{\mathbf{q}}^T \mathbf{R}(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

$$\tilde{\mathbf{q}}^T \mathbf{R} \left[\mathbf{t}\right]_\times \tilde{\mathbf{p}} = 0$$

Fundamental matrix

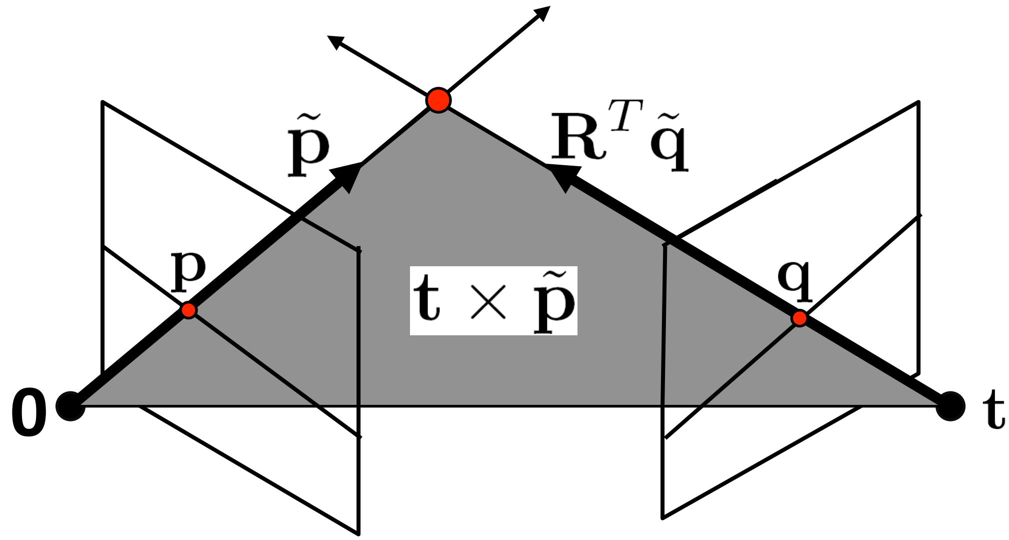


$$\tilde{\mathbf{q}}^T \underbrace{\mathbf{R}\,[\mathbf{t}]_\times}_{\mathbf{E}} \tilde{\mathbf{p}} = 0$$

$$\tilde{\mathbf{q}}^T \mathbf{E} \tilde{\mathbf{p}} = 0$$

$\mathbf{E}$ ← the Essential matrix

Fundamental matrix

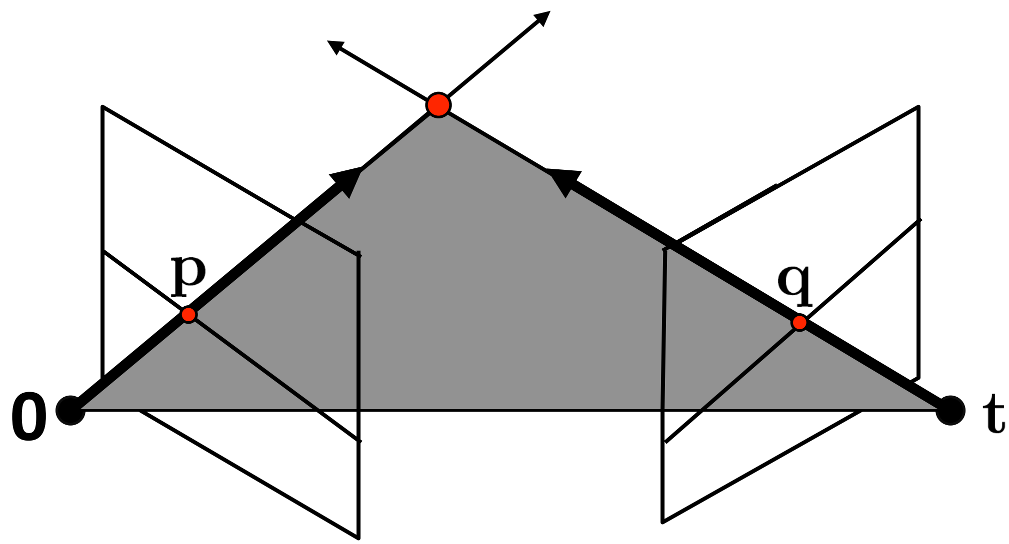


$$\tilde{\mathbf{q}}^T \mathbf{R} \left[\mathbf{t}\right]_\times \tilde{\mathbf{p}} = 0$$

$$\mathbf{q}^T \underbrace{\mathbf{K}_2^{-T} \mathbf{R} \left[\mathbf{t}\right]_\times \mathbf{K}_1^{-1}}_{\mathbf{F}} \mathbf{p} = 0$$

$\mathbf{F}$ ← the Fundamental matrix

Fundamental matrix



$\mathbf{K}_1$: intrinsics of camera 1

$\mathbf{K}_2$: intrinsics of camera 2

$\mathbf{R}$: rotation of image 2 w.r.t. camera 1

$$\mathbf{q}^T \underbrace{\mathbf{K}_2^{-T} \mathbf{R} \, [\mathbf{t}]_\times \, \mathbf{K}_1^{-1}}_{\mathbf{F}} \mathbf{p} = 0$$

$\mathbf{F}$ ← the Fundamental matrix

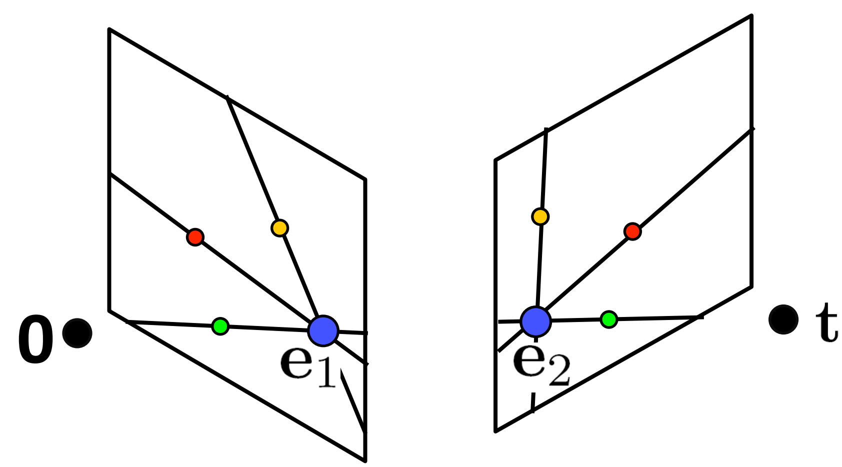
Fundamental matrix result

$$\mathbf{q}^T \mathbf{F} \mathbf{p} = 0$$

(Longuet-Higgins, 1981)

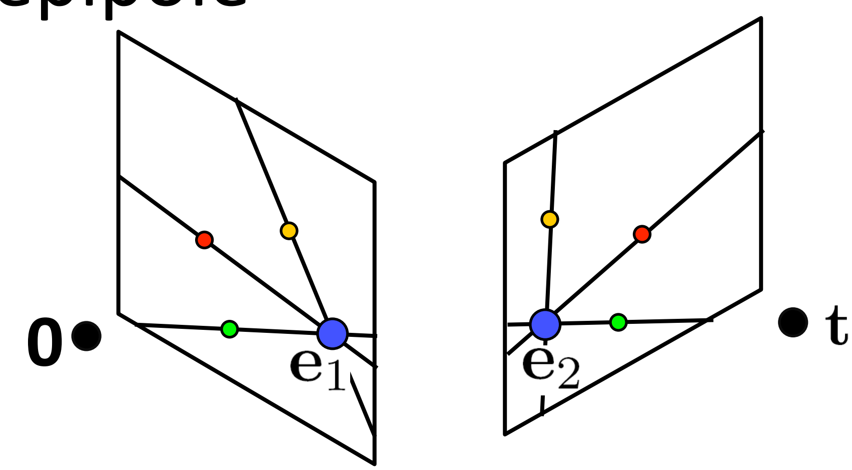
Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$



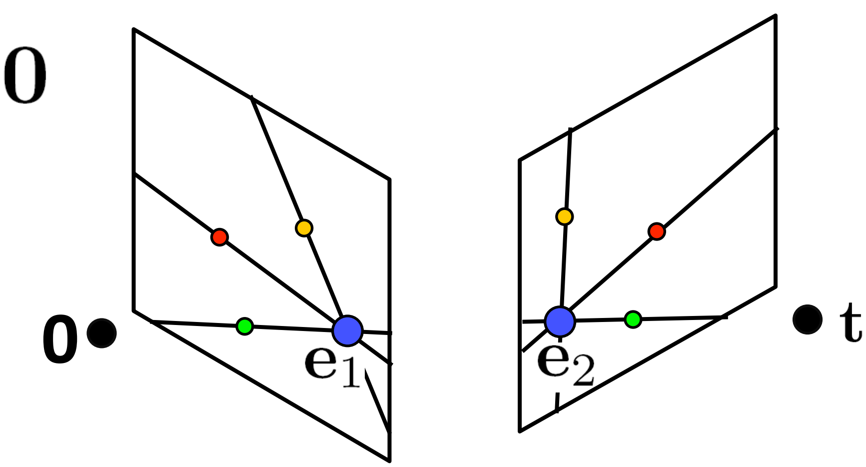
Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$
- $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{F}^T\mathbf{e}_2 = \mathbf{0}$
- All epipolar lines contain epipole



Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$
- $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{F}^T\mathbf{e}_2 = \mathbf{0}$
- $\mathbf{F}$ is rank 2



Why Rank 2

Useful fact: Cross product with a vector **t** can be represented as multiplication with a (*skew-symmetric*) 3x3 matrix

$$[\mathbf{t}]_\times = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

$$\mathbf{t} \times \tilde{\mathbf{p}} = [\mathbf{t}]_\times \tilde{\mathbf{p}}$$

Fundamental matrix song

Questions?

Alternative Formulation

Homogeneous notation for lines

Recall that a point (x, y) in 2D is represented by the homogeneous 3-vector $\mathbf{x} = (x_1, x_2, x_3)^\top$, where $x = x_1/x_3, y = x_2/x_3$

A line in 2D is represented by the homogeneous 3-vector

$$\mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$

which is the line $l_1 x + l_2 y + l_3 = 0$.

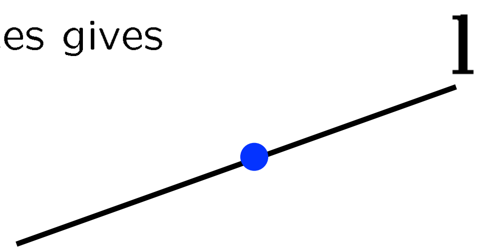
Example represent the line $y = 1$ as a homogeneous vector.

Write the line as $-y + 1 = 0$ then $l_1 = 0, l_2 = -1, l_3 = 1$, and $\mathbf{l} = (0, -1, 1)^\top$.

Note that $\mu(l_1 x + l_2 y + l_3) = 0$ represents the same line (only the ratio of the homogeneous line coordinates is significant).

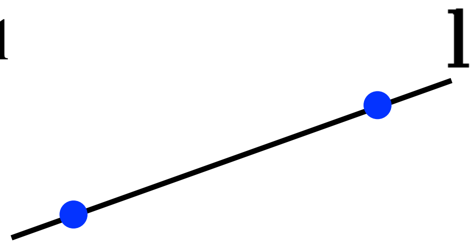
Writing both the point and line in homogeneous coordinates gives

$$l_1 x_1 + l_2 x_2 + l_3 x_3 = 0$$



- point on line $\mathbf{l}.\mathbf{x} = 0$ or $\mathbf{l}^\top \mathbf{x} = 0$ or $\mathbf{x}^\top \mathbf{l} = 0$

- The line $\mathbf{l}$ through the two points $\mathbf{p}$ and $\mathbf{q}$ is $\mathbf{l} = \mathbf{p} \times \mathbf{q}$



Proof

$$\mathbf{l}.\mathbf{p} = (\mathbf{p} \times \mathbf{q}).\mathbf{p} = 0 \qquad \mathbf{l}.\mathbf{q} = (\mathbf{p} \times \mathbf{q}).\mathbf{q} = 0$$

- The intersection of two lines $\mathbf{l}$ and $\mathbf{m}$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{m}$

Algebraic representation of epipolar geometry

$$\mathbf{x} \quad \rightarrow \quad \mathbf{l}'$$

point in first image — epipolar line in second image

- the map ony depends on the cameras $\mathtt{P}, \mathtt{P}'$ (not on structure)

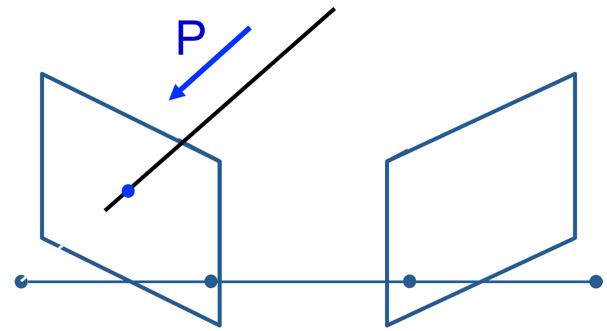
- it will be shown that the map is linear and can be written as $\mathbf{l}' = \mathtt{F}\mathbf{x}$, where $\mathtt{F}$ is a 3×3 matrix called the fundamental matrix

Derivation of the algebraic expression

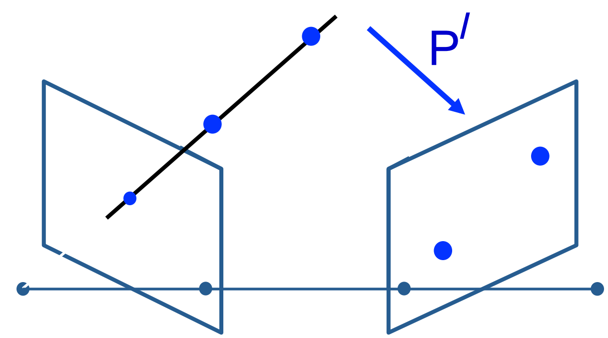
$$\mathbf{l}' = \mathtt{F}\mathbf{x}$$

Outline

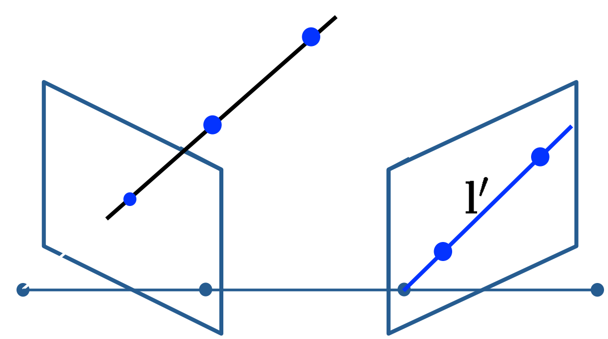
Step 1: for a point x in the first image back project a ray with camera P



Step 2: choose two points on the ray and project into the second image with camera P'



Step 3: compute the line through the two image points using the relation $\mathbf{l}' = \mathbf{p} \times \mathbf{q}$



- choose camera matrices

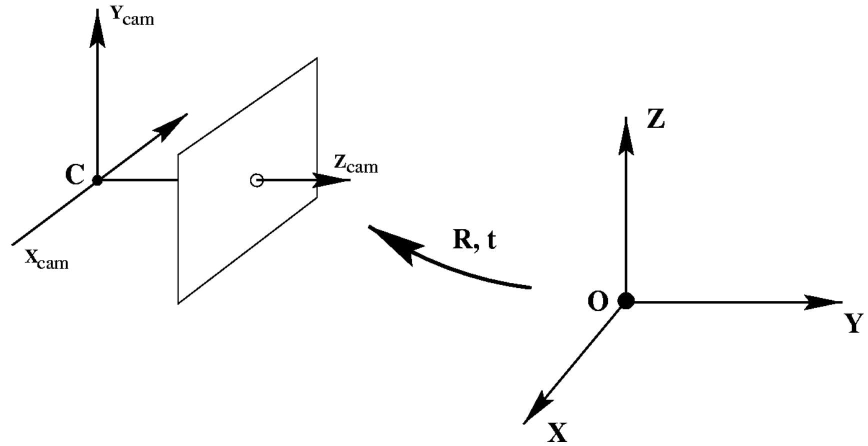
$$P = K[R \mid t]$$

internal calibration

rotation

translation

from world to camera coordinate frame

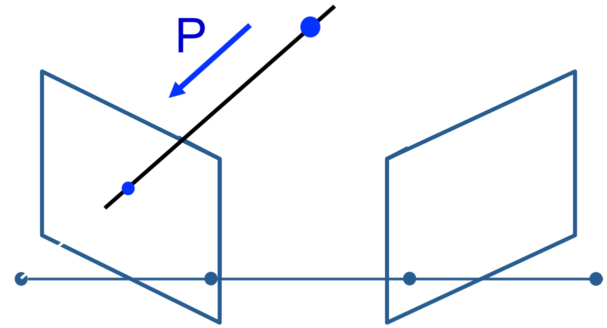


- first camera $$P = K[I \mid 0]$$

world coordinate frame aligned with first camera

- second camera $$P' = K'[R \mid t]$$

Step 1: for a point x in the first image
back project a ray with camera $\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}]$



A point $\mathbf{x}$ back projects to a ray

$$\begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{pmatrix} = \mathtt{z}\mathtt{K}^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \mathtt{z}\mathtt{K}^{-1}\mathbf{x}$$

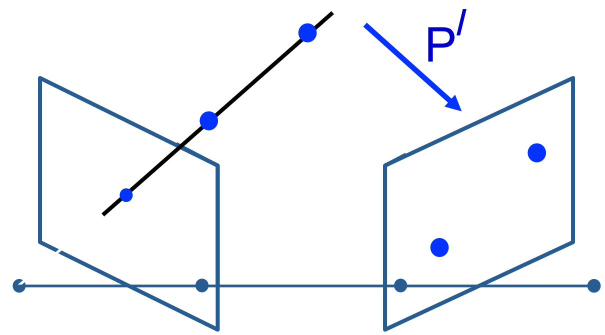
where $\mathbf{Z}$ is the point's depth, since

$$\mathrm{X}(\mathtt{z}) = \begin{pmatrix} \mathtt{z}\mathtt{K}^{-1}\mathbf{x} \\ 1 \end{pmatrix}$$

satisfies

$$\mathtt{P}\mathrm{X}(\mathtt{z}) = \mathtt{K}[\mathtt{I} \mid \mathbf{0}]\mathrm{X}(\mathtt{z}) = \mathbf{x}$$

Step 2: choose two points on the ray and project into the second image with camera P'



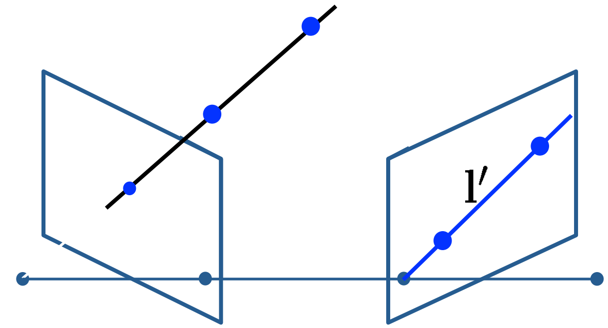
Consider two points on the ray $\mathrm{X}(z) = \begin{pmatrix} z\mathrm{K}^{-1}\mathbf{x} \\ 1 \end{pmatrix}$

- $Z = 0$ is the camera centre $\begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$
- $Z = \infty$ is the point at infinity $\begin{pmatrix} \mathrm{K}^{-1}\mathbf{x} \\ 0 \end{pmatrix}$

Project these two points into the second view

$$\mathrm{P}' \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \mathrm{K}'[\mathrm{R} \mid \mathbf{t}] \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \mathrm{K}'\mathbf{t} \qquad \mathrm{P}' \begin{pmatrix} \mathrm{K}^{-1}\mathbf{x} \\ 0 \end{pmatrix} = \mathrm{K}'[\mathrm{R} \mid \mathbf{t}] \begin{pmatrix} \mathrm{K}^{-1}\mathbf{x} \\ 0 \end{pmatrix} = \mathrm{K}'\mathrm{R}\mathrm{K}^{-1}\mathbf{x}$$

Step 3: compute the line through the two image points using the relation $\mathbf{l}' = \mathbf{p} \times \mathbf{q}$



Compute the line through the points $\mathbf{l}' = (\mathtt{K}'\mathbf{t}) \times (\mathtt{K}'\mathtt{R}\mathtt{K}^{-1}\mathbf{x})$

Using the identity $(\mathtt{M}\mathbf{a}) \times (\mathtt{M}\mathbf{b}) = \mathtt{M}^{-\top}(\mathbf{a} \times \mathbf{b})$ where $\mathtt{M}^{-\top} = (\mathtt{M}^{-1})^{\top} = (\mathtt{M}^{\top})^{-1}$

$$\mathbf{l}' = \mathtt{K}'^{-\top}\left(\mathbf{t} \times (\mathtt{R}\mathtt{K}^{-1}\mathbf{x})\right) = \underbrace{\mathtt{K}'^{-\top}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1}}_{\mathtt{F}}\mathbf{x}$$

F is the fundamental matrix

$$\mathbf{l}' = \mathtt{F}\mathbf{x} \qquad \mathtt{F} = \mathtt{K}'^{-\top}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1}$$

Points $\mathbf{x}$ and $\mathbf{x}'$ correspond $(\mathbf{x} \leftrightarrow \mathbf{x}')$ then $\mathbf{x}'^{\top}\mathbf{l}' = 0$

$$\mathbf{x}'^{\top}\mathtt{F}\mathbf{x} = 0$$