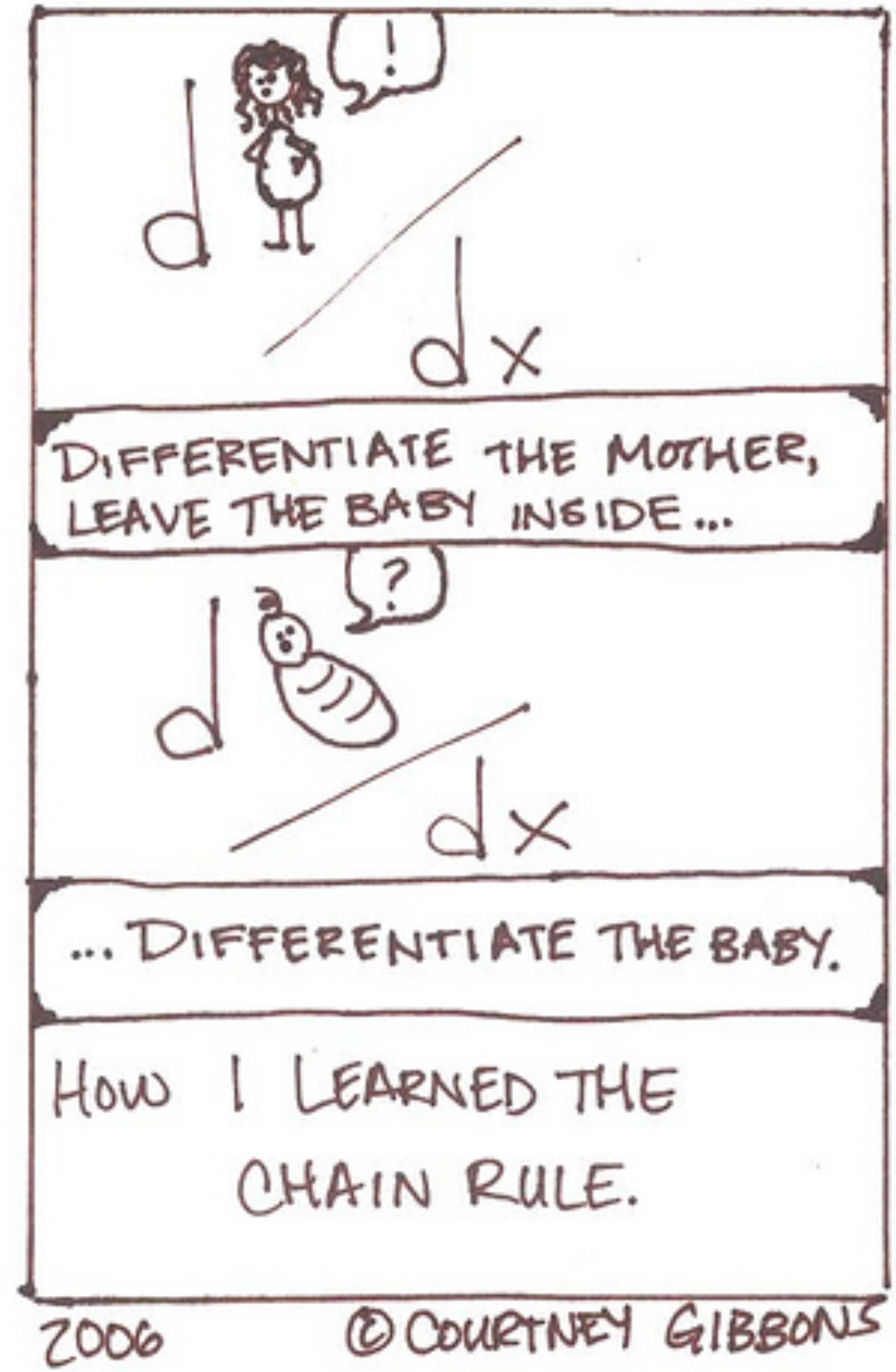
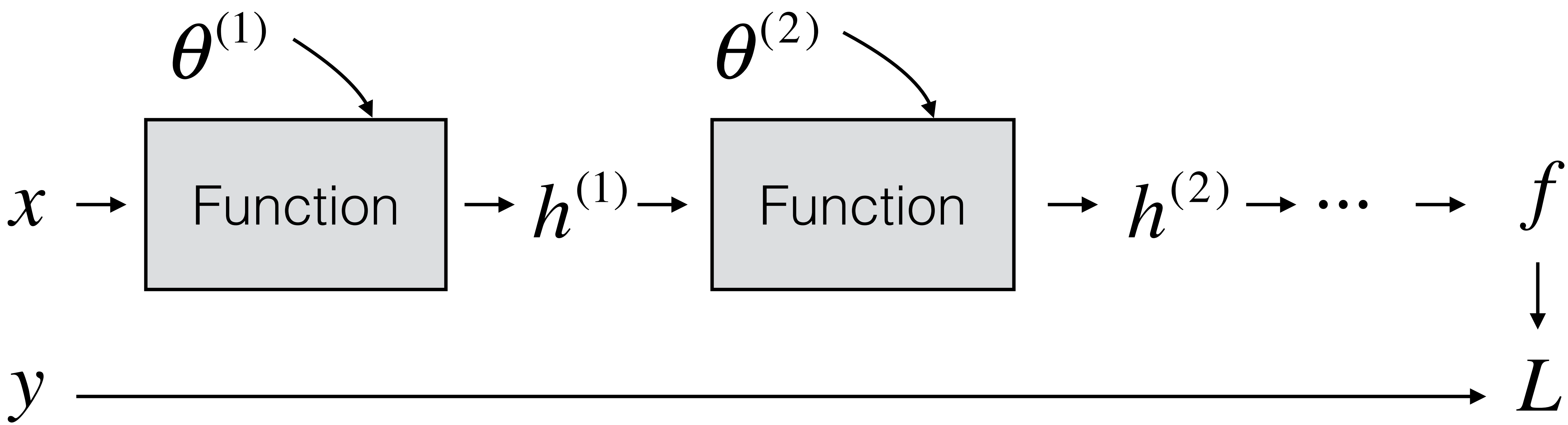
Convolutional Neural Networks

CS 4670
Sean Bell



http://brownsharpie.courtneygibbons.org/?p=90

Review: Setup



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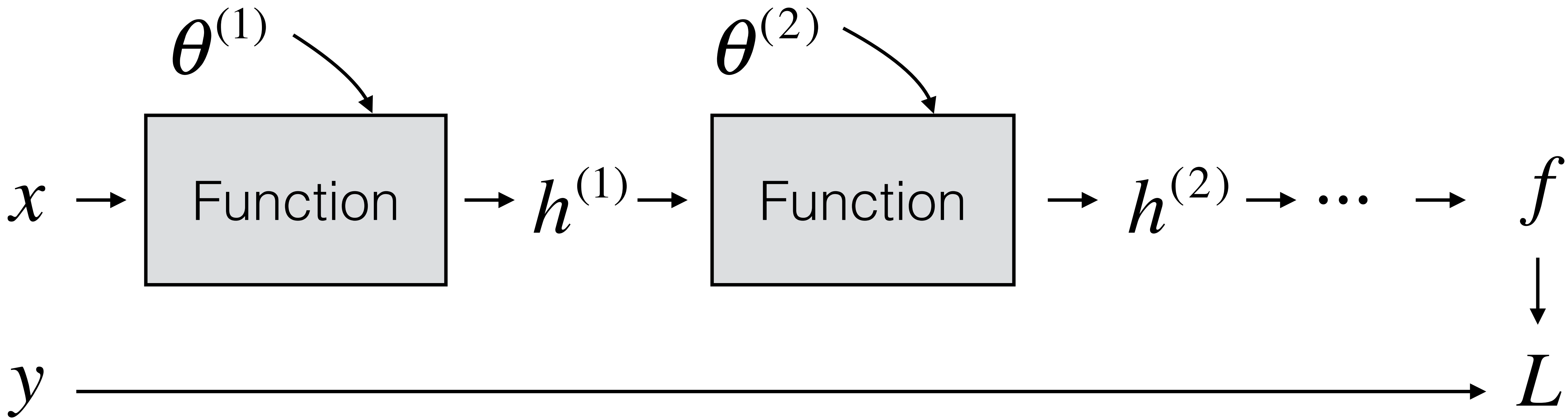

- **Goal:** Find a value for parameters ($\theta^{(1)}$, $\theta^{(2)}$, …), so that the loss (L) is small

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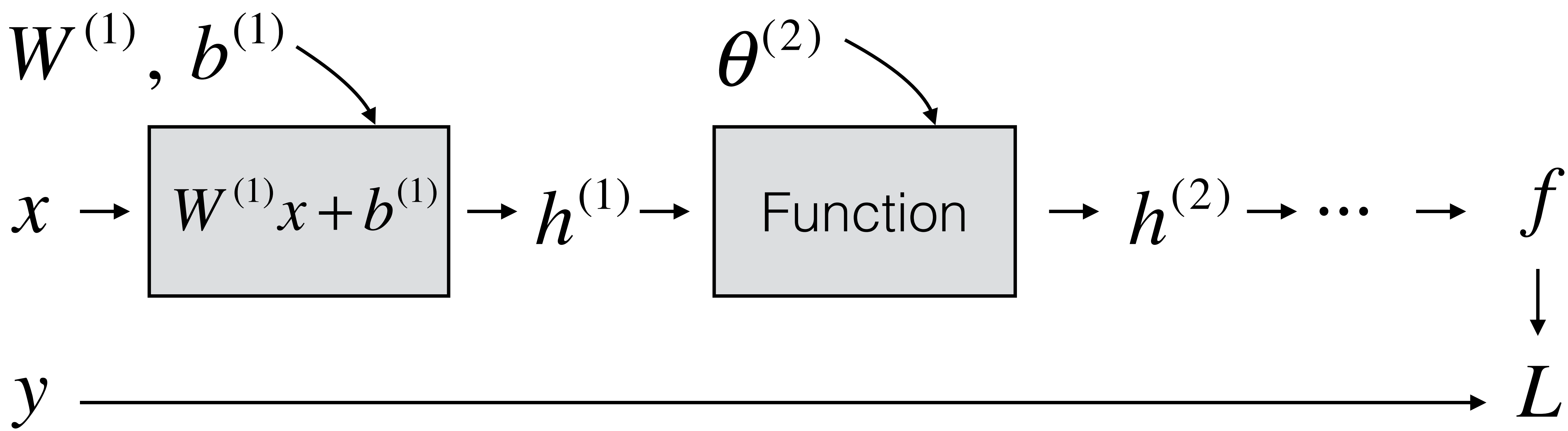

Toy Example:

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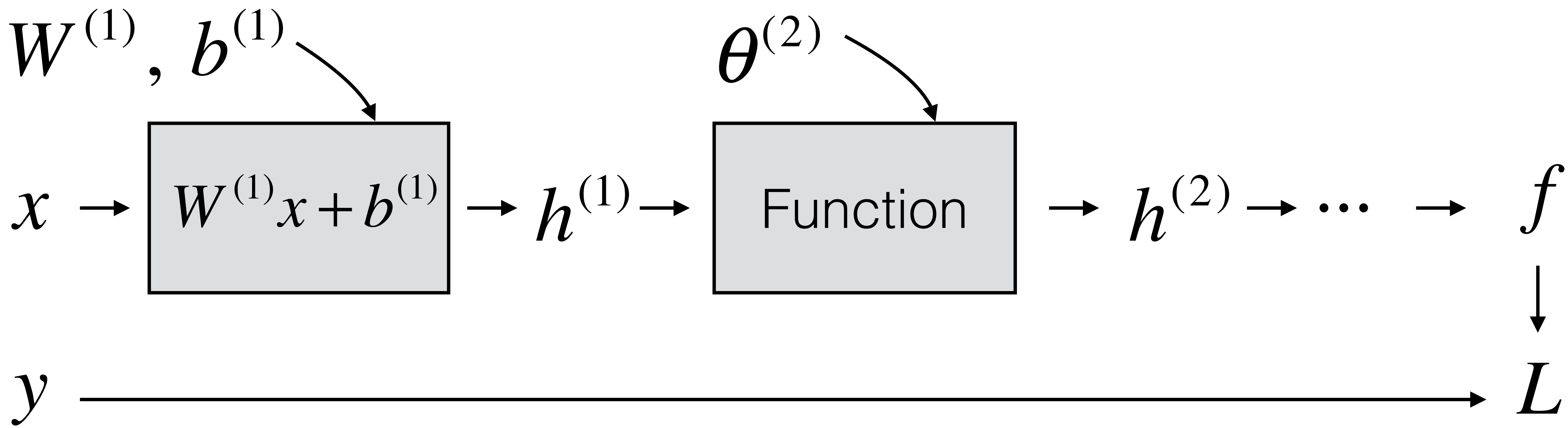

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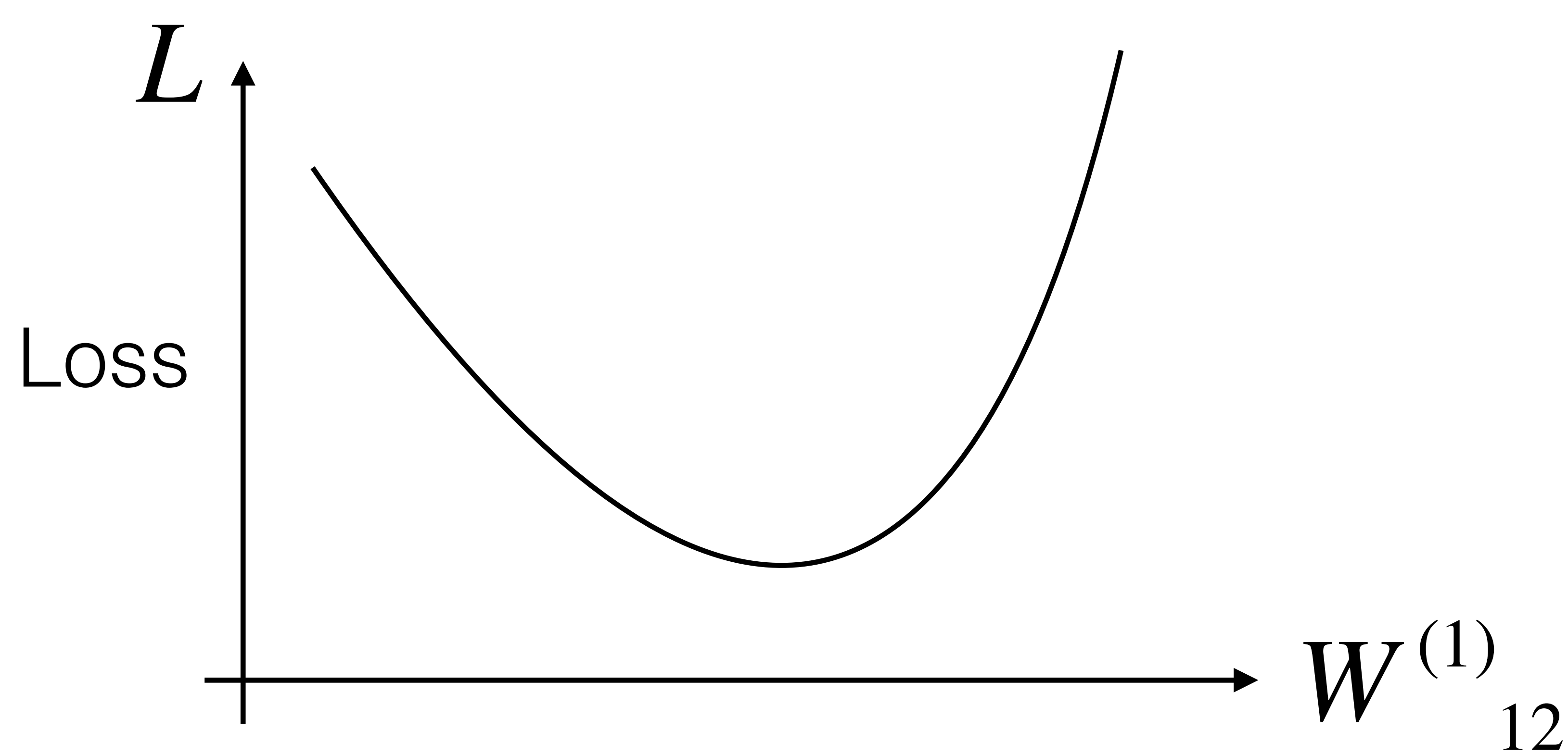
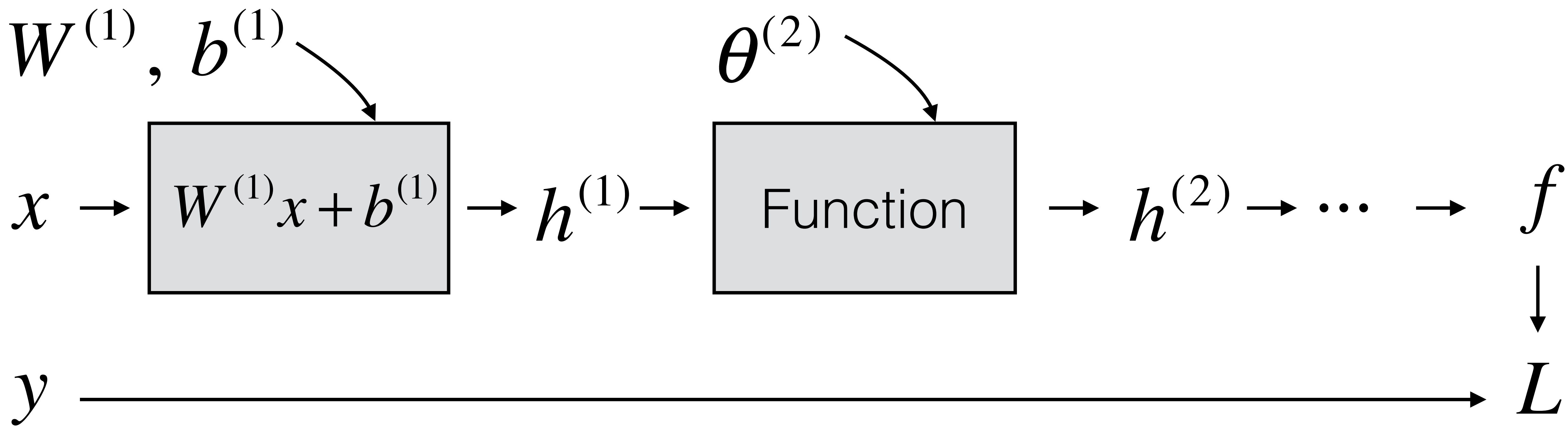


A weight somewhere in the network

Review: Setup



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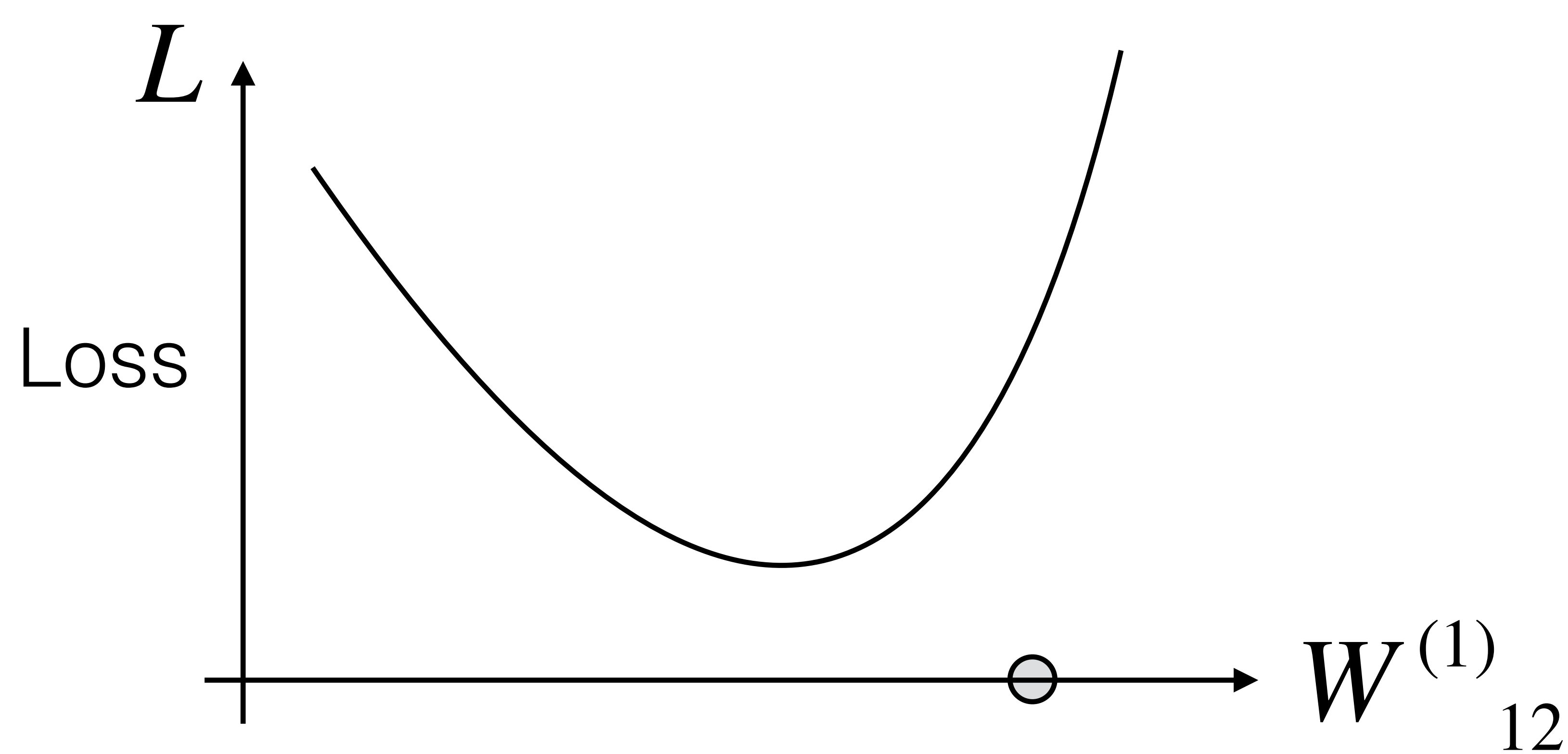

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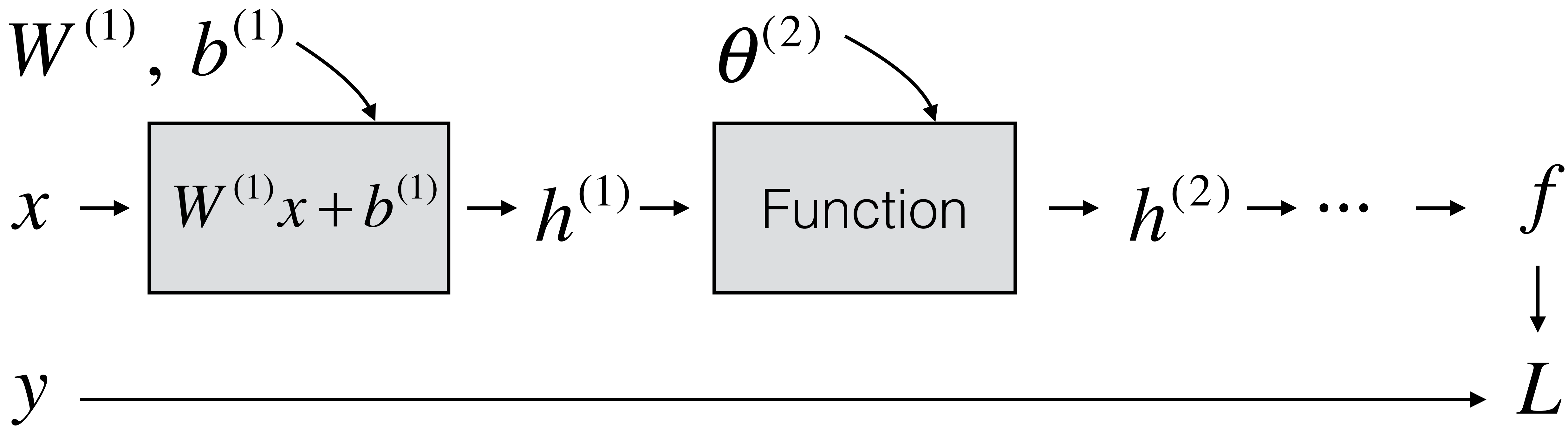

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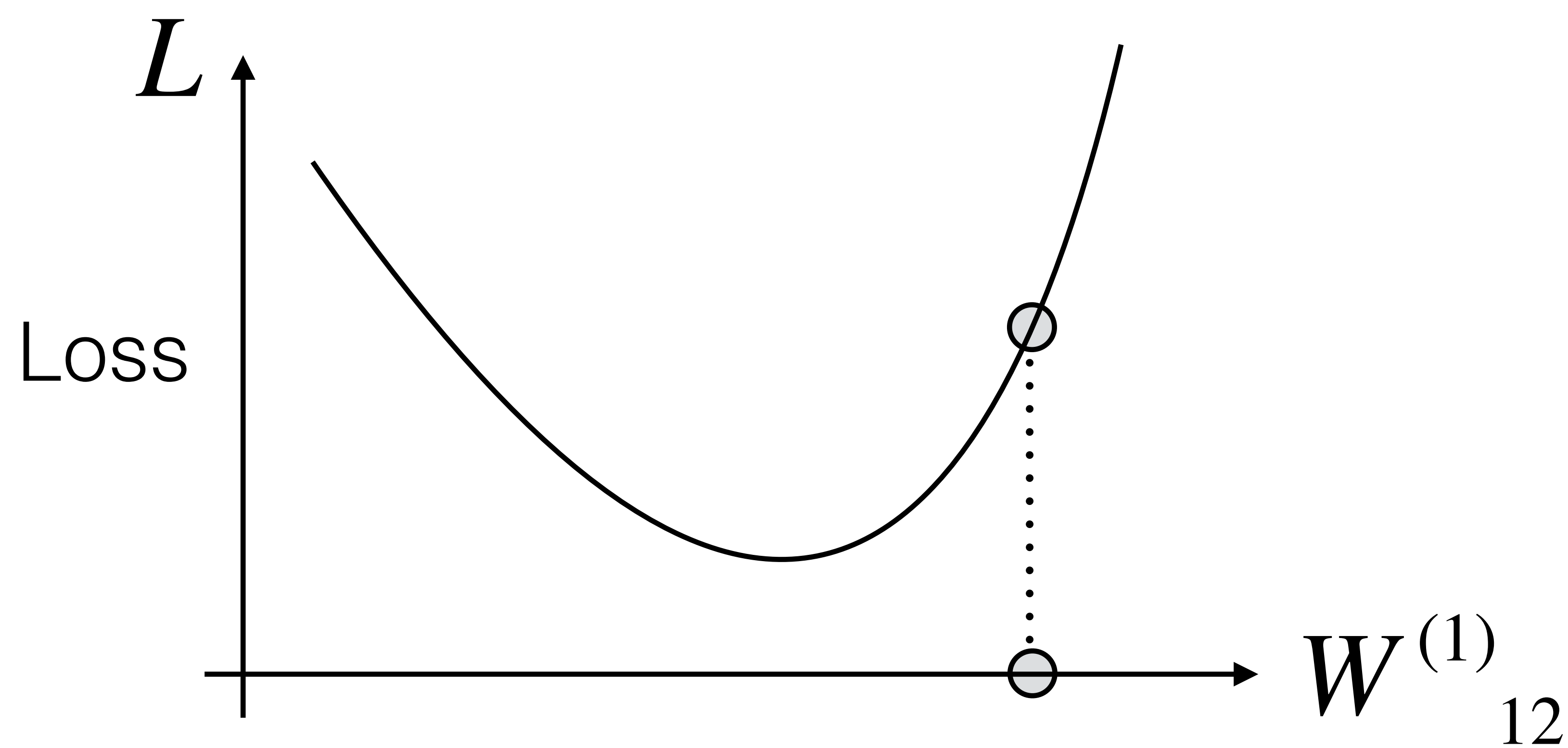

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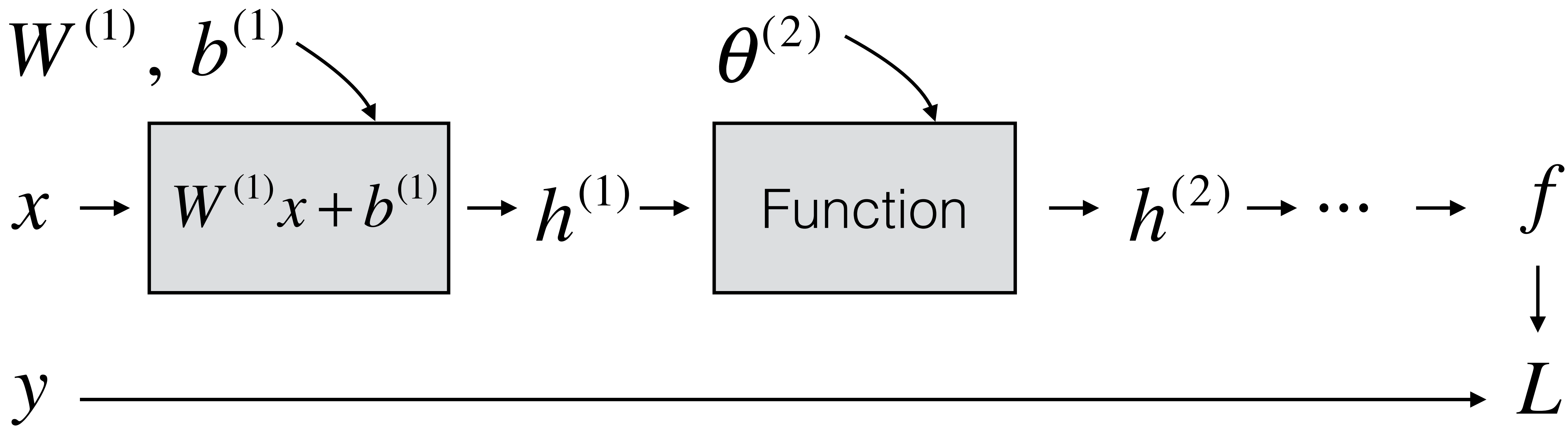

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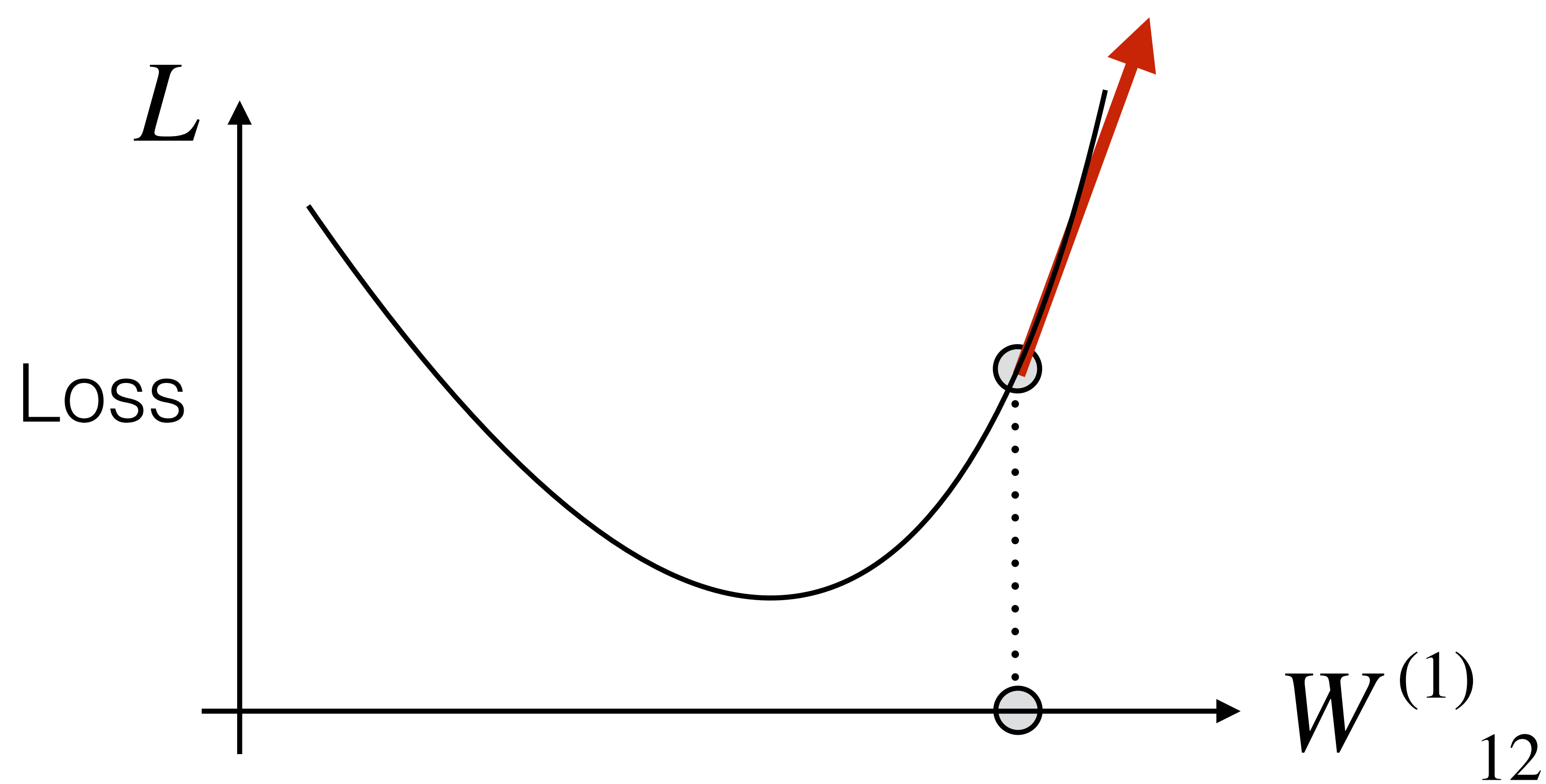

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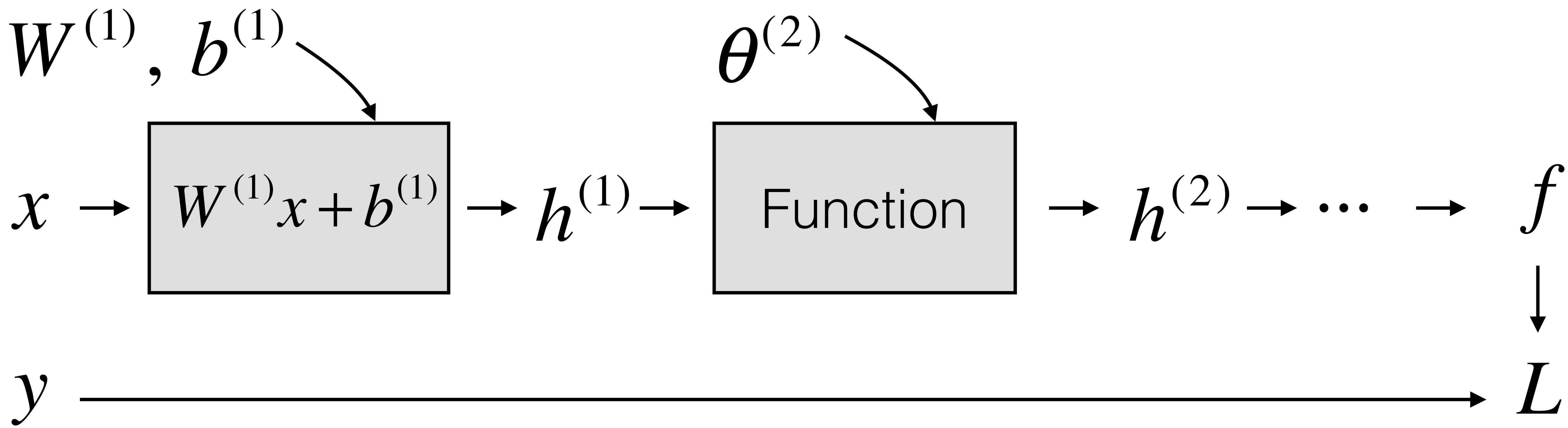

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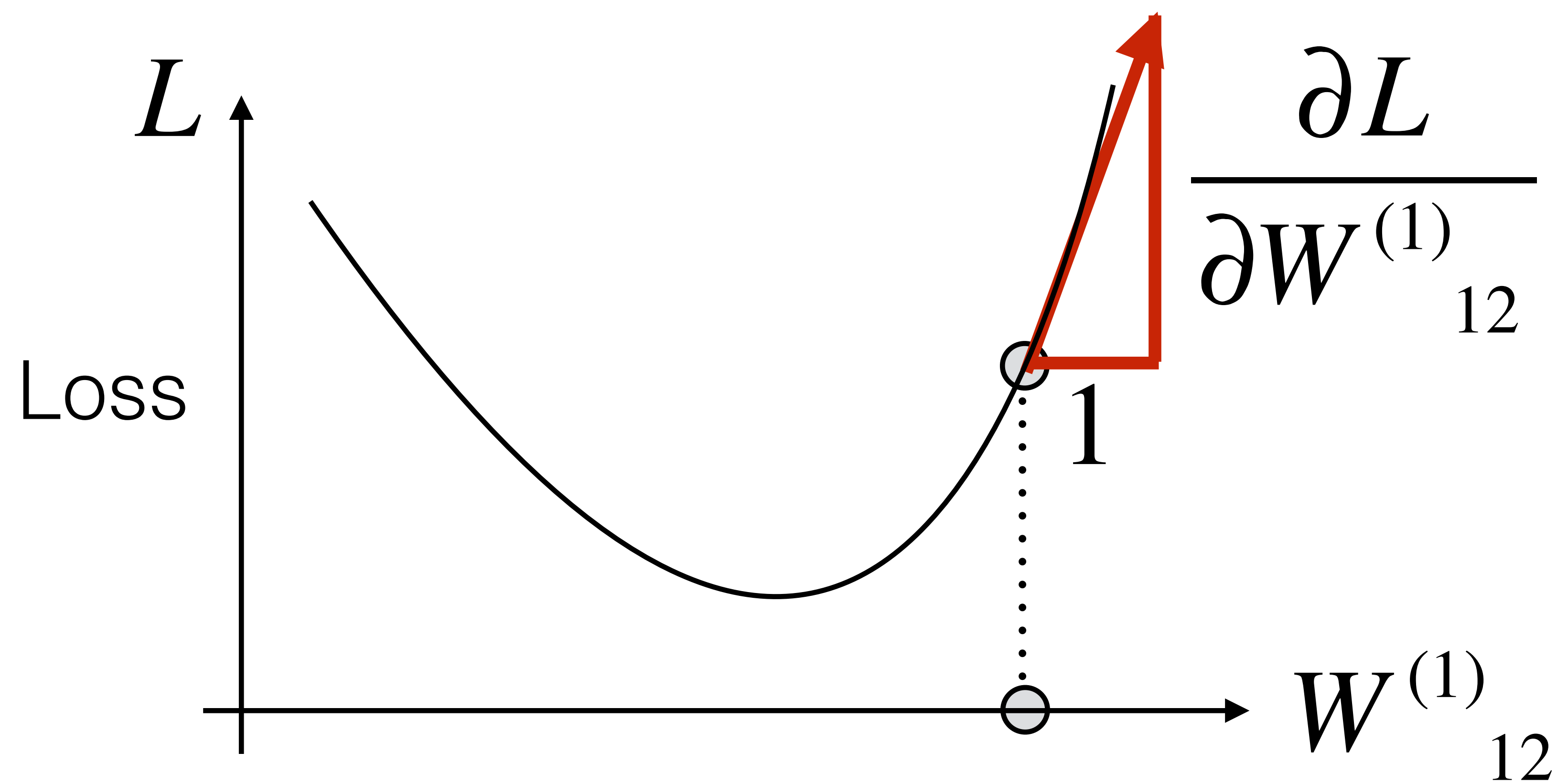

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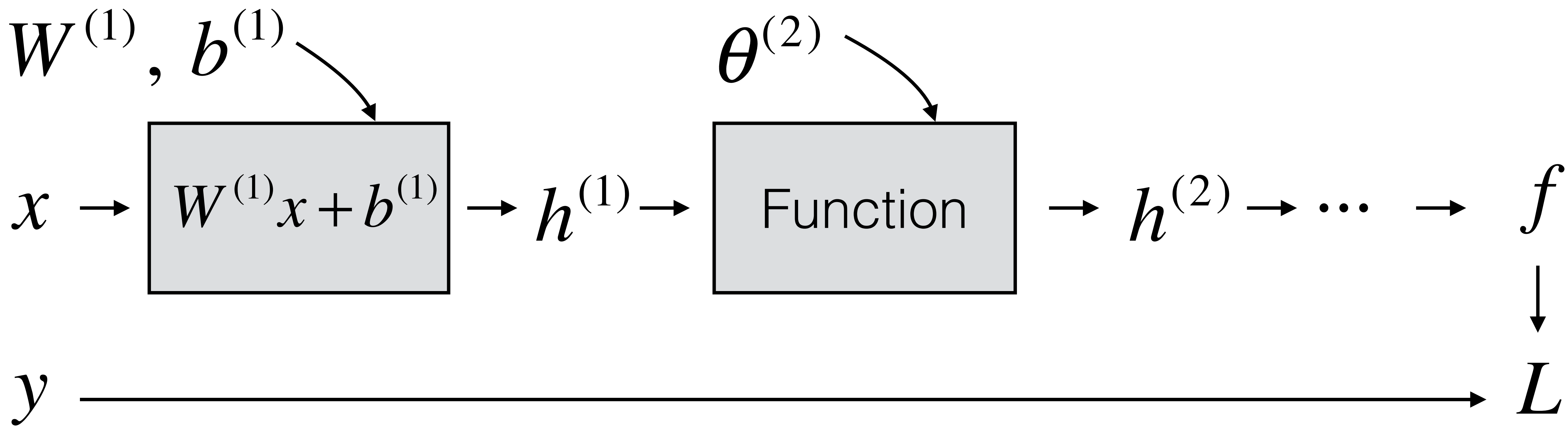

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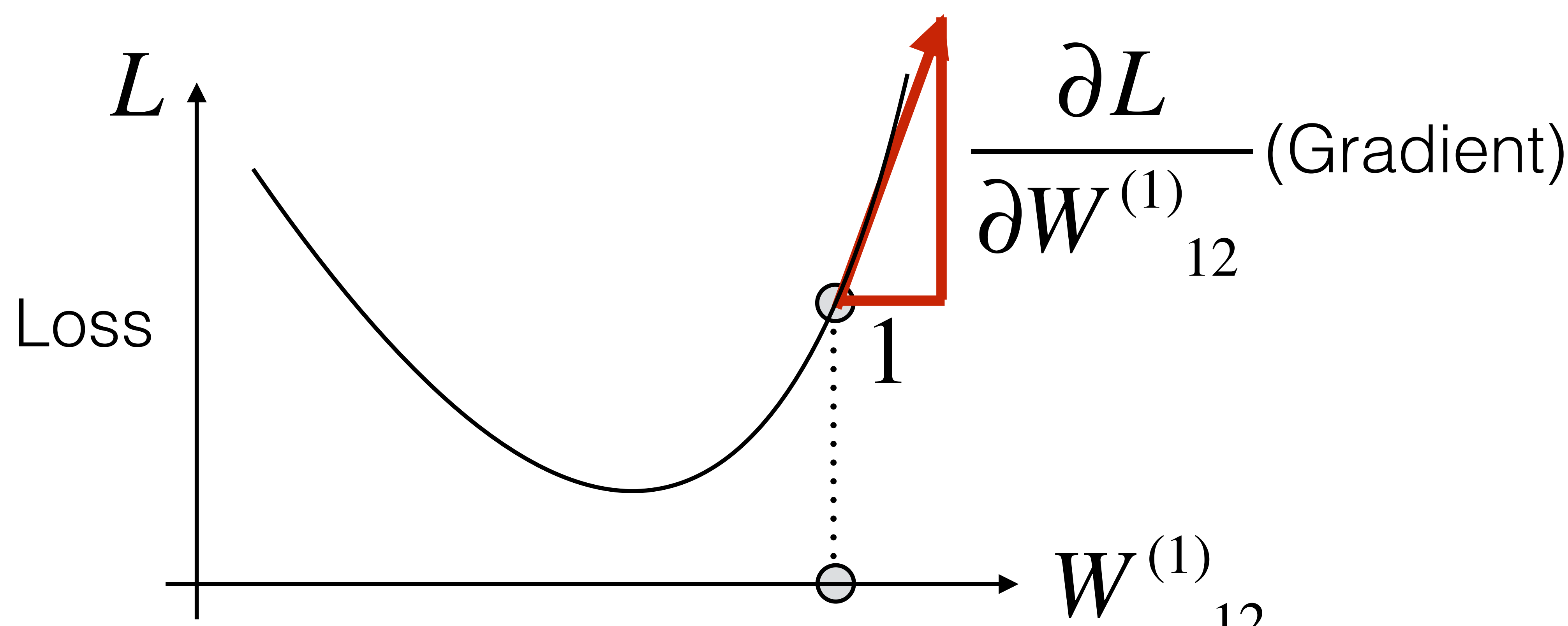

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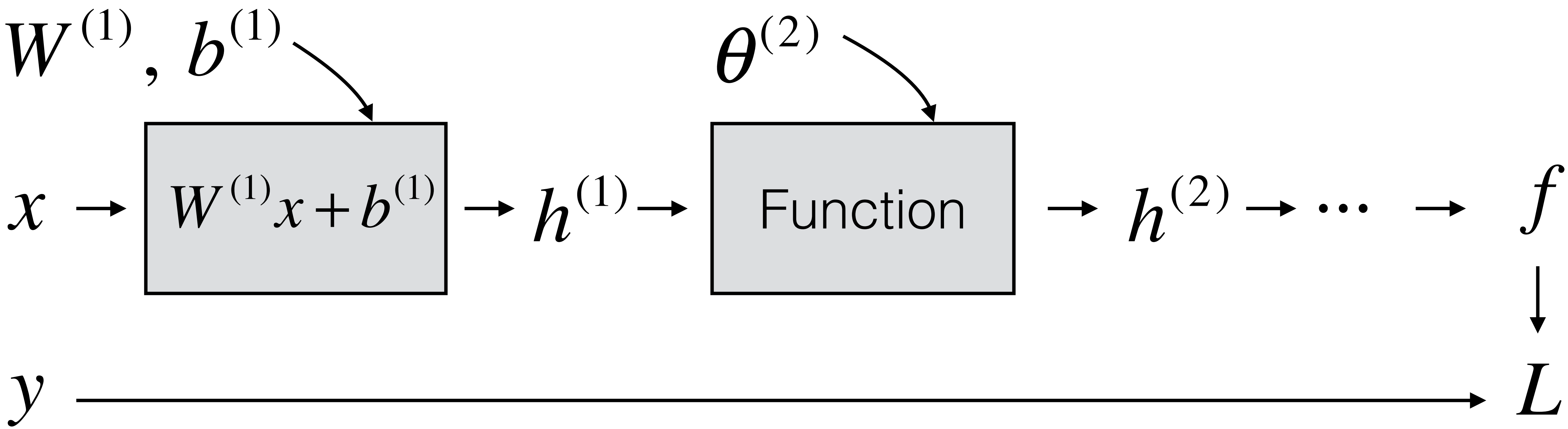

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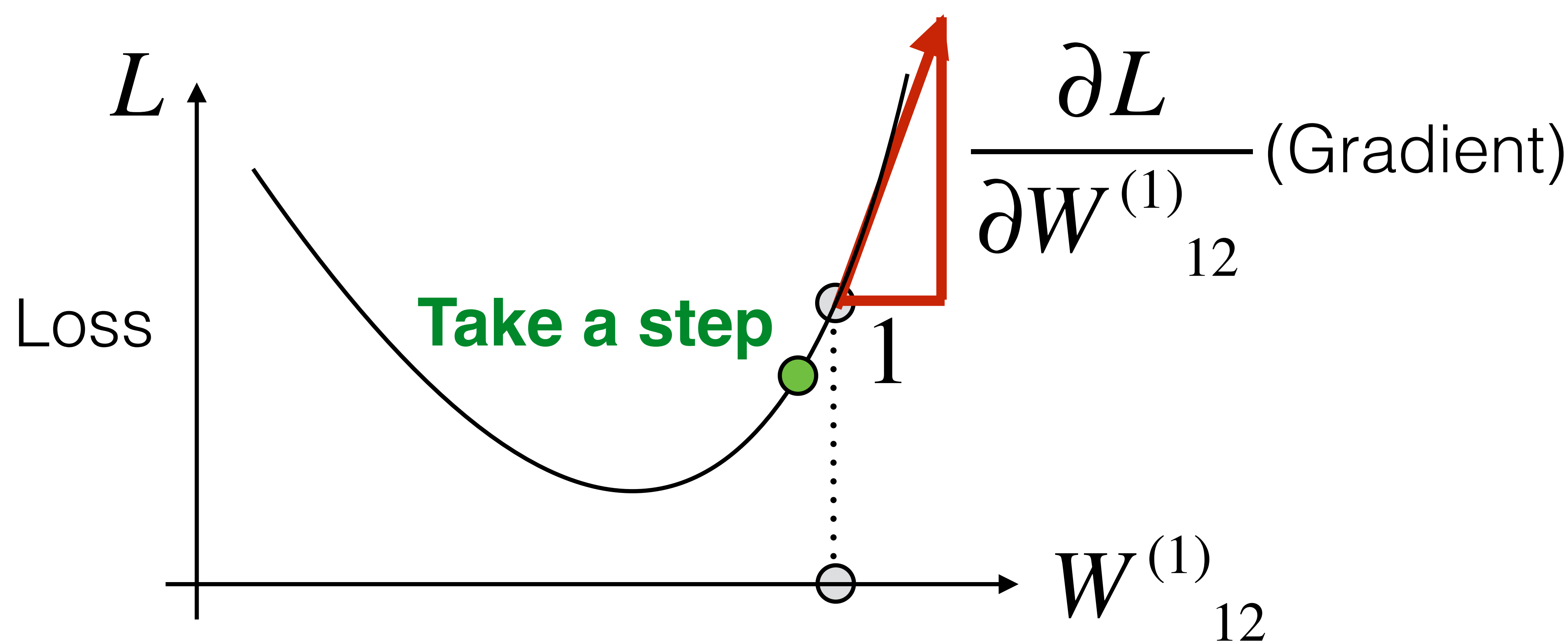

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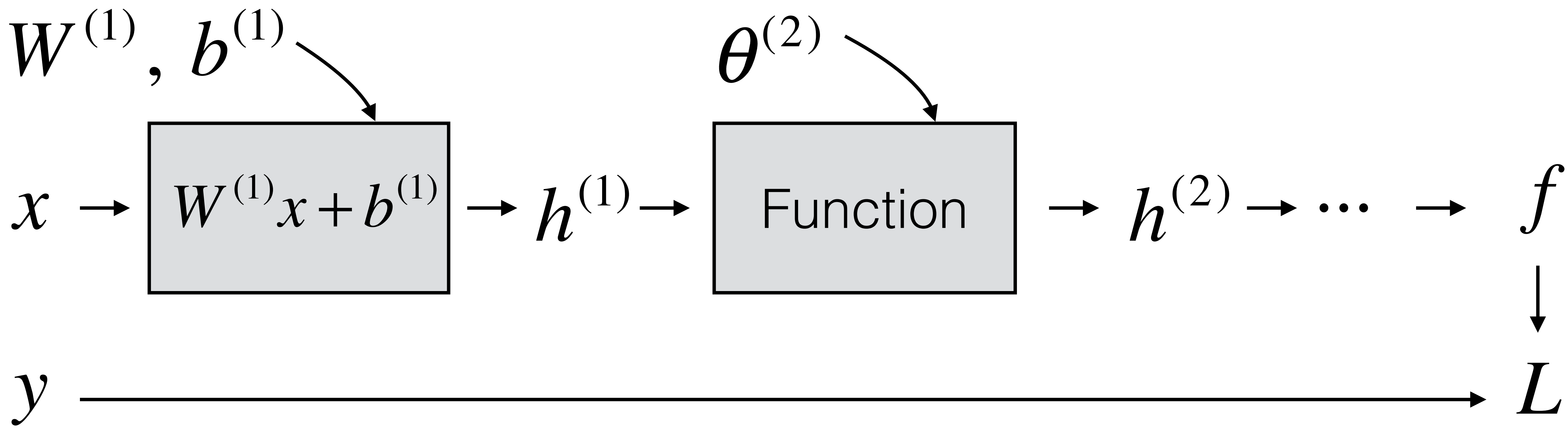

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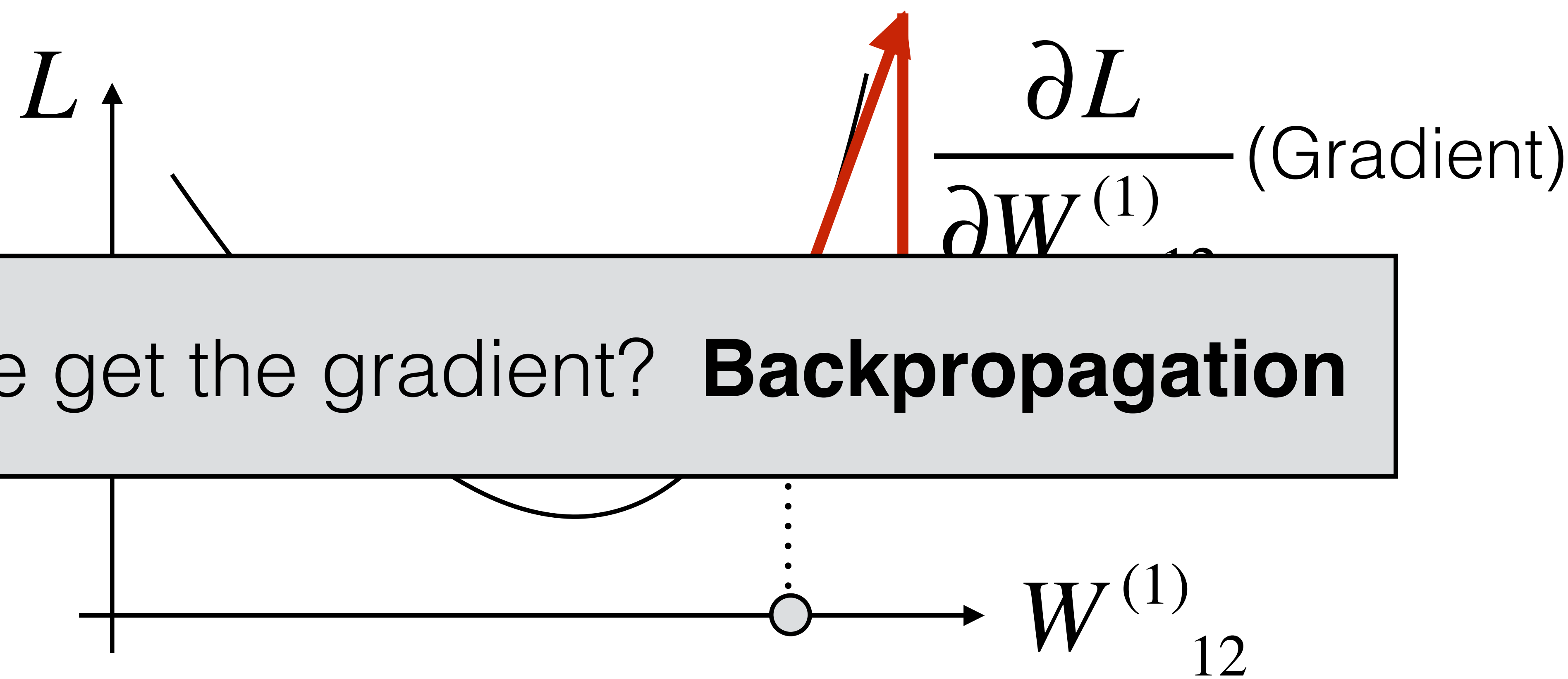


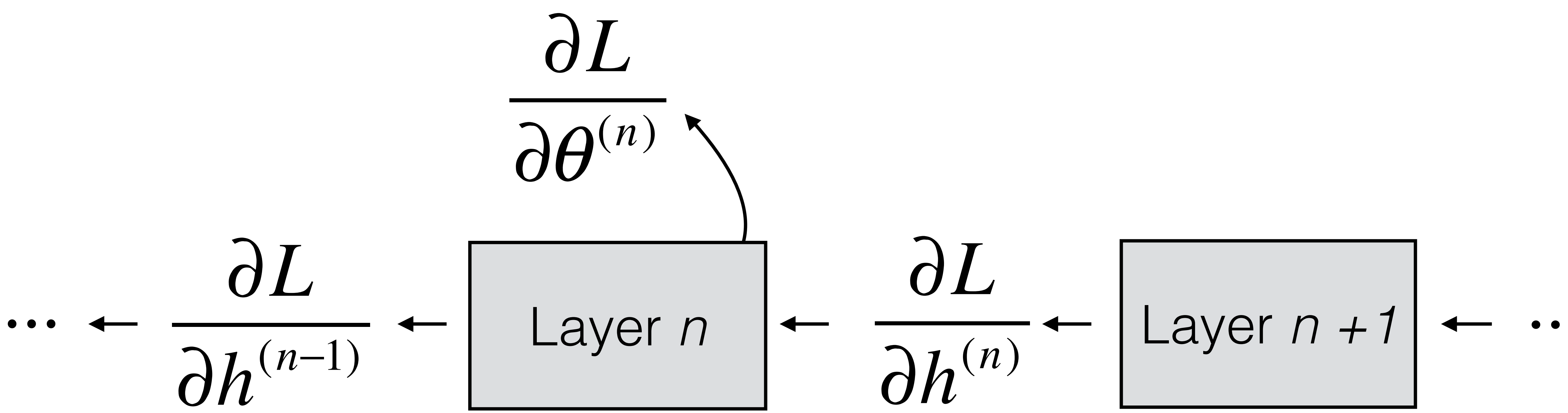
How do we get the gradient? **Backpropagation**

A weight somewhere in the network

Backprop

It’s just the chain rule

Backprop


$\frac{\partial L}{\partial \theta^{(n)}}$
$\cdots$
$\frac{\partial L}{\partial h^{(n-1)}}$
Layer n
$\frac{\partial L}{\partial h^{(n)}}$
Layer $n+1$
$\cdots$

Backprop

This is what we want for each layer

$$\boxed{\frac{\partial L}{\partial \theta^{(n)}}}$$

$$\cdots \leftarrow \frac{\partial L}{\partial h^{(n-1)}} \leftarrow \boxed{\text{Layer } n} \leftarrow \frac{\partial L}{\partial h^{(n)}} \leftarrow \boxed{\text{Layer } n+1} \leftarrow \cdots$$

Backprop

This is what we want for each layer

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To compute it, we need to propagate this gradient

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For each layer:

Backprop

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$$\cdots \leftarrow \frac{\partial L}{\partial h^{(n-1)}} \leftarrow \boxed{\text{Layer } n} \leftarrow \boxed{\frac{\partial L}{\partial h^{(n)}}} \leftarrow \boxed{\text{Layer } n+1} \leftarrow \cdots$$

For each layer:

$$\boxed{\frac{\partial L}{\partial \theta^{(n)}}} = \frac{\partial L}{\partial h^{(n)}} \cdot \frac{\partial h^{(n)}}{\partial \theta^{(n)}}$$

What we want

Backprop

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given to us

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What we want

This is just the local gradient of layer *n*

Backprop

This is what we want for each layer

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To compute it, we need to propagate this gradient

$$\cdots \leftarrow \frac{\partial L}{\partial h^{(n-1)}} \leftarrow \boxed{\text{Layer } n} \leftarrow \boxed{\frac{\partial L}{\partial h^{(n)}}} \leftarrow \boxed{\text{Layer } n+1} \leftarrow \cdots$$

given to us

For each layer:

$$\boxed{\frac{\partial L}{\partial \theta^{(n)}}} = \boxed{\frac{\partial L}{\partial h^{(n)}}} \cdot \boxed{\frac{\partial h^{(n)}}{\partial \theta^{(n)}}}$$

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given to us

For each layer:

$$\boxed{\frac{\partial L}{\partial \theta^{(n)}}} = \boxed{\frac{\partial L}{\partial h^{(n)}}} \cdot \boxed{\frac{\partial h^{(n)}}{\partial \theta^{(n)}}} \qquad \frac{\partial L}{\partial h^{(n-1)}} = \boxed{\frac{\partial L}{\partial h^{(n)}}} \cdot \frac{\partial h^{(n)}}{\partial h^{(n-1)}}$$

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What we want

This is just the local gradient of layer *n*

Backprop

For each layer, we compute:

$$\left[\text{Propagated gradient to the left}\right] = \left[\text{Propagated gradient from right}\right] \cdot \left[\text{Local gradient}\right]$$

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(Can compute immediately)

Backprop

For each layer, we compute:

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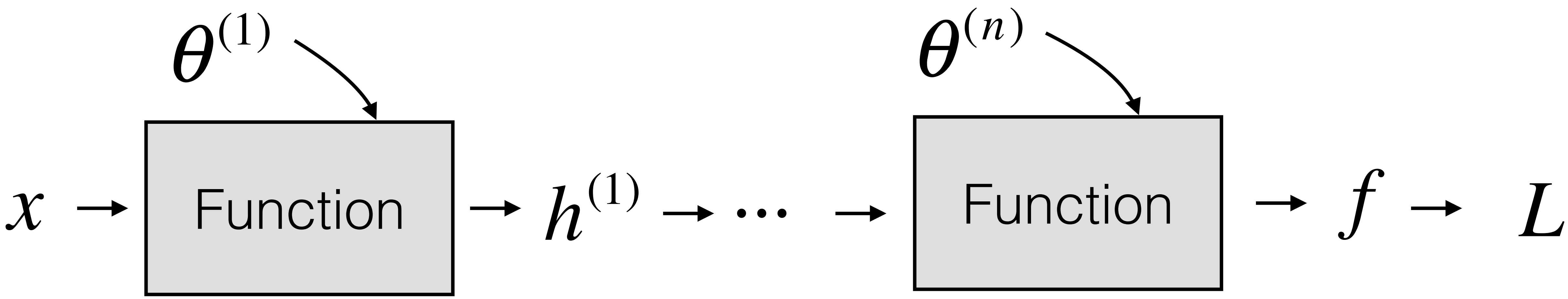
(Received during backprop)

(Can compute immediately)

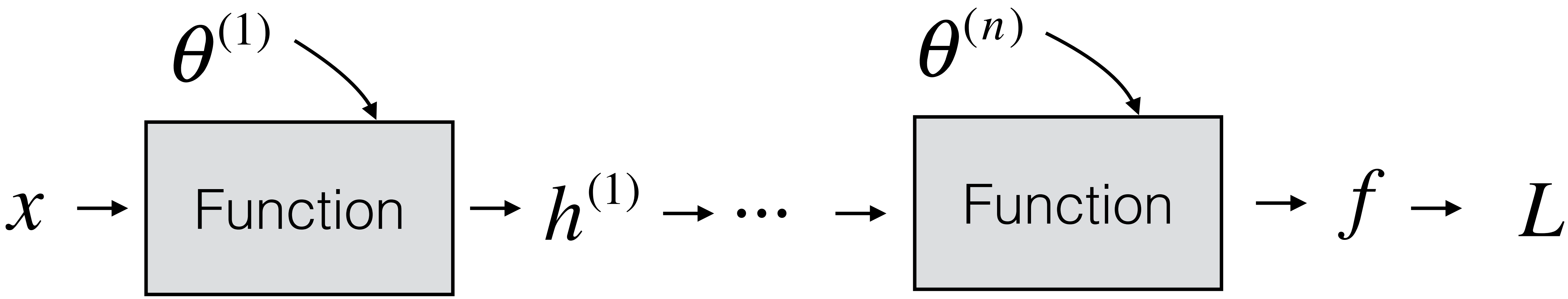
Backprop

Forward Propagation:



Backprop

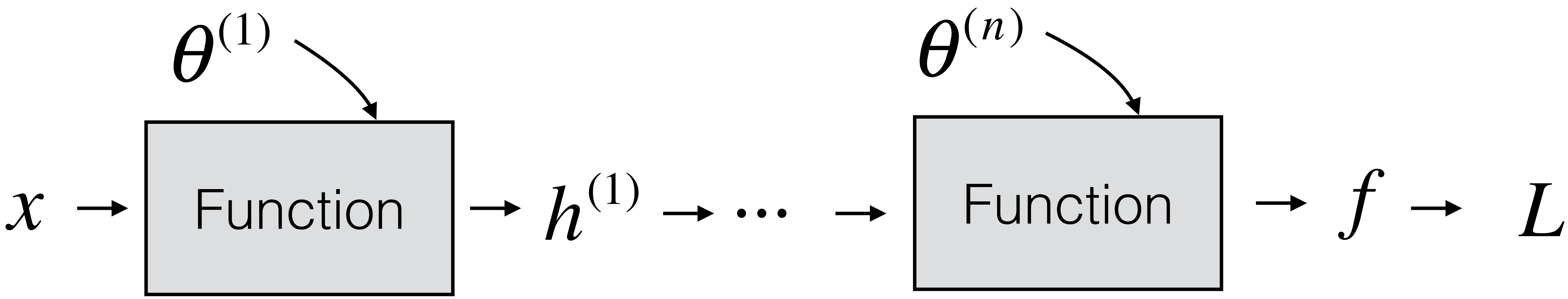
Forward Propagation:



Backward Propagation:

Backprop

Forward Propagation:

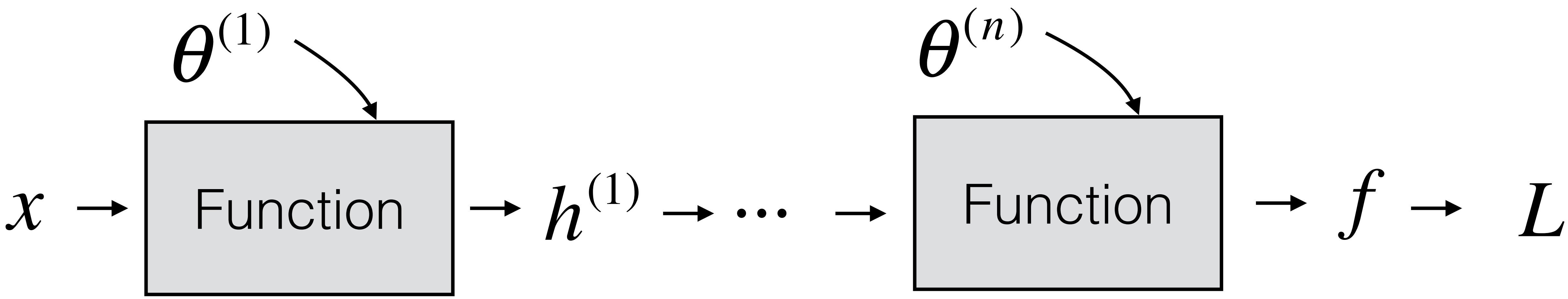


Backward Propagation:

L

Backprop

Forward Propagation:



Backward Propagation:

$$\frac{\partial L}{\partial f} \leftarrow L$$

Backprop

Forward Propagation:

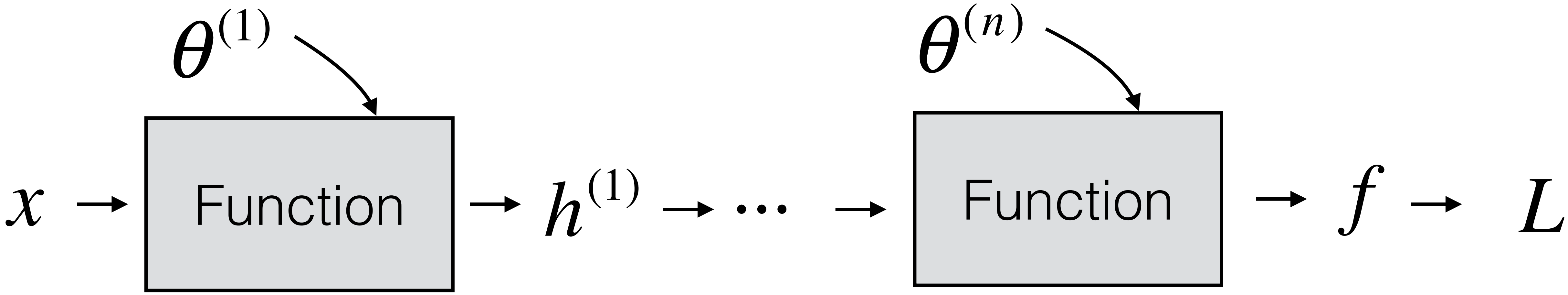
$$\begin{tikzcd} & \theta^{(1)} \arrow[d, bend left] & & & \theta^{(n)} \arrow[d, bend left] & & \\ x \arrow[r] & \boxed{\text{Function}} \arrow[r] & h^{(1)} \arrow[r] & \cdots \arrow[r] & \boxed{\text{Function}} \arrow[r] & f \arrow[r] & L \end{tikzcd}$$

Backward Propagation:

$$\begin{tikzcd} \frac{\partial L}{\partial \theta^{(n)}} & & \\ \boxed{\text{Function}} \arrow[u, bend right] & \frac{\partial L}{\partial f} \arrow[l] & L \arrow[l] \end{tikzcd}$$

Backprop

Forward Propagation:



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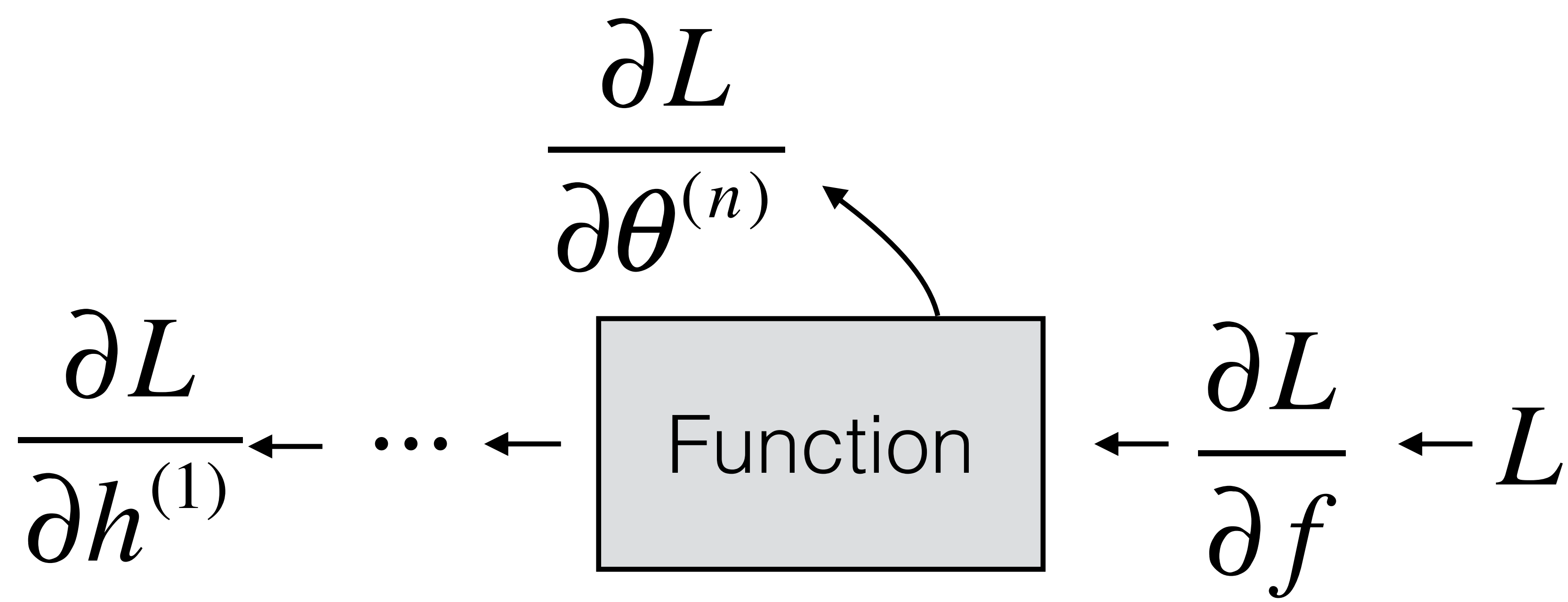

Backprop

Forward Propagation:

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Backward Propagation:

$$\begin{tikzcd} & \dfrac{\partial L}{\partial \theta^{(1)}} & & & \dfrac{\partial L}{\partial \theta^{(n)}} & & \\ \dfrac{\partial L}{\partial x} & \boxed{\text{Function}} \arrow[l] \arrow[u, bend right] & \dfrac{\partial L}{\partial h^{(1)}} \arrow[l] & \cdots \arrow[l] & \boxed{\text{Function}} \arrow[l] \arrow[u, bend right] & \dfrac{\partial L}{\partial f} \arrow[l] & L \arrow[l] \end{tikzcd}$$

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$$\frac{\partial L}{\partial x} = \frac{\partial L}{\partial h}\frac{\partial h}{\partial x}$$

x, h scalars
(L is always scalar)

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(L is always scalar)

$$\frac{\partial L}{\partial x_j} = \sum_i \frac{\partial L}{\partial h_i}\frac{\partial h_i}{\partial x_j}$$

x, h 1D arrays (vectors)

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x, h 1D arrays (vectors)

$$\frac{\partial L}{\partial x_{ab}} = \sum_i \sum_j \frac{\partial L}{\partial h_{ij}}\frac{\partial h_{ij}}{\partial x_{ab}}$$

x, h 2D arrays

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just add more subscripts and more summations

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x, h 1D arrays (vectors)

$$\frac{\partial L}{\partial x_{ab}} = \sum_i \sum_j \frac{\partial L}{\partial h_{ij}}\frac{\partial h_{ij}}{\partial x_{ab}}$$

x, h 2D arrays

$$\frac{\partial L}{\partial x_{abc}} = \sum_i \sum_j \sum_k \frac{\partial L}{\partial h_{ijk}}\frac{\partial h_{ijk}}{\partial x_{abc}}$$

x, h 3D arrays

Example: Mean Subtraction

(for a single input)

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(for a single input)

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(here, "i" and "k" are channels)

Example: Mean Subtraction
(for a single input)

- Ok, so how do we actually derive the backwards pass? Let's walk through an example together.

- Example layer: mean subtraction:

$$h_i = x_i - \frac{1}{D}\sum_k x_k$$

(here, "i" and "k" are channels)

- For backprop, we just need the local derivative

Example: Mean Subtraction

(for a single input)

Example: Mean Subtraction
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Example: Mean Subtraction
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- Taking the derivative of the layer:

Example: Mean Subtraction
(for a single input)

- Forward: $h_i = x_i - \frac{1}{D}\sum_k x_k$
- Taking the derivative of the layer: $\frac{\partial h_i}{\partial x_j} = \delta_{ij} - \frac{1}{D}$

Example: Mean Subtraction
(for a single input)

- Forward: $h_i = x_i - \frac{1}{D}\sum_k x_k$
- Taking the derivative of the layer: $\frac{\partial h_i}{\partial x_j} = \delta_{ij} - \frac{1}{D}$

$$\left(\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases} \right)$$

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$$\frac{\partial L}{\partial x_j} = \sum_i \frac{\partial L}{\partial h_i}\frac{\partial h_i}{\partial x_j}$$

(backprop aka chain rule)

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Example: Mean Subtraction

(for a single input)

- Forward: $h_i = x_i - \frac{1}{D}\sum_k x_k$
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$$\frac{\partial L}{\partial x_j} = \sum_i \frac{\partial L}{\partial h_i}\frac{\partial h_i}{\partial x_j}$$ (backprop aka chain rule)

$$= \sum_i \frac{\partial L}{\partial h_i}\left(\delta_{ij} - \frac{1}{D}\right)$$

$$\left(\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}\right)$$

Example: Mean Subtraction

(for a single input)

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(backprop aka chain rule)

$$= \sum_i \frac{\partial L}{\partial h_i}\left(\delta_{ij} - \frac{1}{D} \right)$$

$$= \sum_i \frac{\partial L}{\partial h_i}\delta_{ij} - \frac{1}{D}\sum_i \frac{\partial L}{\partial h_i}$$

Example: Mean Subtraction
(for a single input)

- Forward: $h_i = x_i - \frac{1}{D}\sum_k x_k$
- Taking the derivative of the layer: $\frac{\partial h_i}{\partial x_j} = \delta_{ij} - \frac{1}{D}$

$$\left(\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases} \right)$$

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$$= \sum_i \frac{\partial L}{\partial h_i}\left(\delta_{ij} - \frac{1}{D} \right)$$

$$= \sum_i \frac{\partial L}{\partial h_i}\delta_{ij} - \frac{1}{D}\sum_i \frac{\partial L}{\partial h_i}$$

$$= \frac{\partial L}{\partial h_j} - \frac{1}{D}\sum_i \frac{\partial L}{\partial h_i}$$

Example: Mean Subtraction
(for a single input)

- Forward: $$h_i = x_i - \frac{1}{D}\sum_k x_k$$
- Taking the derivative of the layer: $$\frac{\partial h_i}{\partial x_j} = \delta_{ij} - \frac{1}{D}$$

$$\left(\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases} \right)$$

$$\frac{\partial L}{\partial x_j} = \sum_i \frac{\partial L}{\partial h_i}\frac{\partial h_i}{\partial x_j}$$ (backprop aka chain rule)

$$= \sum_i \frac{\partial L}{\partial h_i}\left(\delta_{ij} - \frac{1}{D}\right)$$

$$= \sum_i \frac{\partial L}{\partial h_i}\delta_{ij} - \frac{1}{D}\sum_i \frac{\partial L}{\partial h_i}$$

$$= \frac{\partial L}{\partial h_j} - \frac{1}{D}\sum_i \frac{\partial L}{\partial h_i}$$

Done!

Example: Mean Subtraction

(for a single input)

$$h_i = x_i - \frac{1}{D}\sum_k x_k$$

$$\frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial h_i} - \frac{1}{D}\sum_k \frac{\partial L}{\partial h_k}$$

Example: Mean Subtraction

(for a single input)

- Forward: $$h_i = x_i - \frac{1}{D}\sum_k x_k$$
- Backward: $$\frac{\partial L}{\partial x_i} = \frac{\partial L}{\partial h_i} - \frac{1}{D}\sum_k \frac{\partial L}{\partial h_k}$$

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This also works:

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Example: Mean Subtraction
(for a single input)

The backward pass is easy:

```
def backward(dh):
    return forward(dh)
```

(Remember they're usually not the same)

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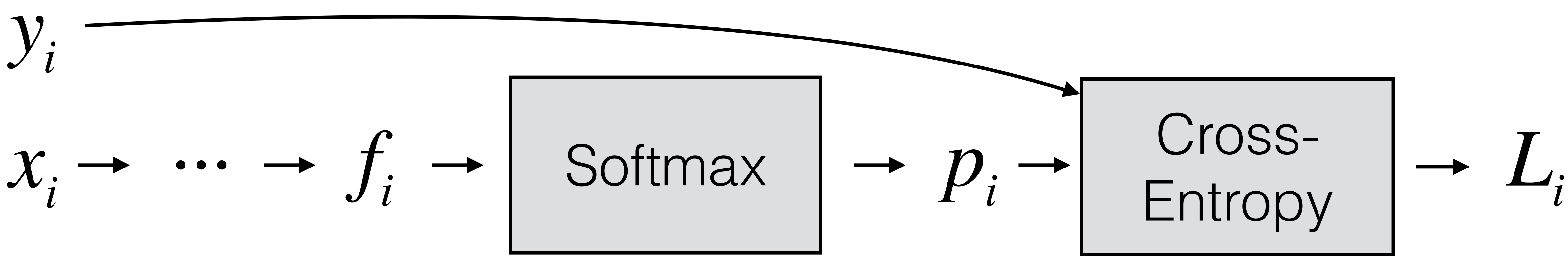

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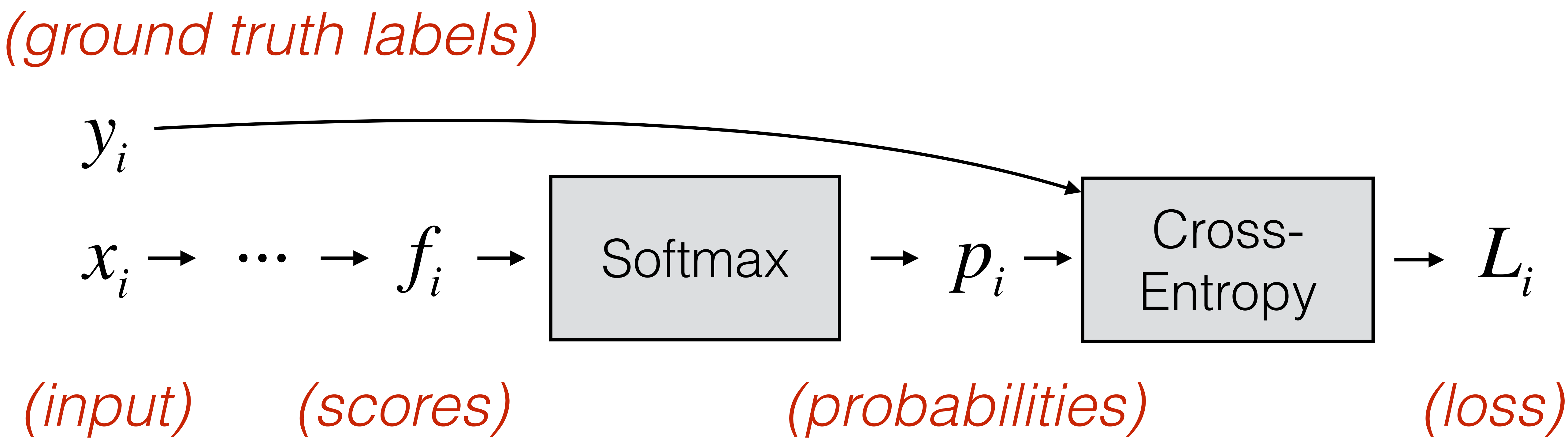

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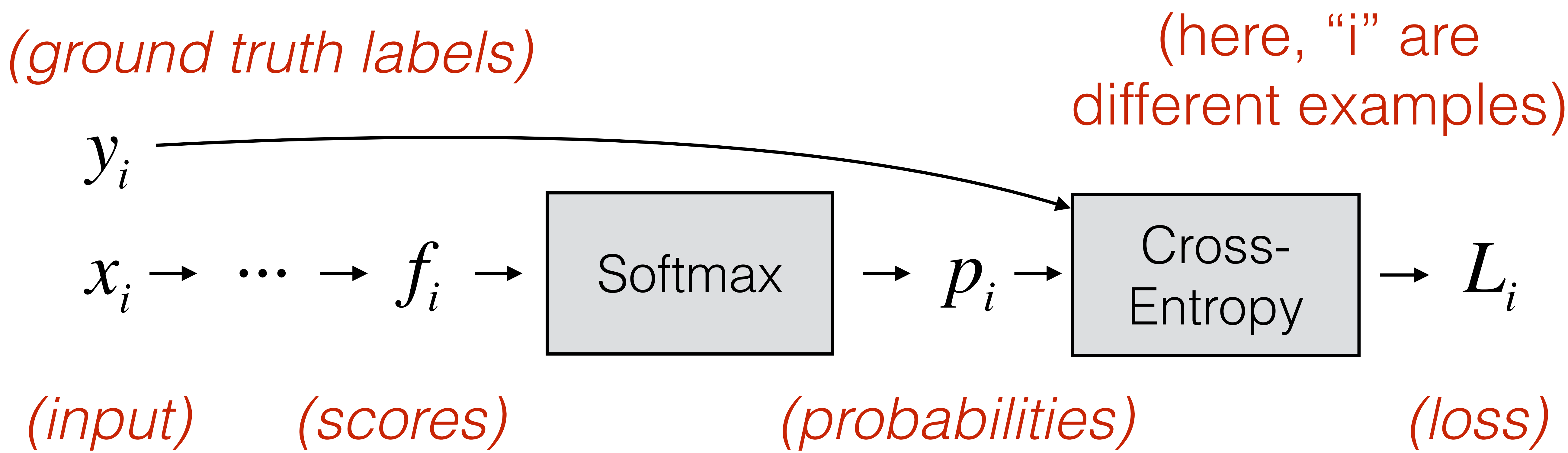

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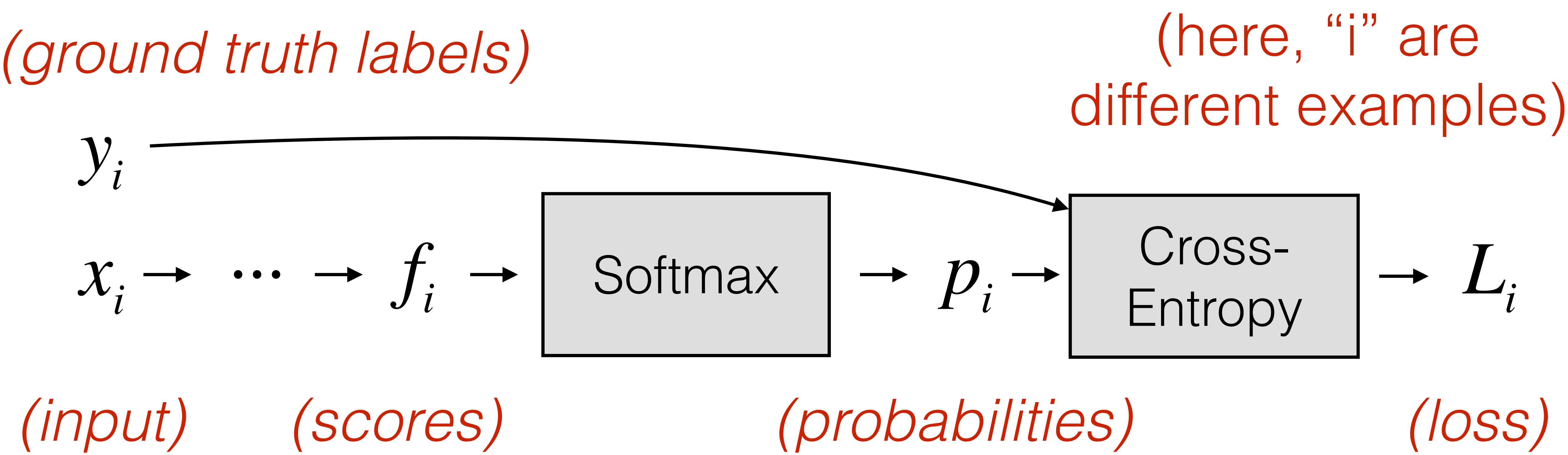

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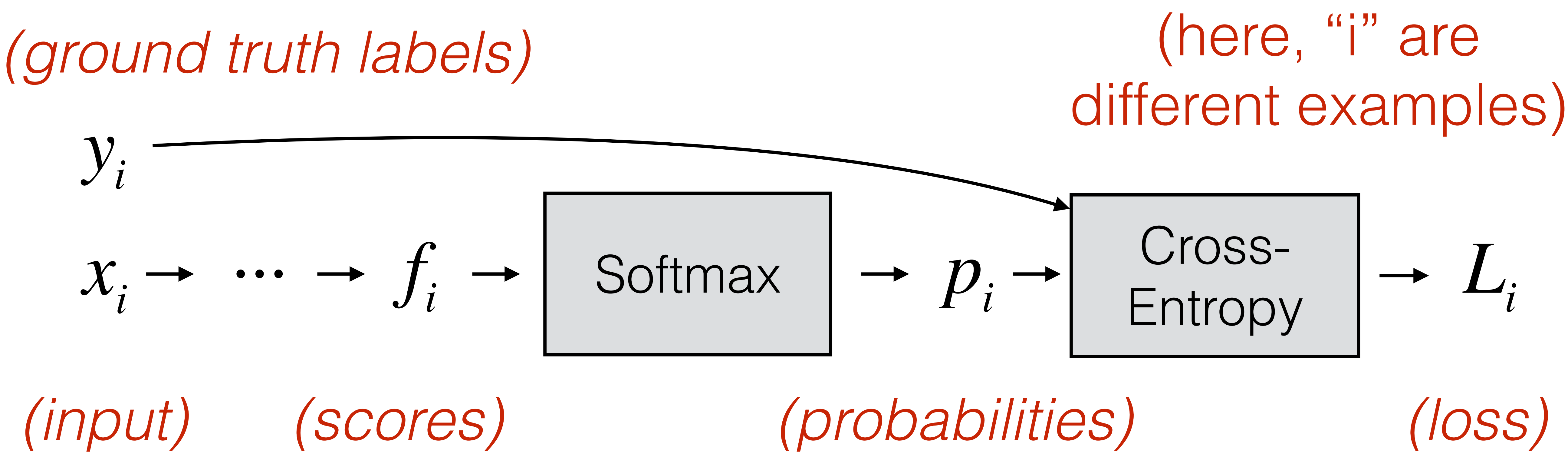

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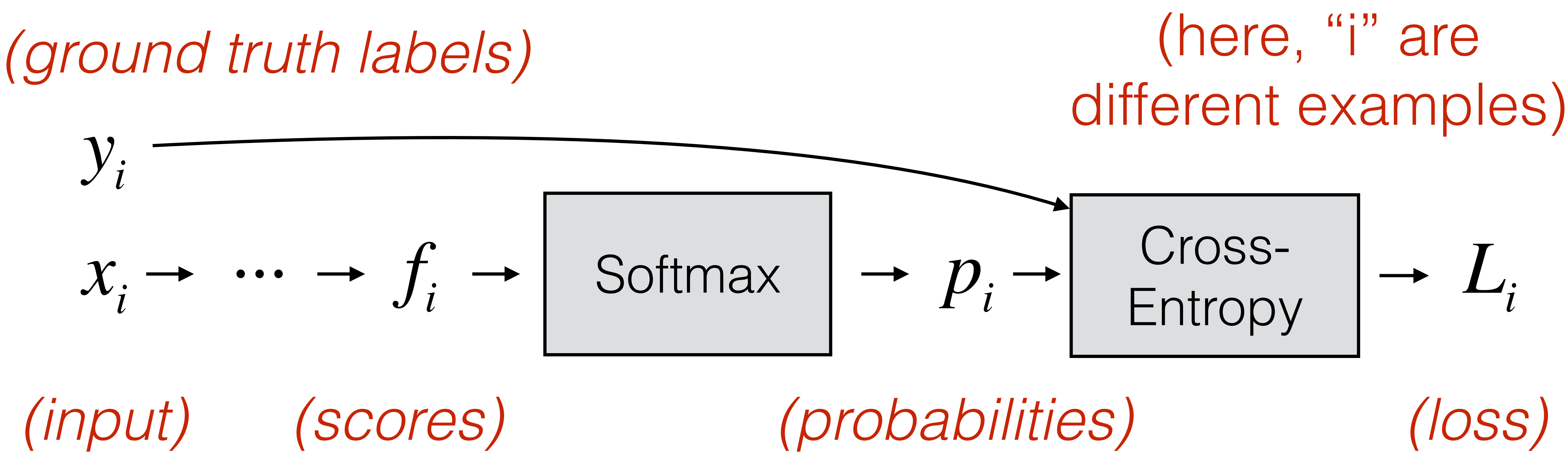

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(Avg. over examples)

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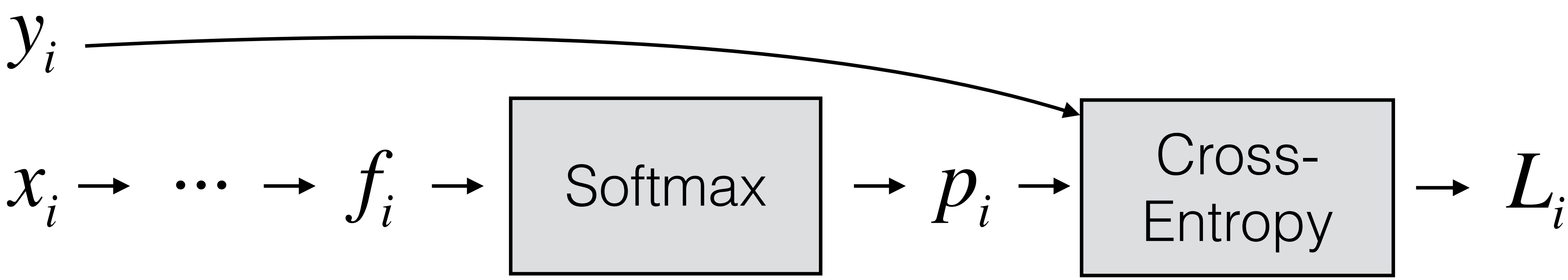

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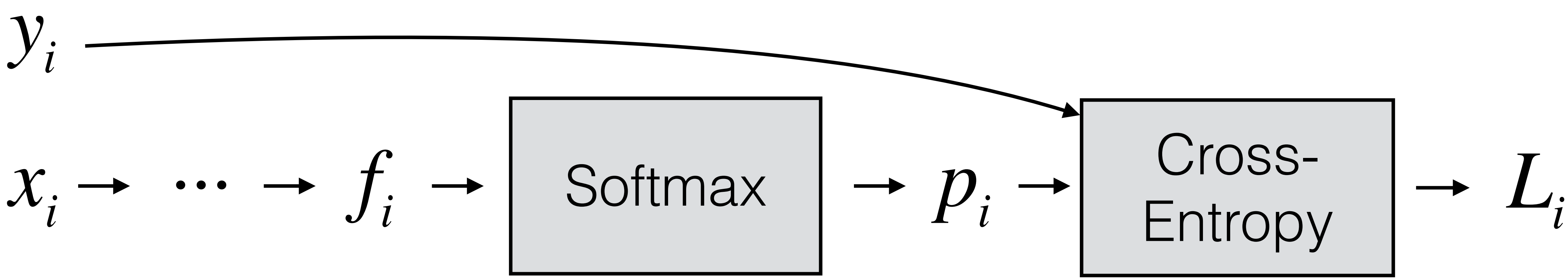

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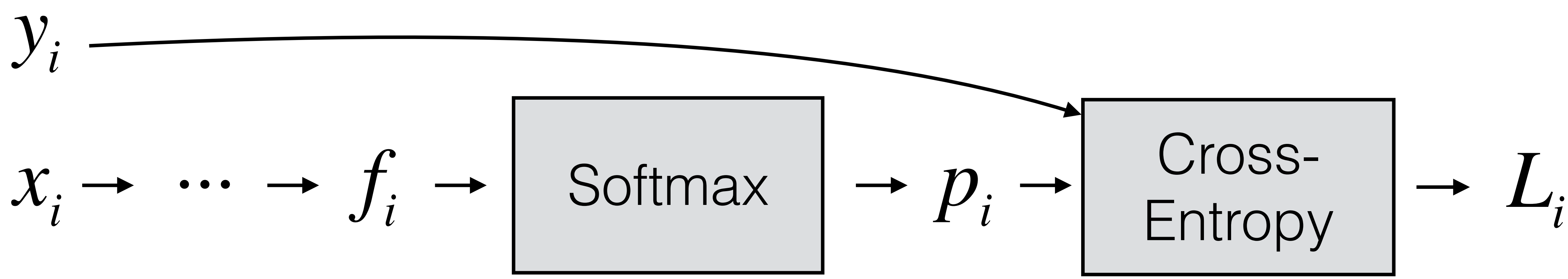

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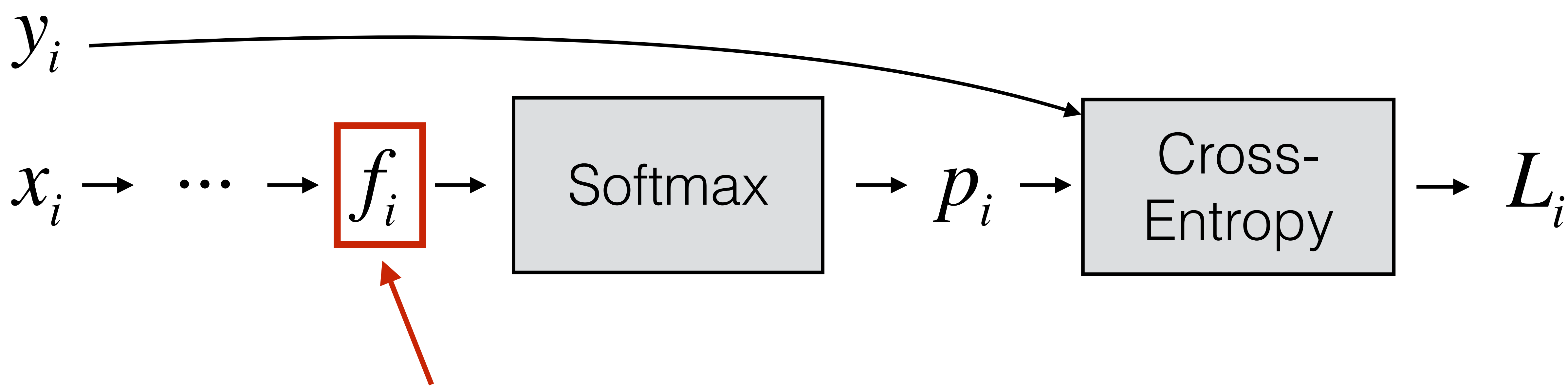

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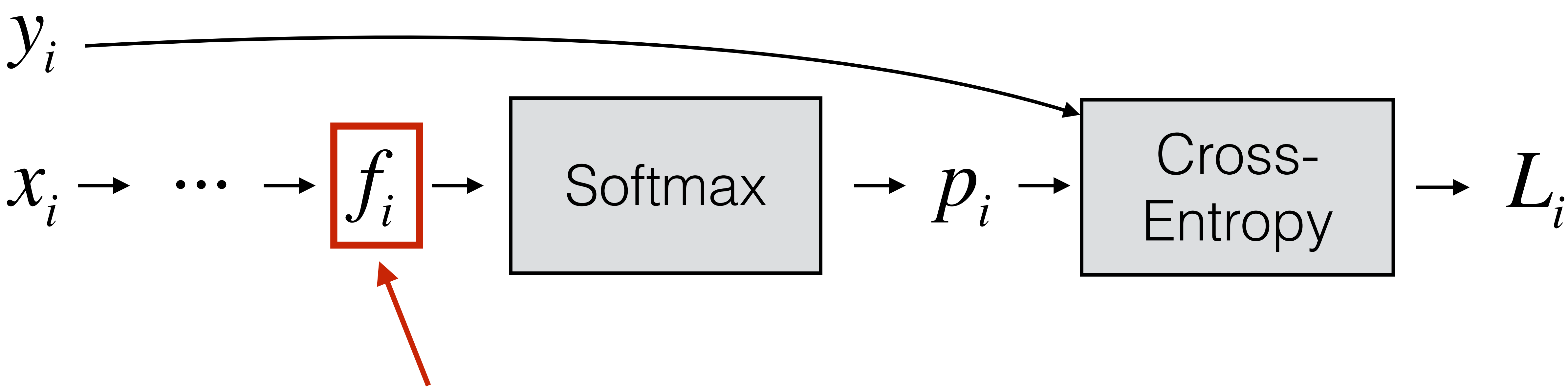

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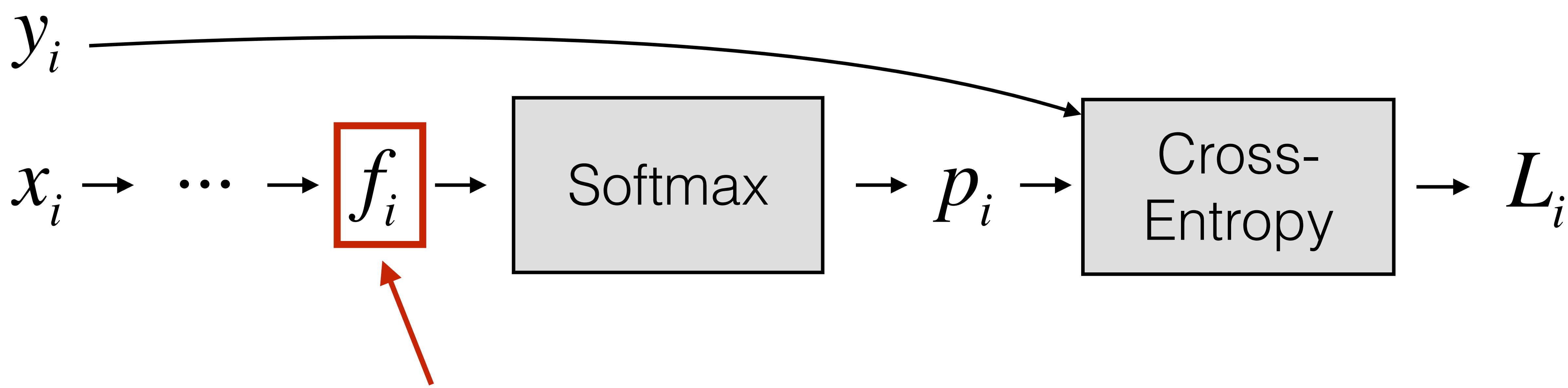

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Now we can continue backpropagating to the layer before "f"

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This expression is numerically unstable

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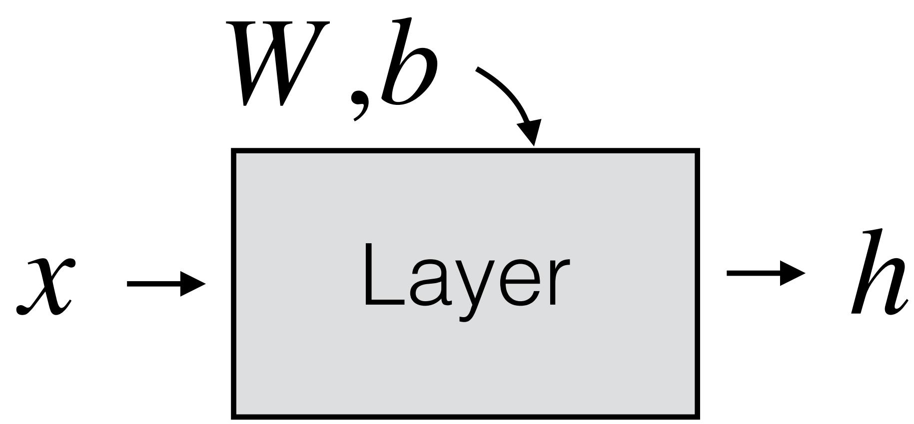

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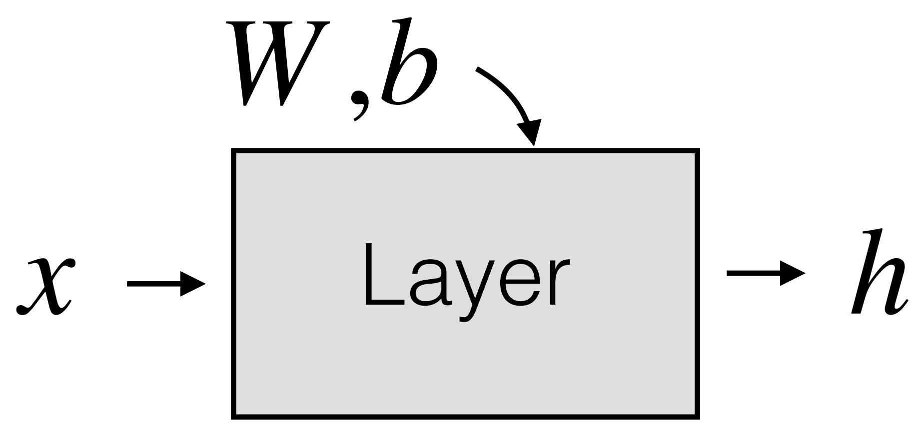

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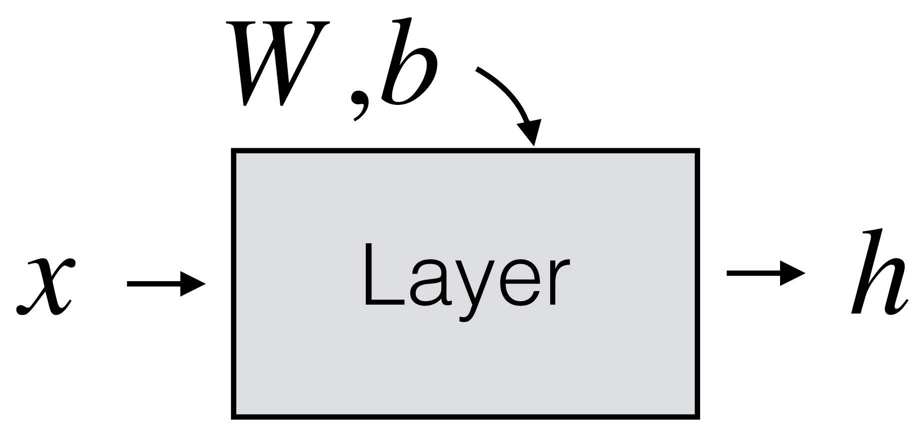

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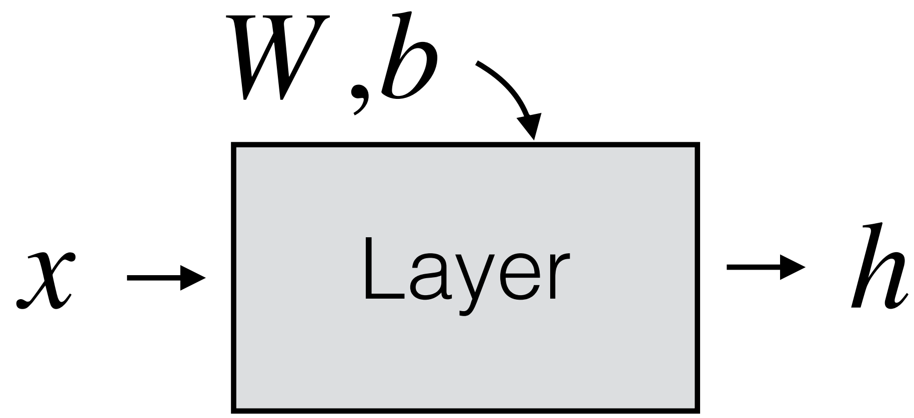

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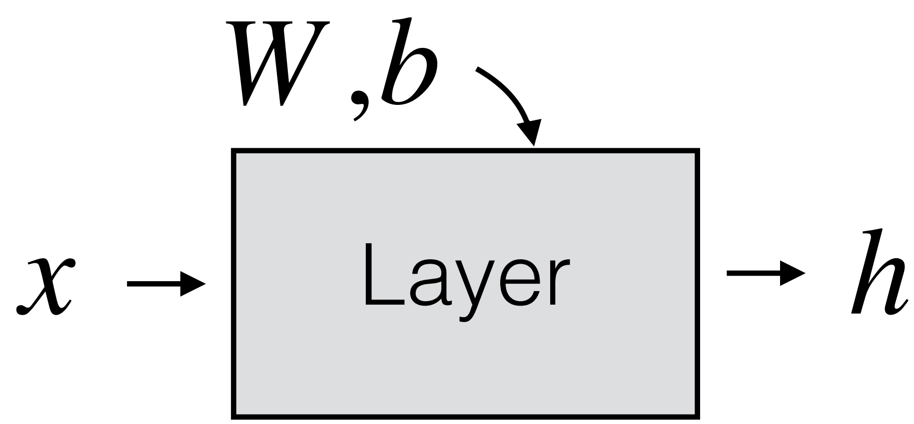

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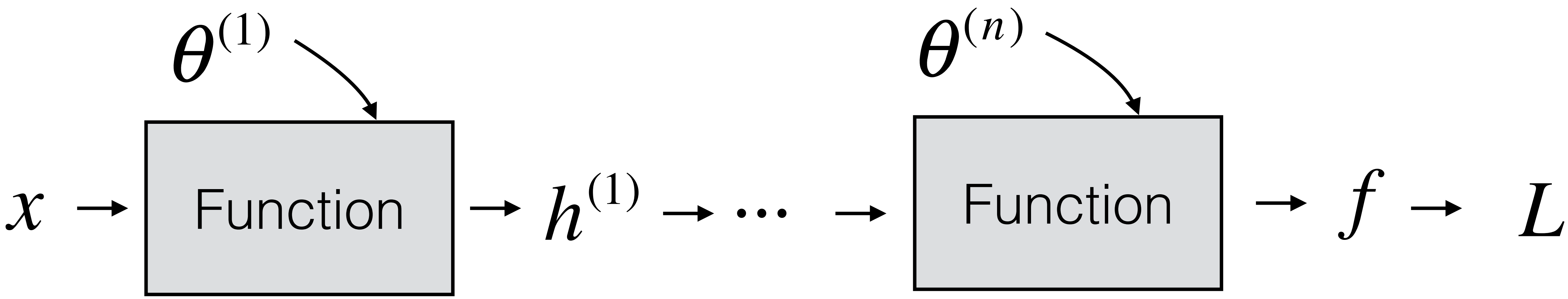
HW2: you will derive this for various layers.

Recap

Forward Propagation:



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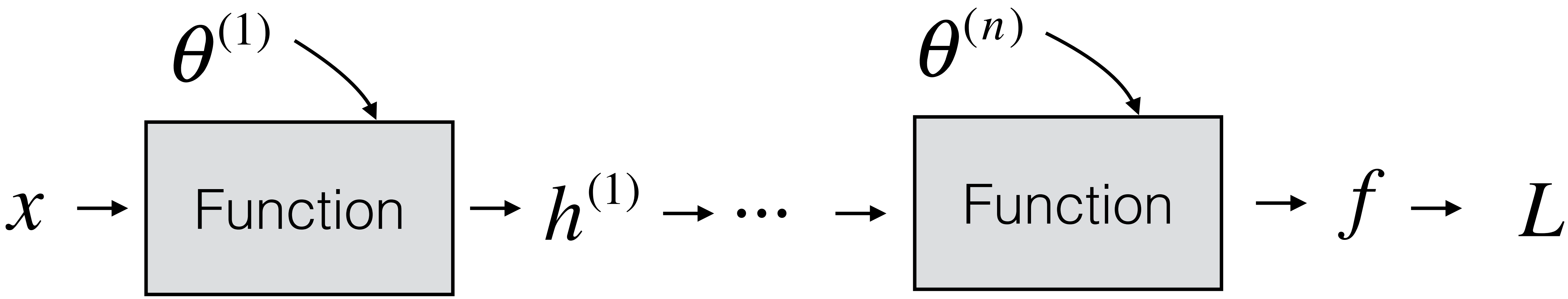

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Recap

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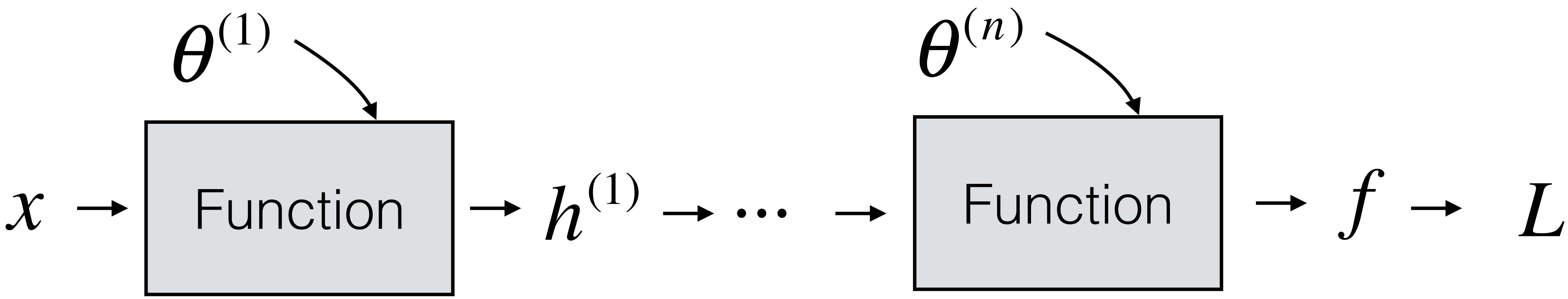

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L

Recap

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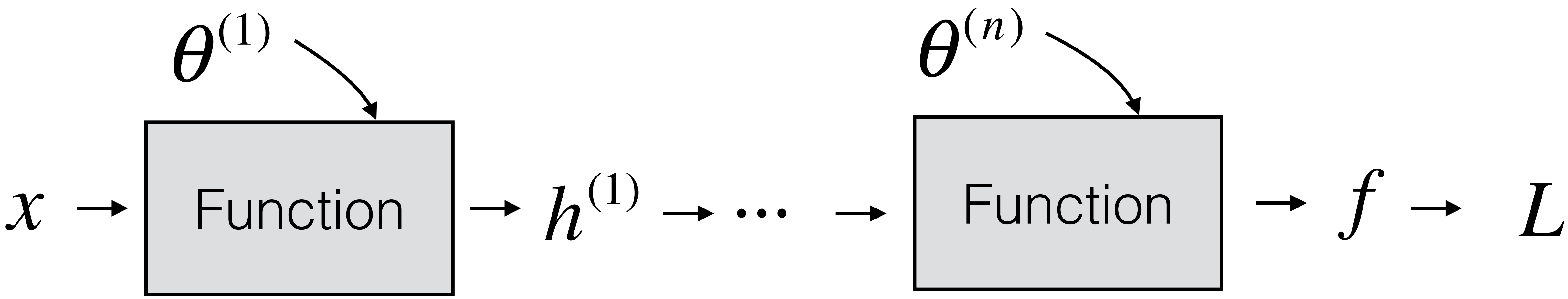

Backward Propagation:

$$\frac{\partial L}{\partial f} \leftarrow L$$

Recap

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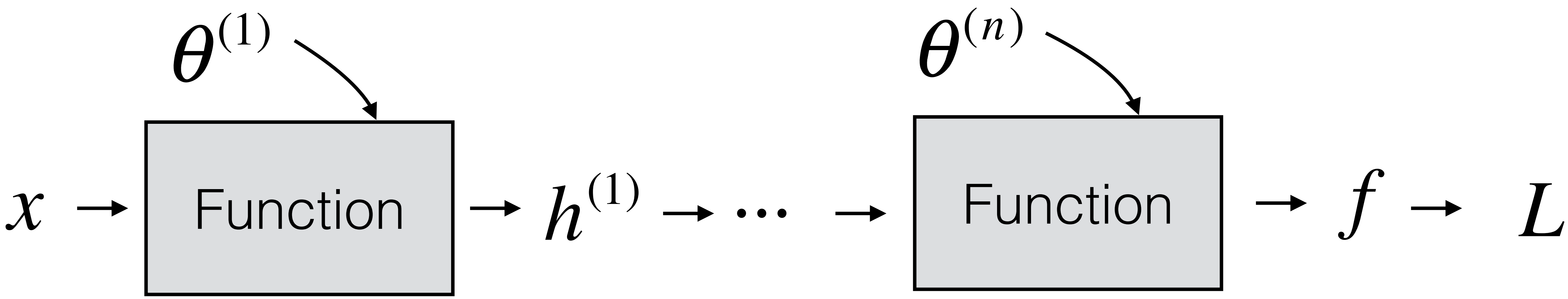

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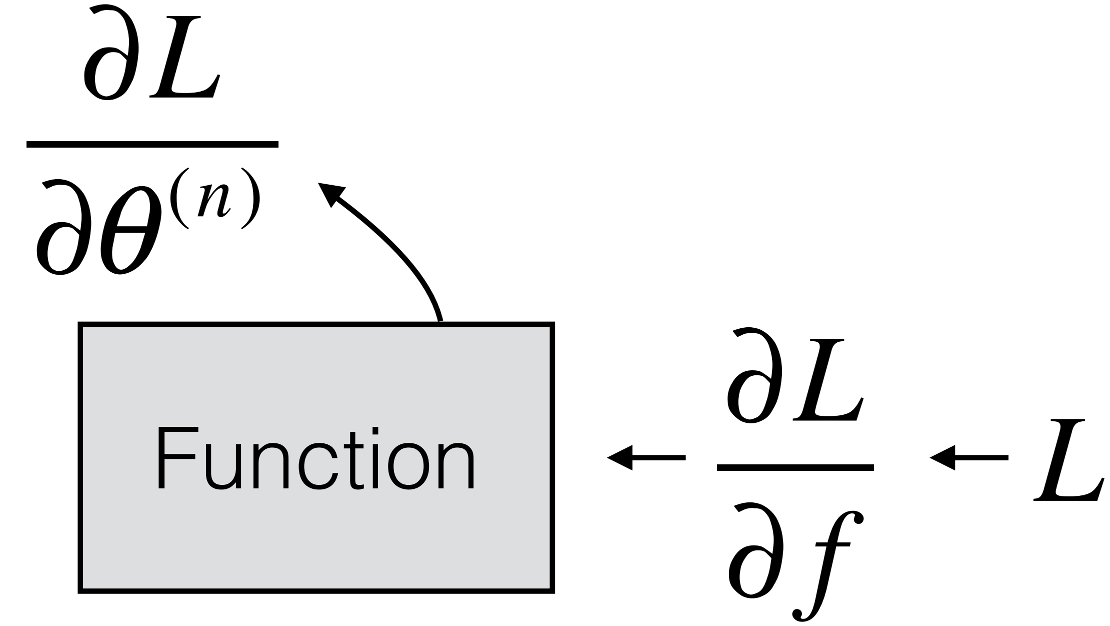

Recap

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$$\begin{tikzcd} & \theta^{(1)} \arrow[d, bend left] & & & \theta^{(n)} \arrow[d, bend left] & & \\ x \arrow[r] & \boxed{\text{Function}} \arrow[r] & h^{(1)} \arrow[r] & \cdots \arrow[r] & \boxed{\text{Function}} \arrow[r] & f \arrow[r] & L \end{tikzcd}$$

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Questions?

30s cat picture break

http://stylonica.com/cat-pictures/

CNNs

It’s just neural networks
with 3D activations

What shape should the activations have?

$$x \rightarrow \boxed{\text{Layer}} \rightarrow h^{(1)} \rightarrow \boxed{\text{Layer}} \rightarrow h^{(2)} \rightarrow \cdots \rightarrow f$$

- The input is an image, which is 3D (RGB channel, height, width)

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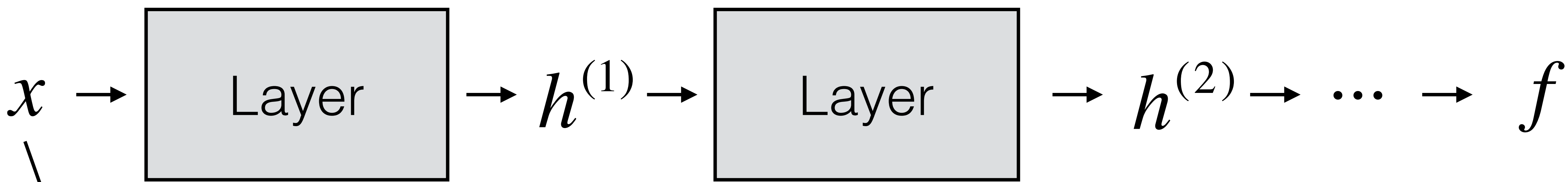

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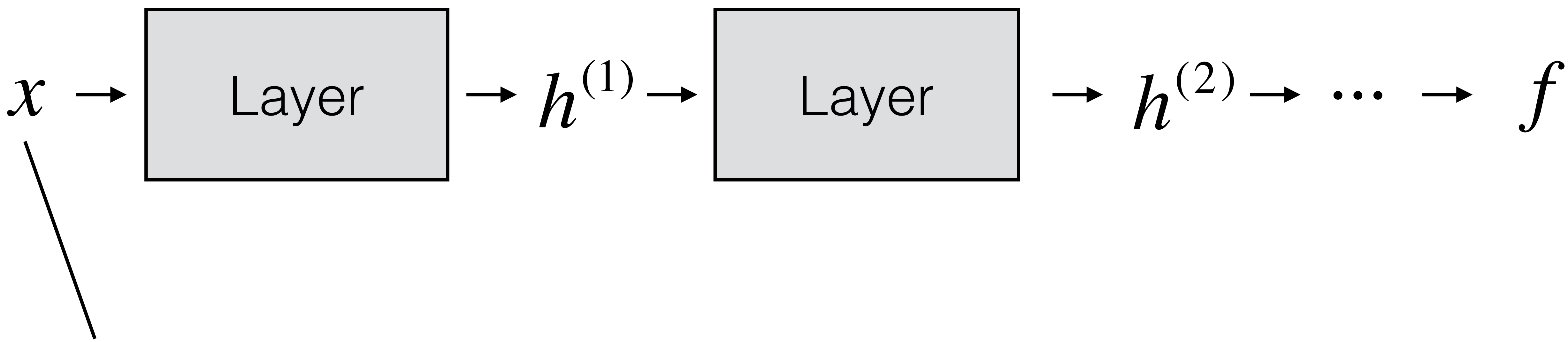

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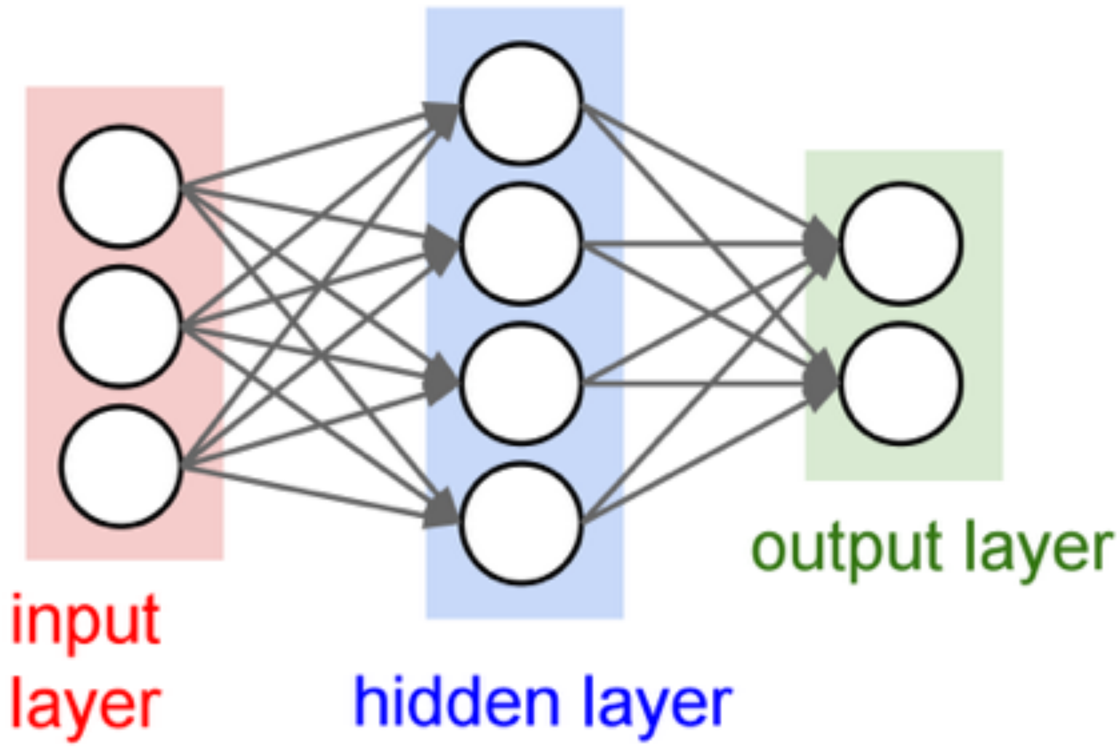
- What about keeping everything in 3D?

3D Activations

before:



(1D vectors)

Figure: Andrej Karpathy

3D Activations

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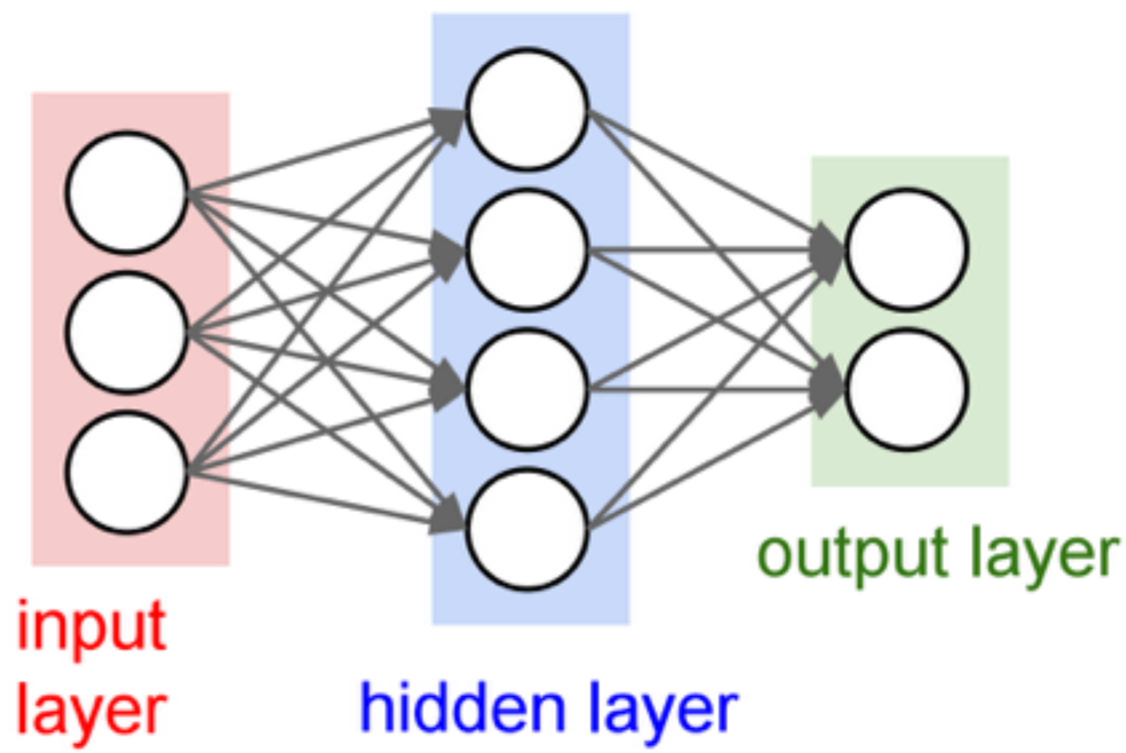
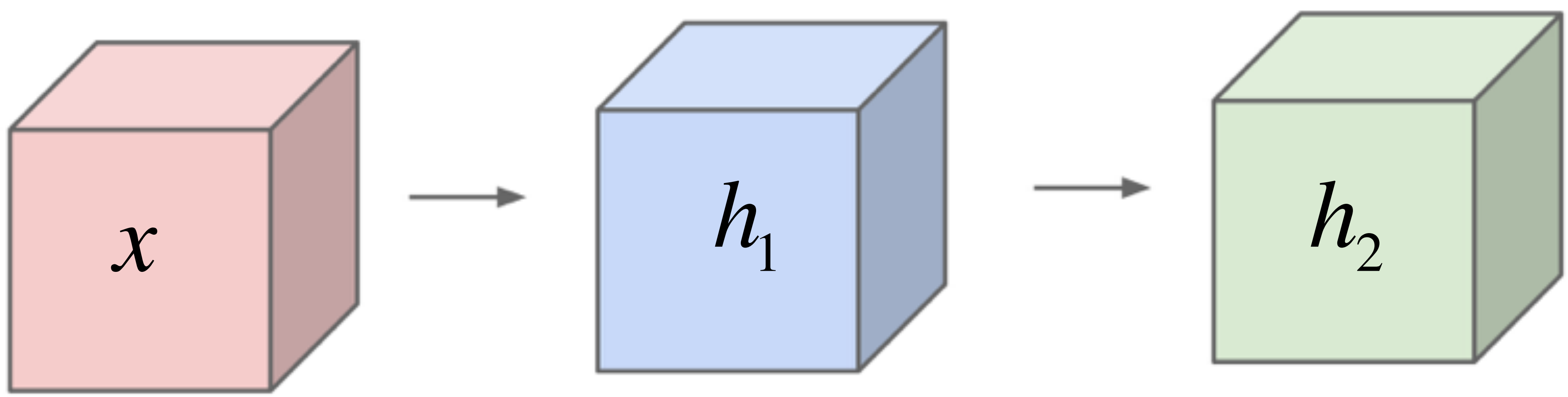


(1D vectors)

now:



(3D arrays)

Figure: Andrej Karpathy

3D Activations

All Neural Net activations arranged in **3 dimensions:**

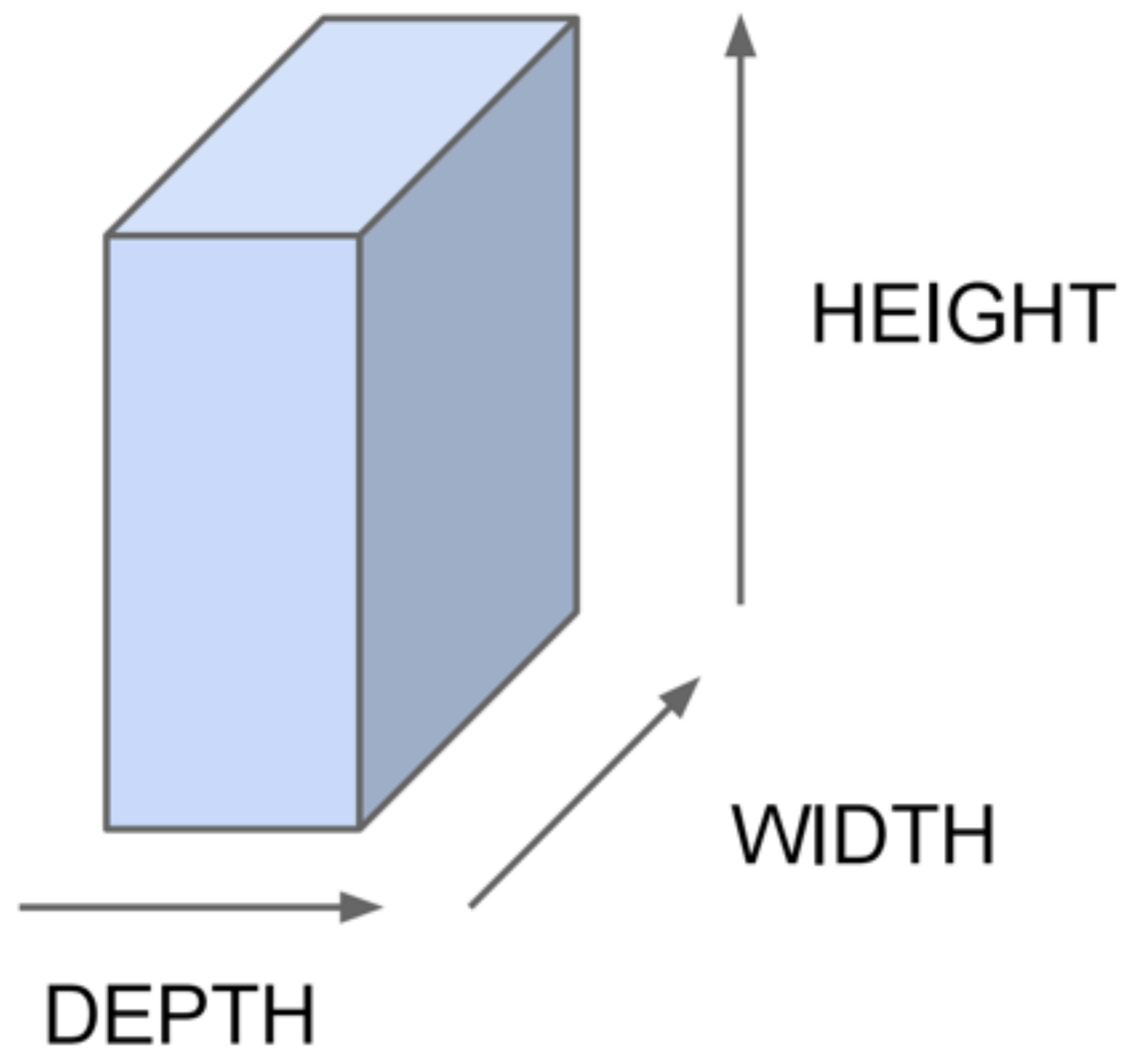
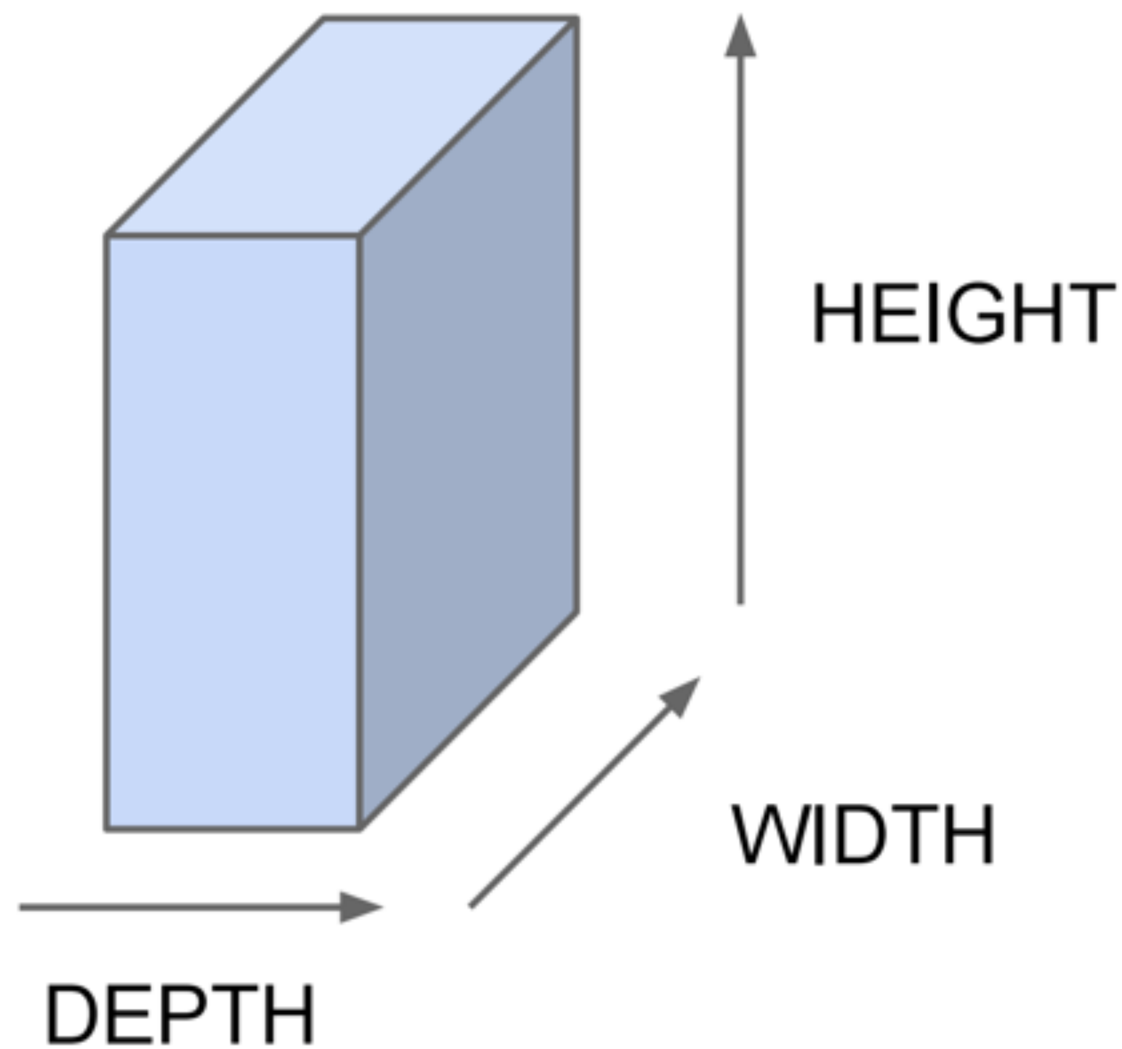


Figure: Andrej Karpathy

3D Activations

All Neural Net activations arranged in **3 dimensions:**



For example, a CIFAR-10 image is a 3x32x32 volume
(3 depth — RGB channels, 32 height, 32 width)

Figure: Andrej Karpathy

3D Activations

1D Activations:

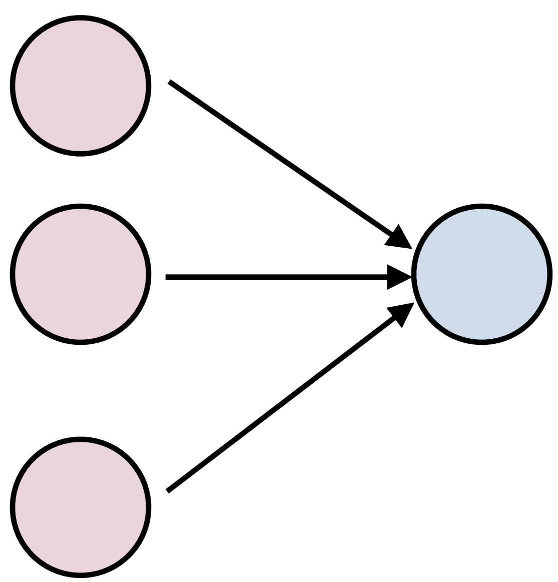
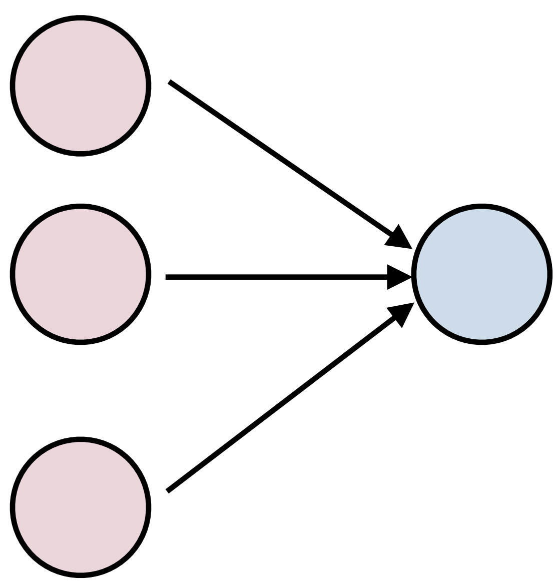


Figure: Andrej Karpathy

3D Activations

1D Activations:



3D Activations:

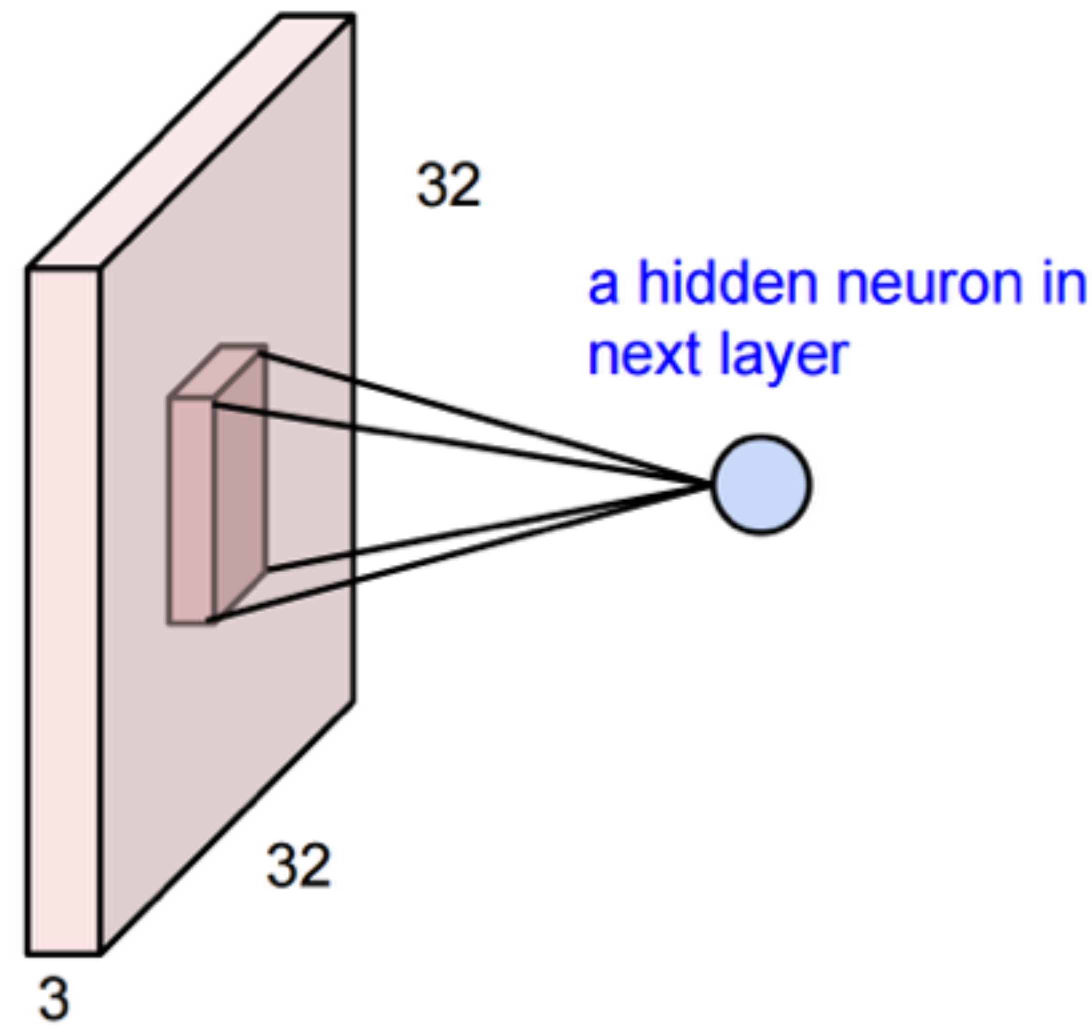
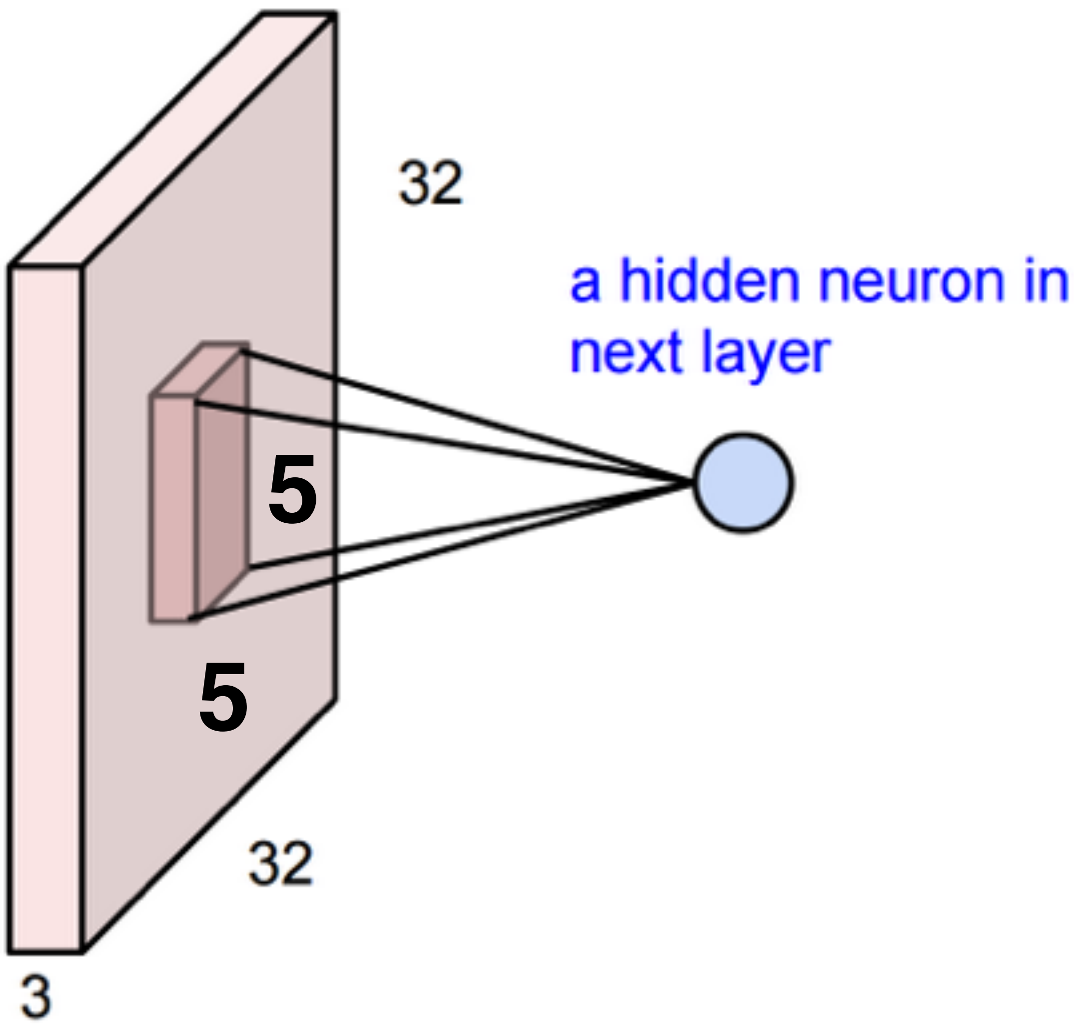


Figure: Andrej Karpathy

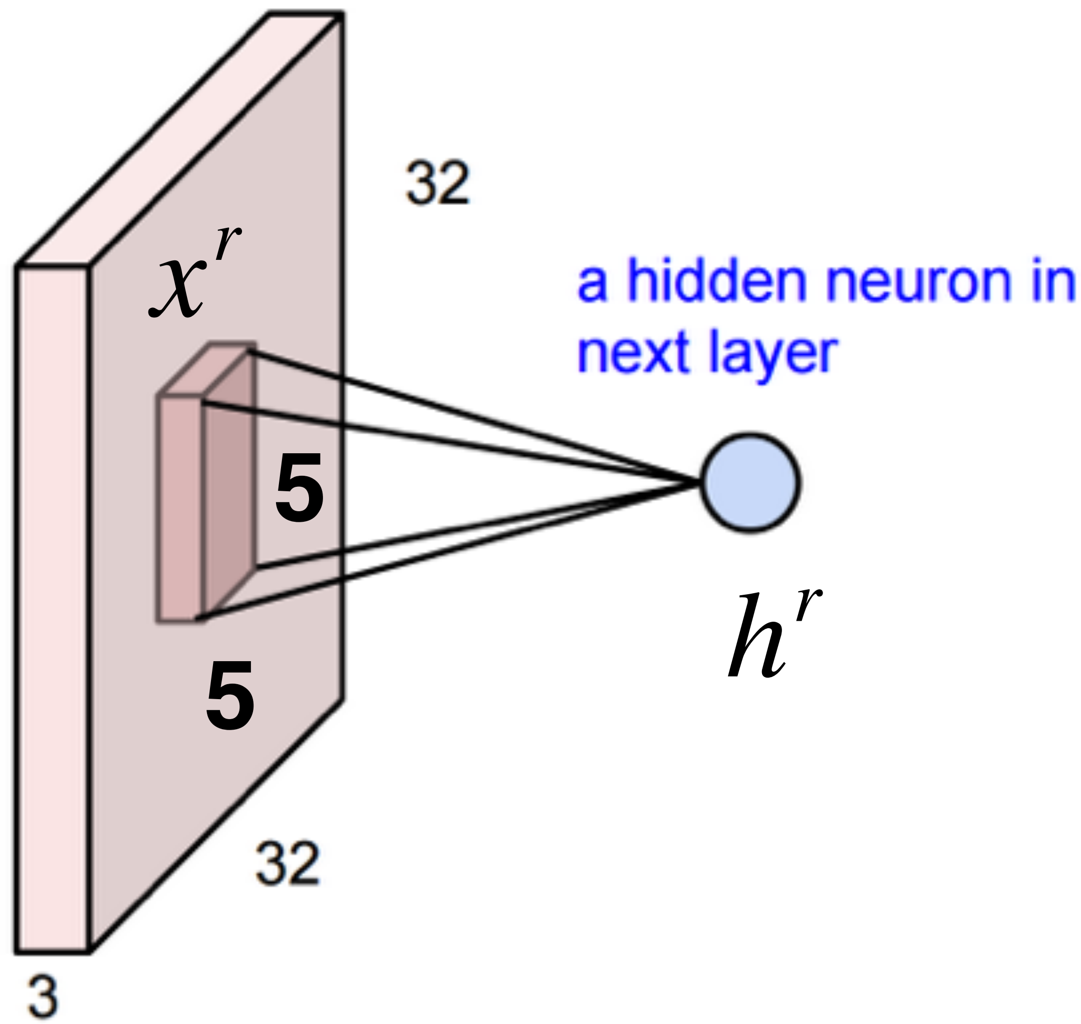
3D Activations



- The input is 3x32x32
- This neuron depends on a 3x5x5 chunk of the input
- The neuron also has a 3x5x5 set of weights and a bias (scalar)

Figure: Andrej Karpathy

3D Activations

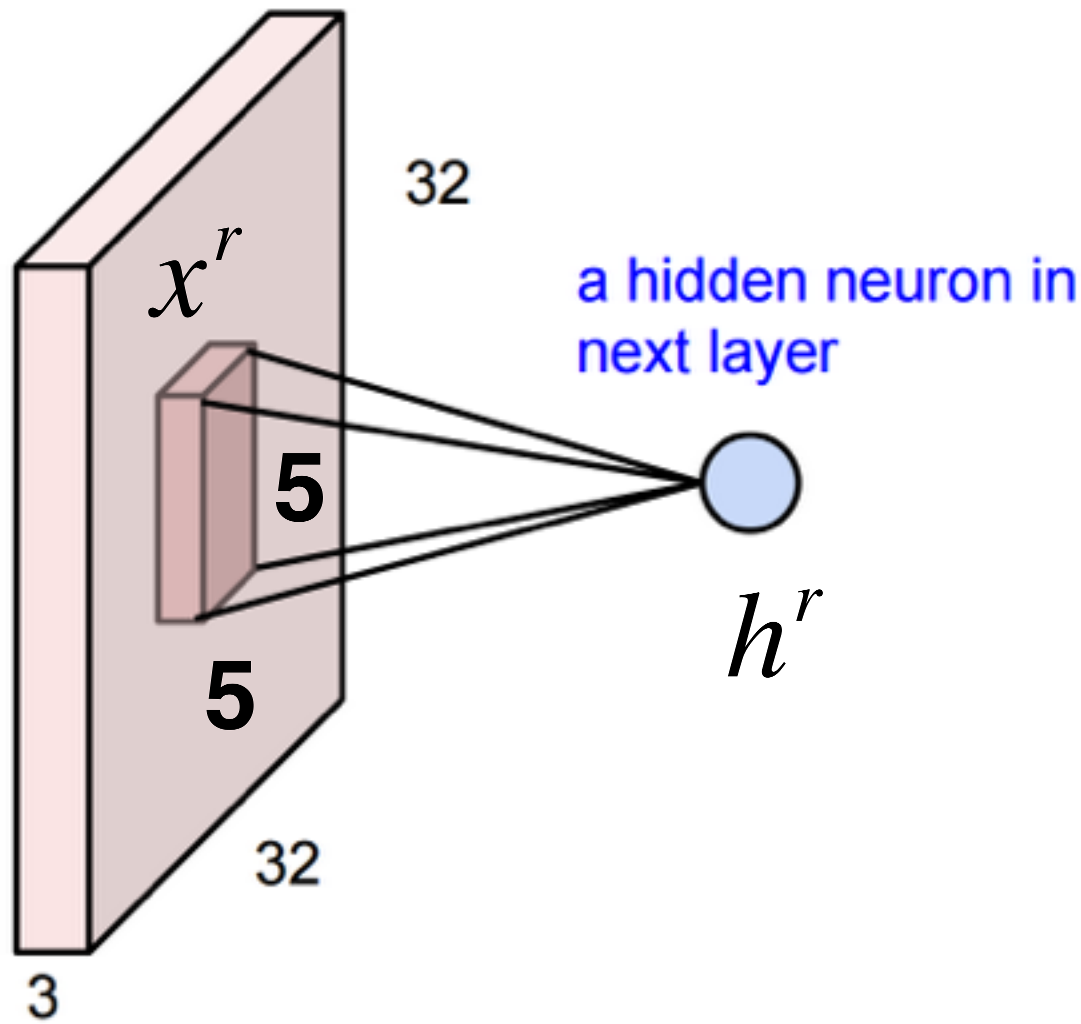


Example: consider the region of the input "x^r"

With output neuron h^r

Figure: Andrej Karpathy

3D Activations



Example: consider the region of the input "x^r"

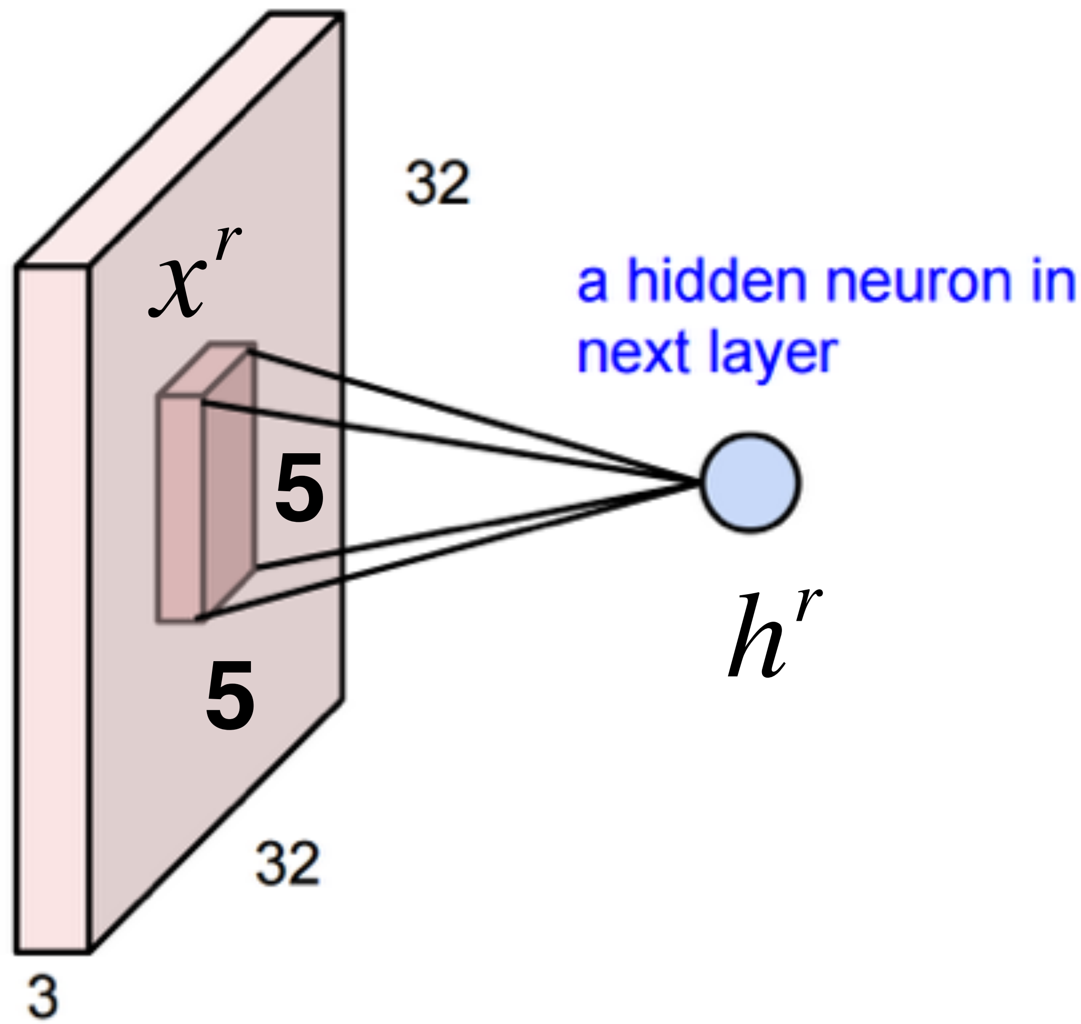
With output neuron h^r

Then the output is:

$$h^r = \sum_{ijk} x^r{}_{ijk} W_{ijk} + b$$

Figure: Andrej Karpathy

3D Activations



Example: consider the region of the input "x^r"

With output neuron h^r

Then the output is:

$$h^r = \sum_{ijk} x^r{}_{ijk} W_{ijk} + b$$

Sum over 3 axes

Figure: Andrej Karpathy

3D Activations

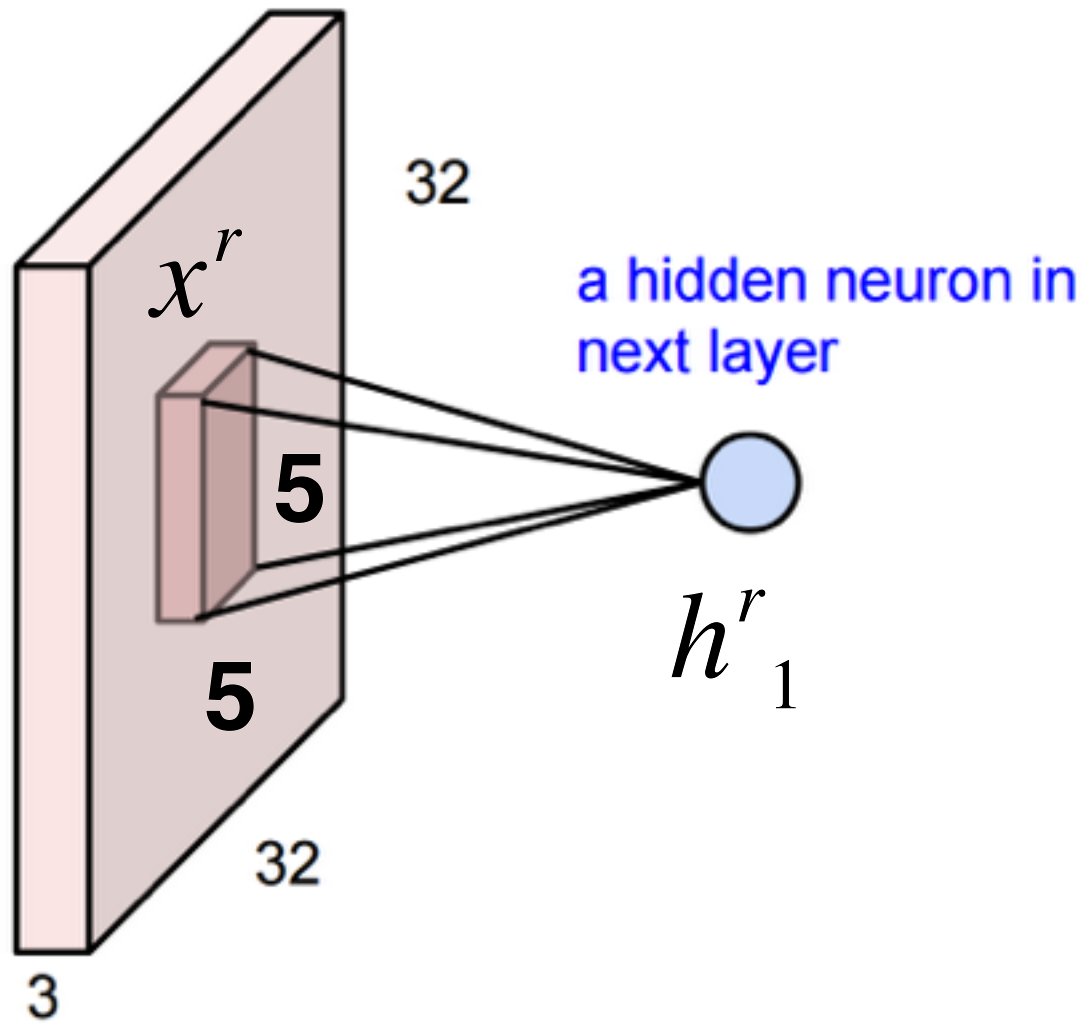


Figure: Andrej Karpathy

3D Activations

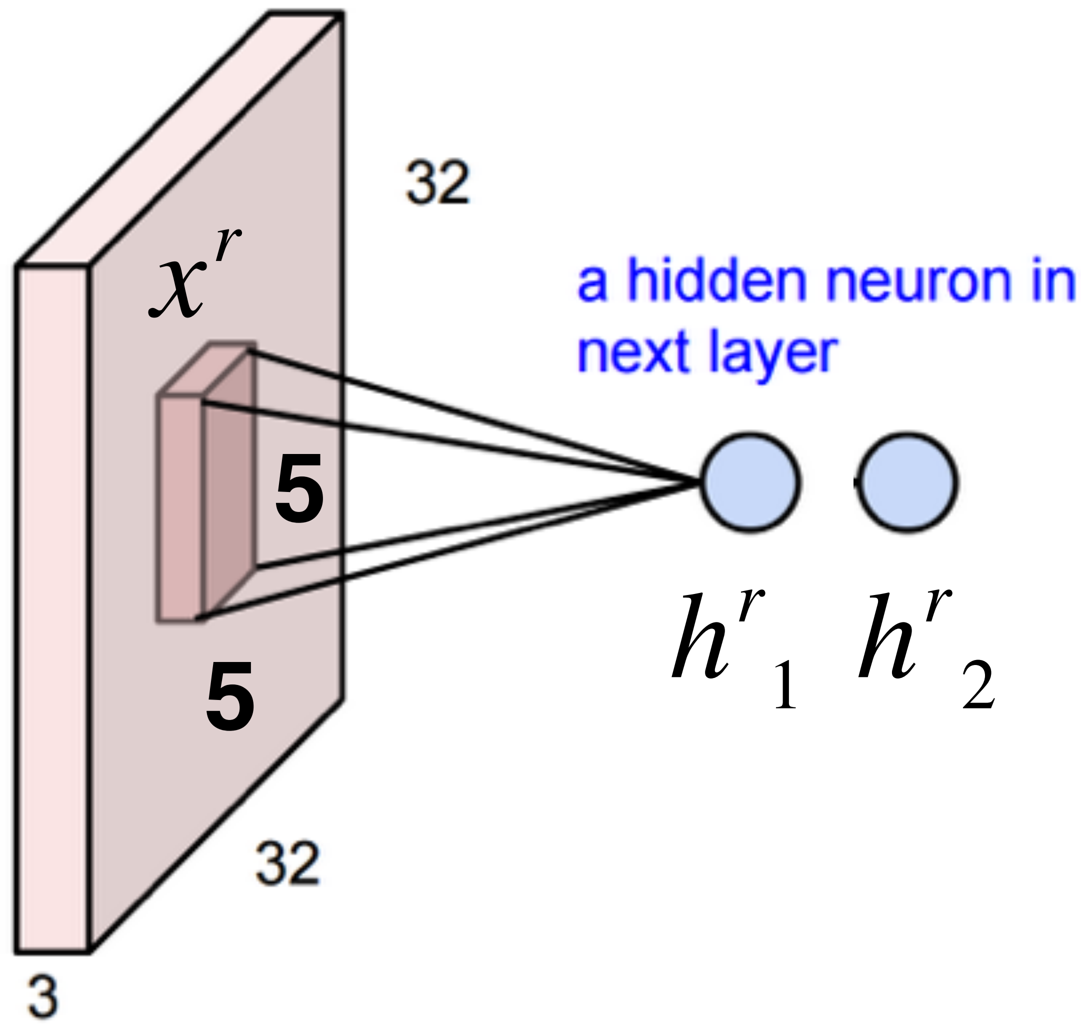
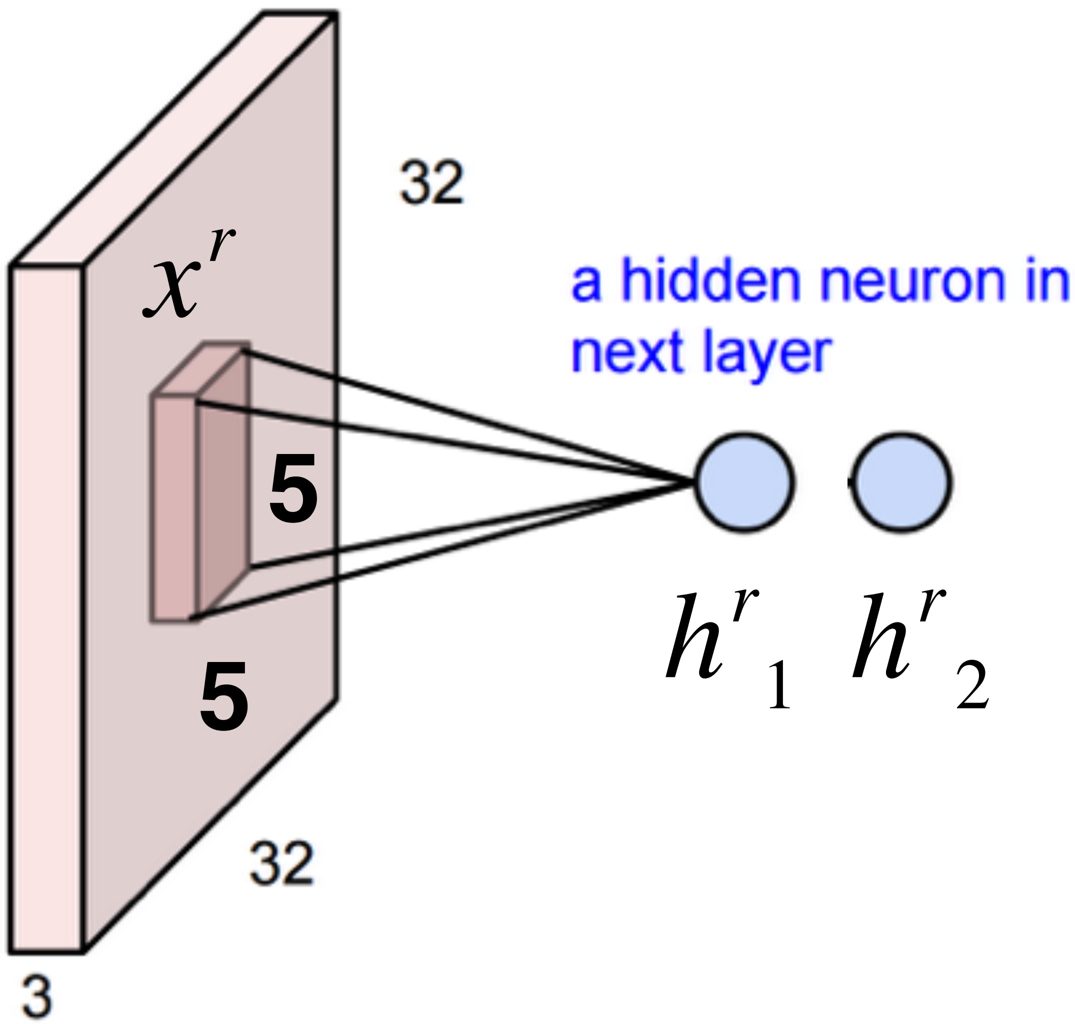


Figure: Andrej Karpathy

3D Activations



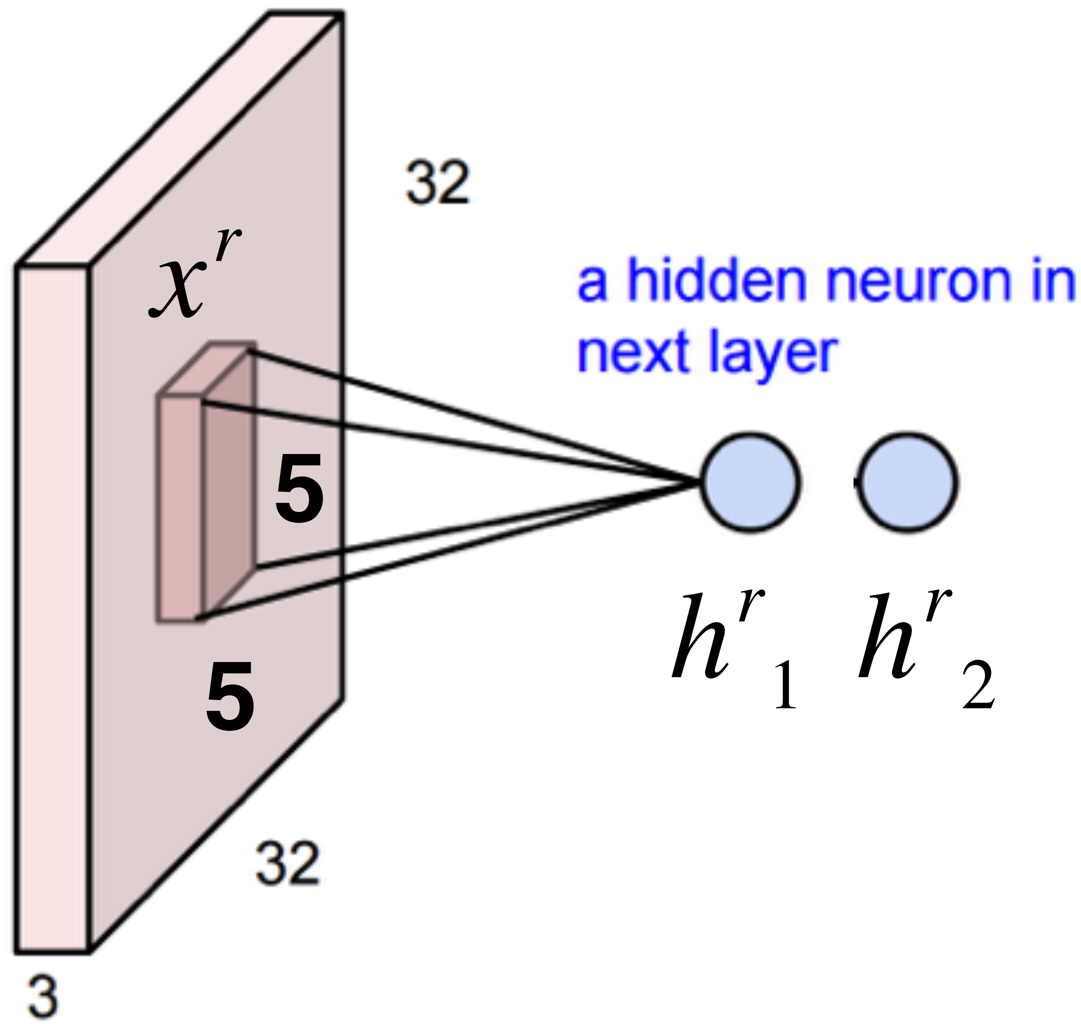
With **2** output neurons

$$h^r{}_1 = \sum_{ijk} x^r{}_{ijk} W_{1ijk} + b_1$$

$$h^r{}_2 = \sum_{ijk} x^r{}_{ijk} W_{2ijk} + b_2$$

Figure: Andrej Karpathy

3D Activations



With **2** output neurons

$$h^r{}_1 = \sum_{ijk} x^r{}_{ijk} W_{\boxed{1}ijk} + b_{\boxed{1}}$$

$$h^r{}_2 = \sum_{ijk} x^r{}_{ijk} W_{\boxed{2}ijk} + b_{\boxed{2}}$$

Figure: Andrej Karpathy

3D Activations

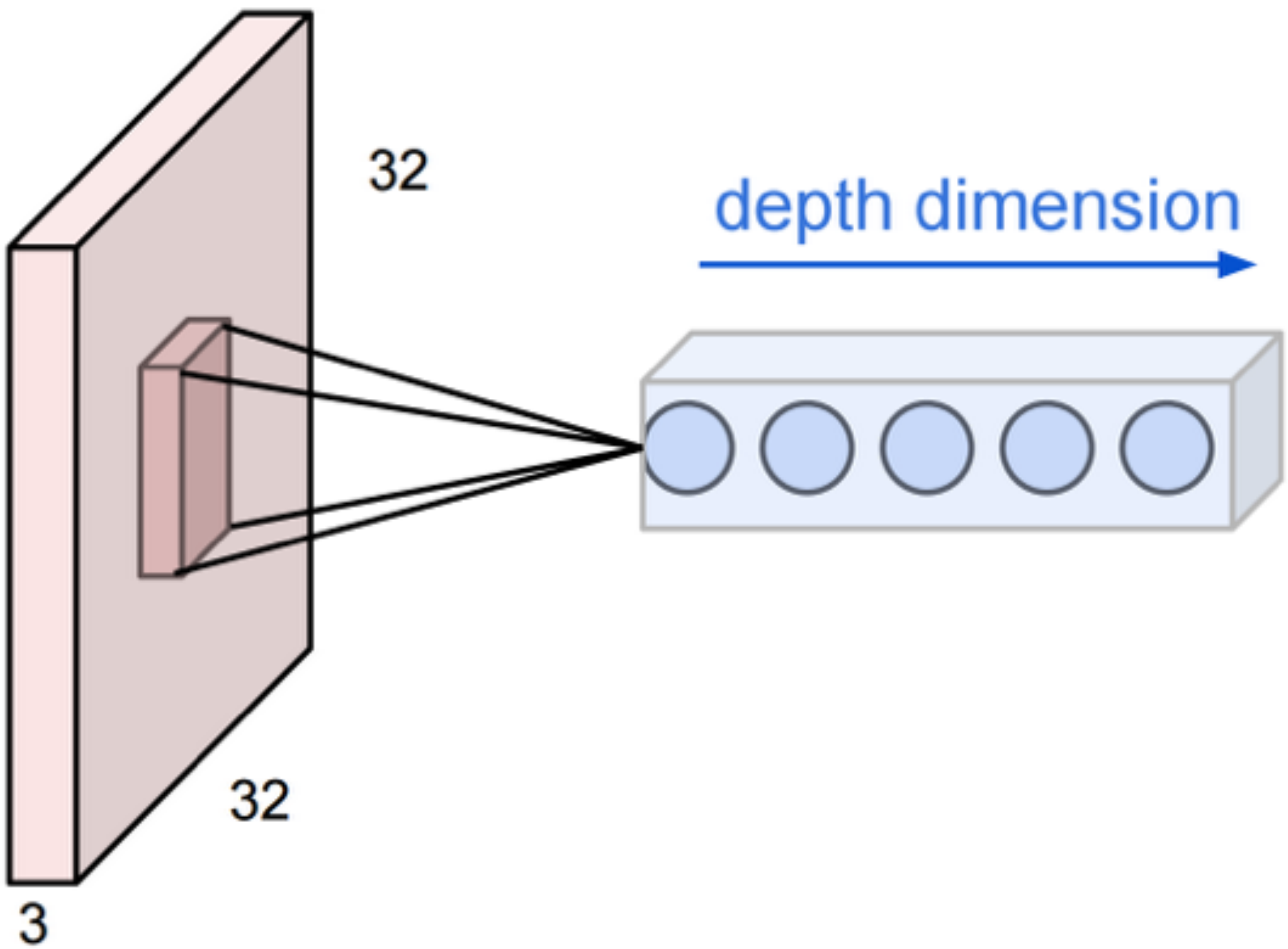
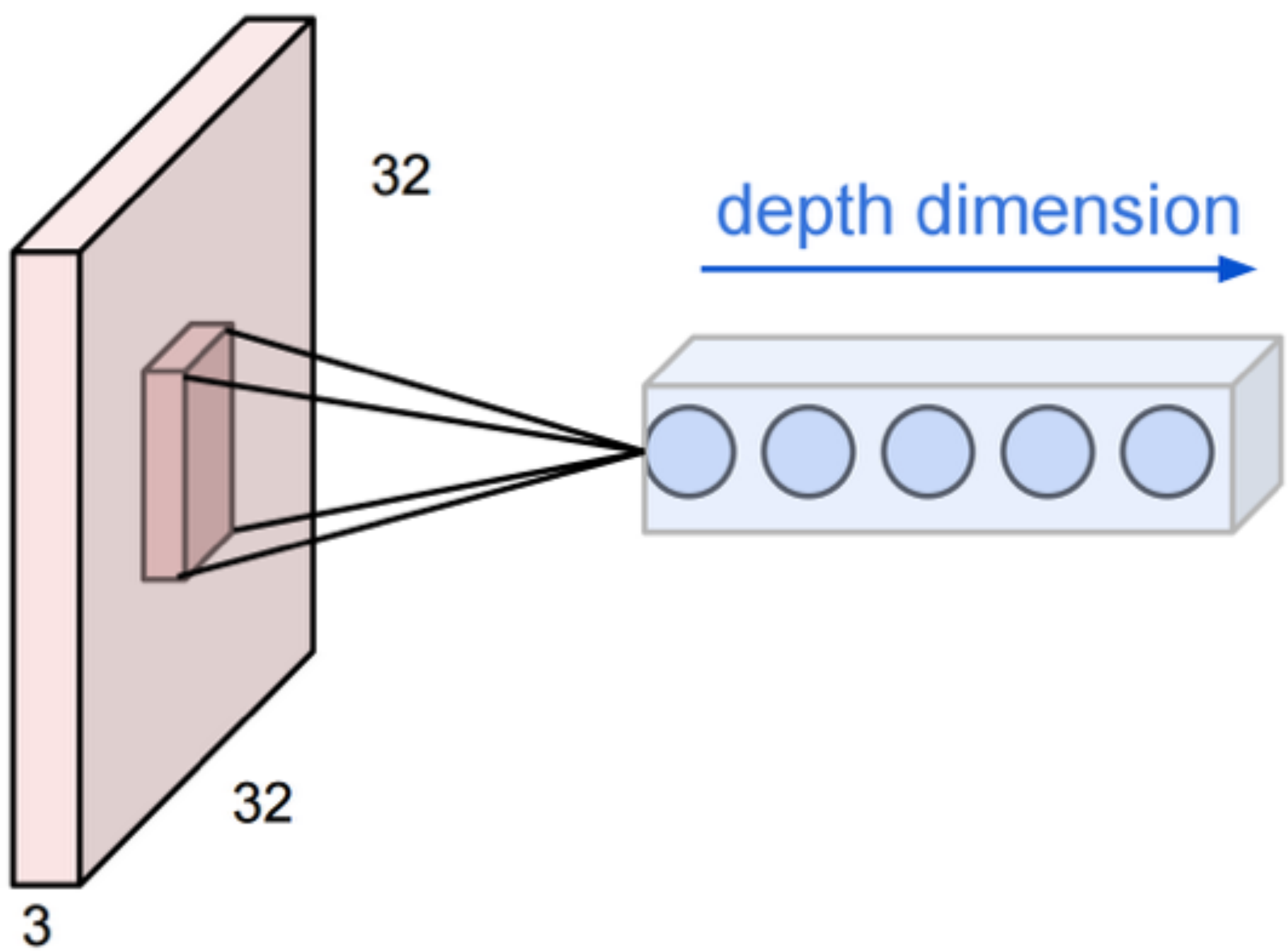


Figure: Andrej Karpathy

3D Activations

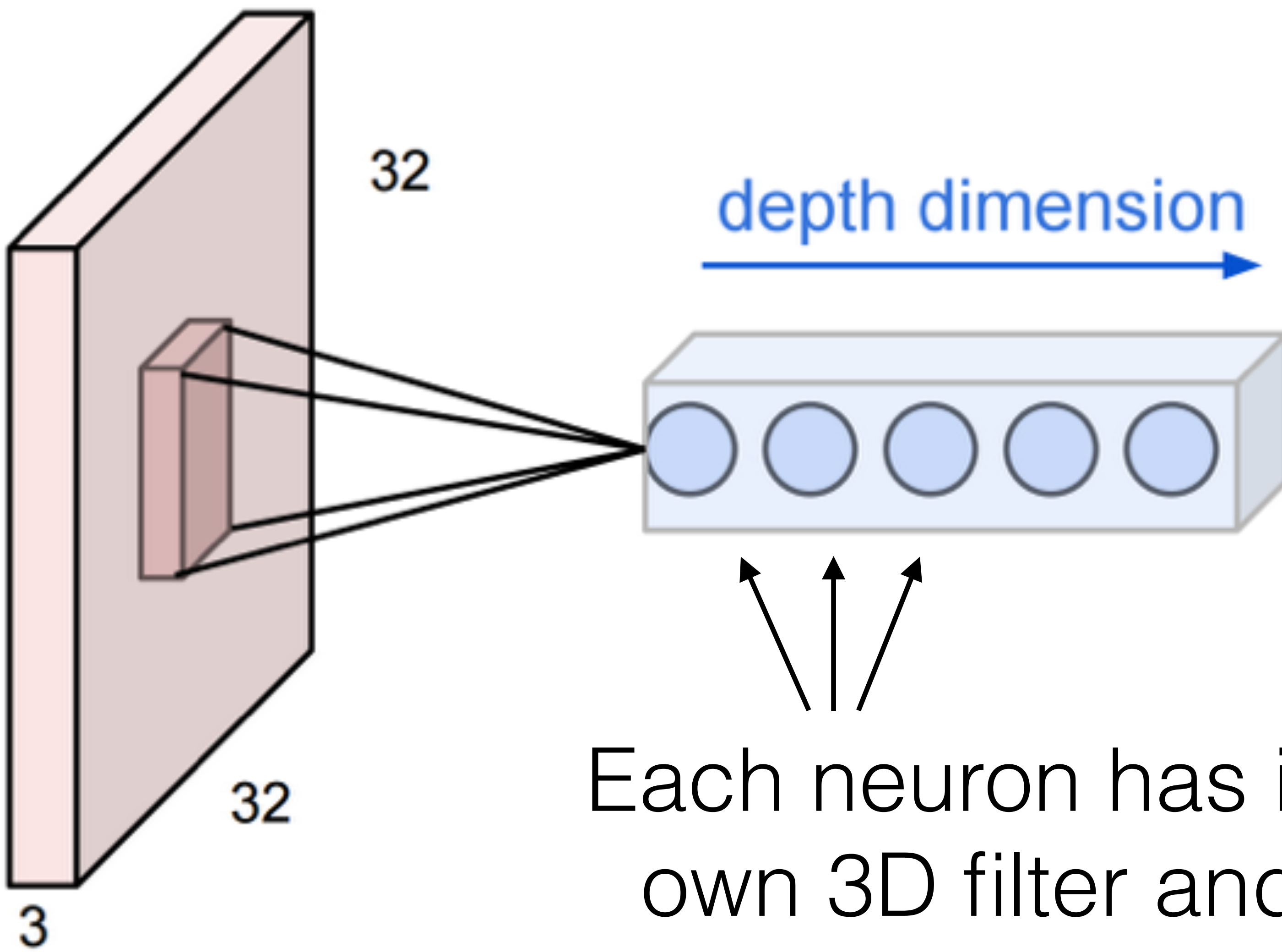


We can keep adding more outputs

These form a column in the output volume: [depth x 1 x 1]

Figure: Andrej Karpathy

3D Activations



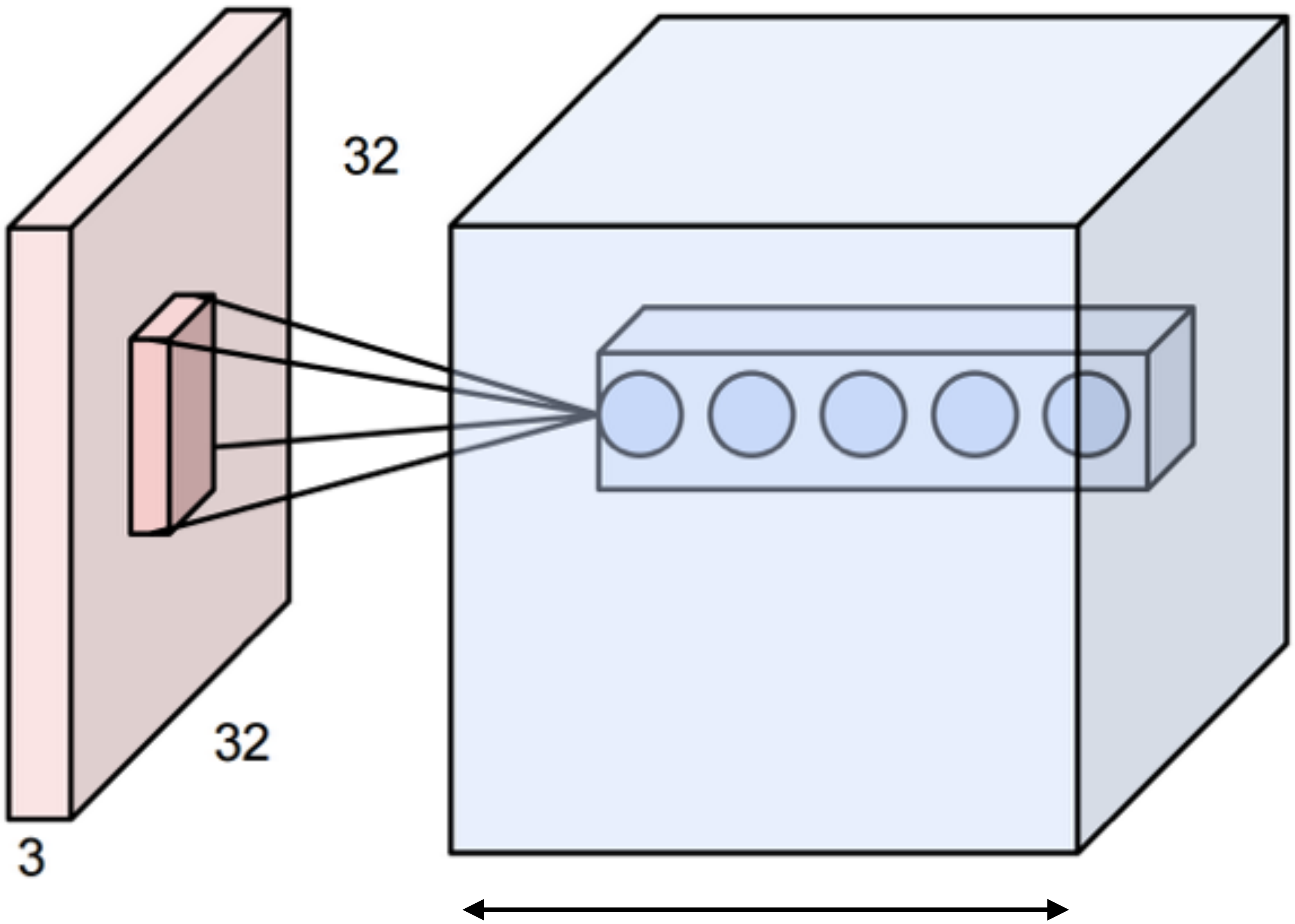
Each neuron has its own 3D filter and own (scalar) bias

We can keep adding more outputs

These form a column in the output volume: [depth x 1 x 1]

Figure: Andrej Karpathy

3D Activations

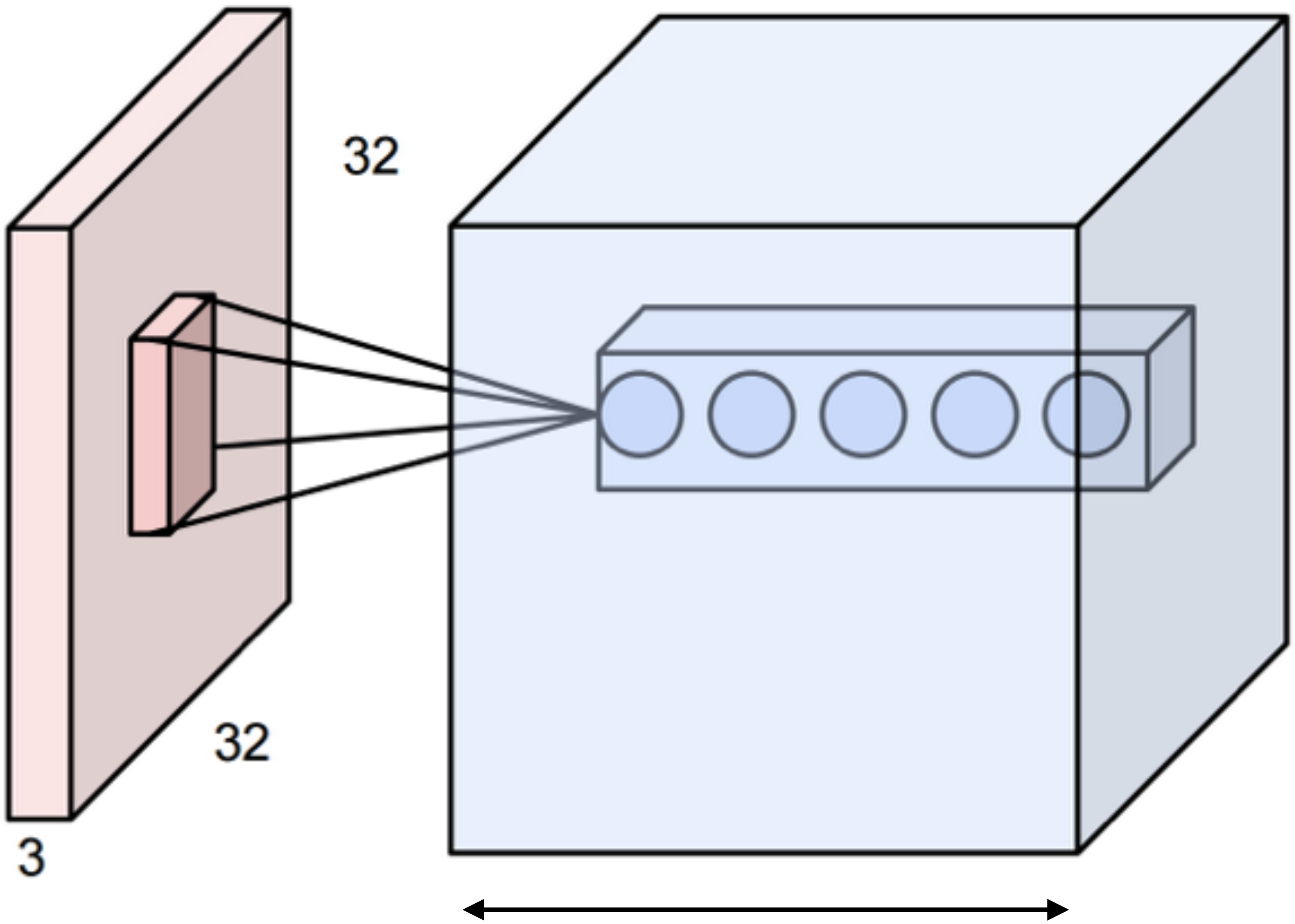


D sets of weights
(also called filters)

Now repeat this across the input

Figure: Andrej Karpathy

3D Activations



D sets of weights
(also called filters)

Now repeat this across the input

Weight sharing:
Each filter shares the same weights (but each depth index has its own set of weights)

Figure: Andrej Karpathy

3D Activations

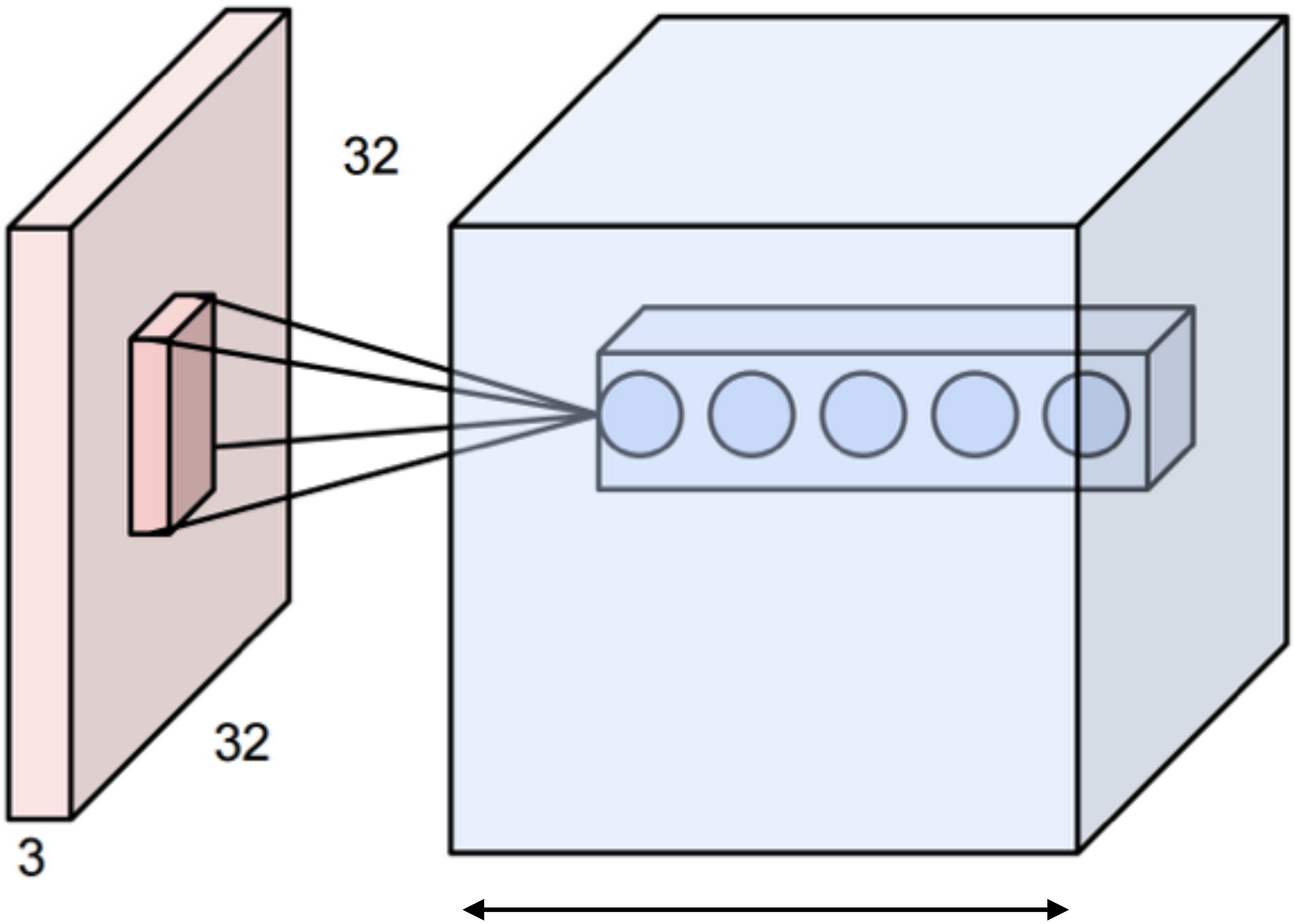


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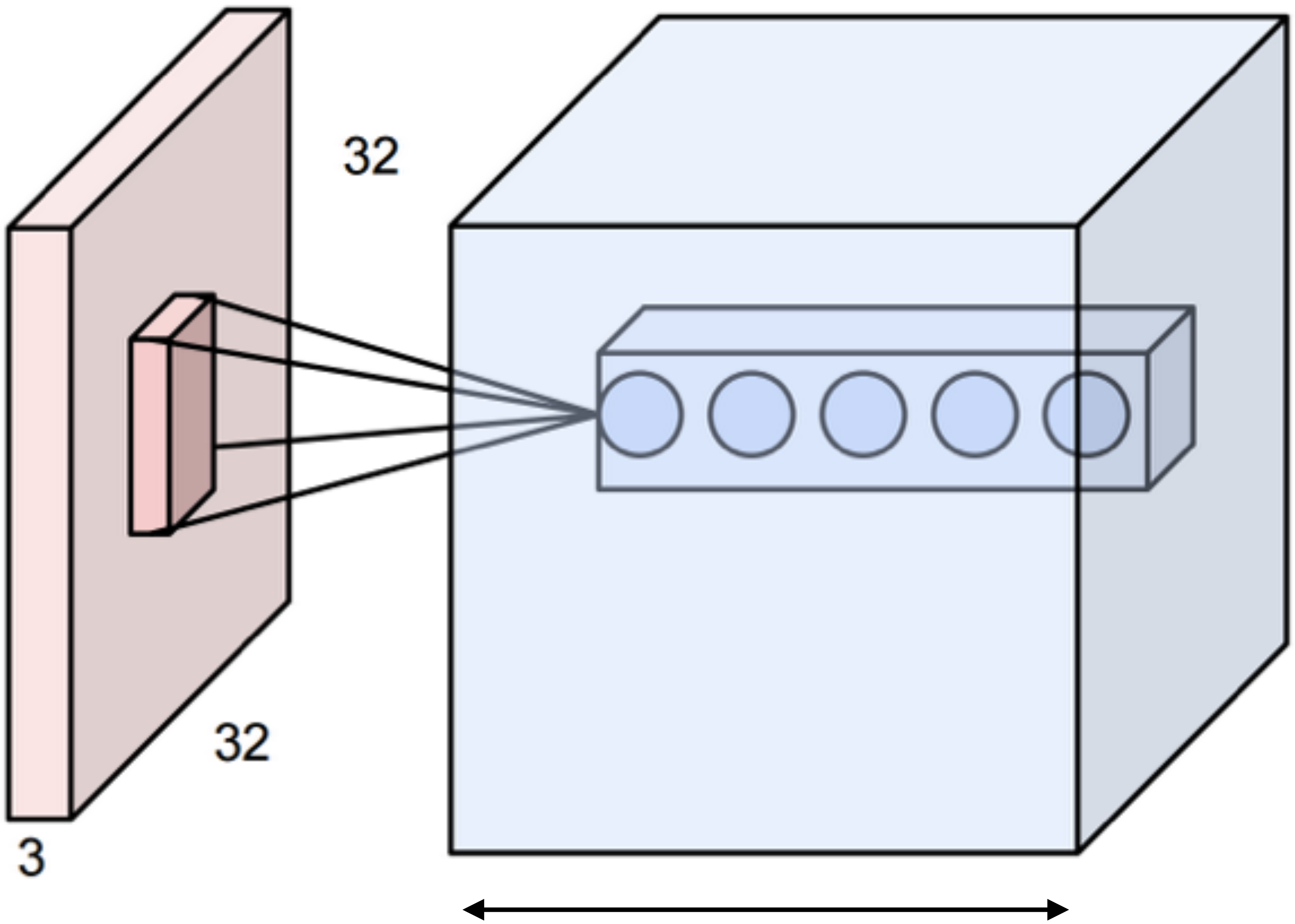
Figure: Andrej Karpathy

3D Activations

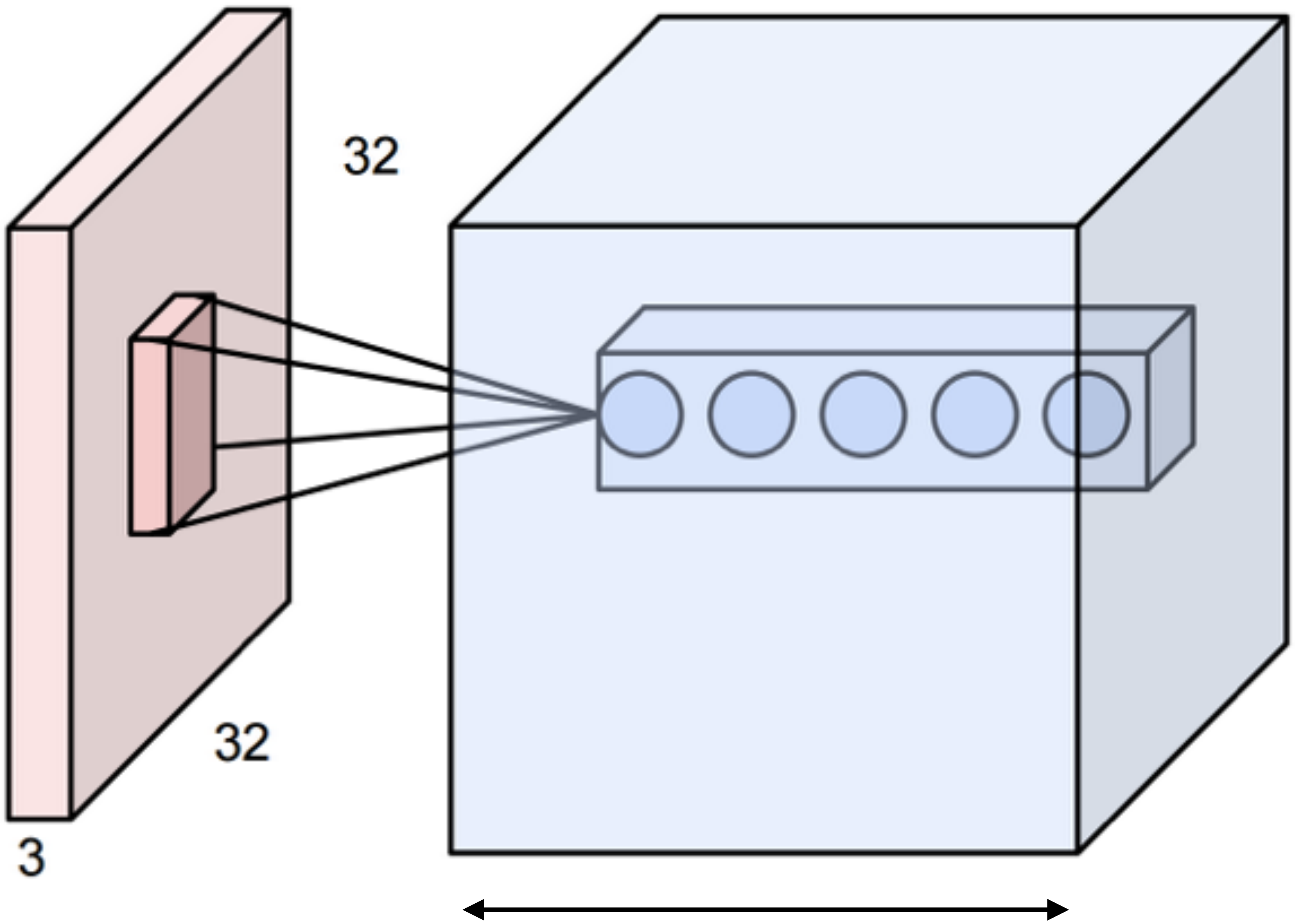


D sets of weights
(also called filters)

With weight sharing, this is called **convolution**

Figure: Andrej Karpathy

3D Activations



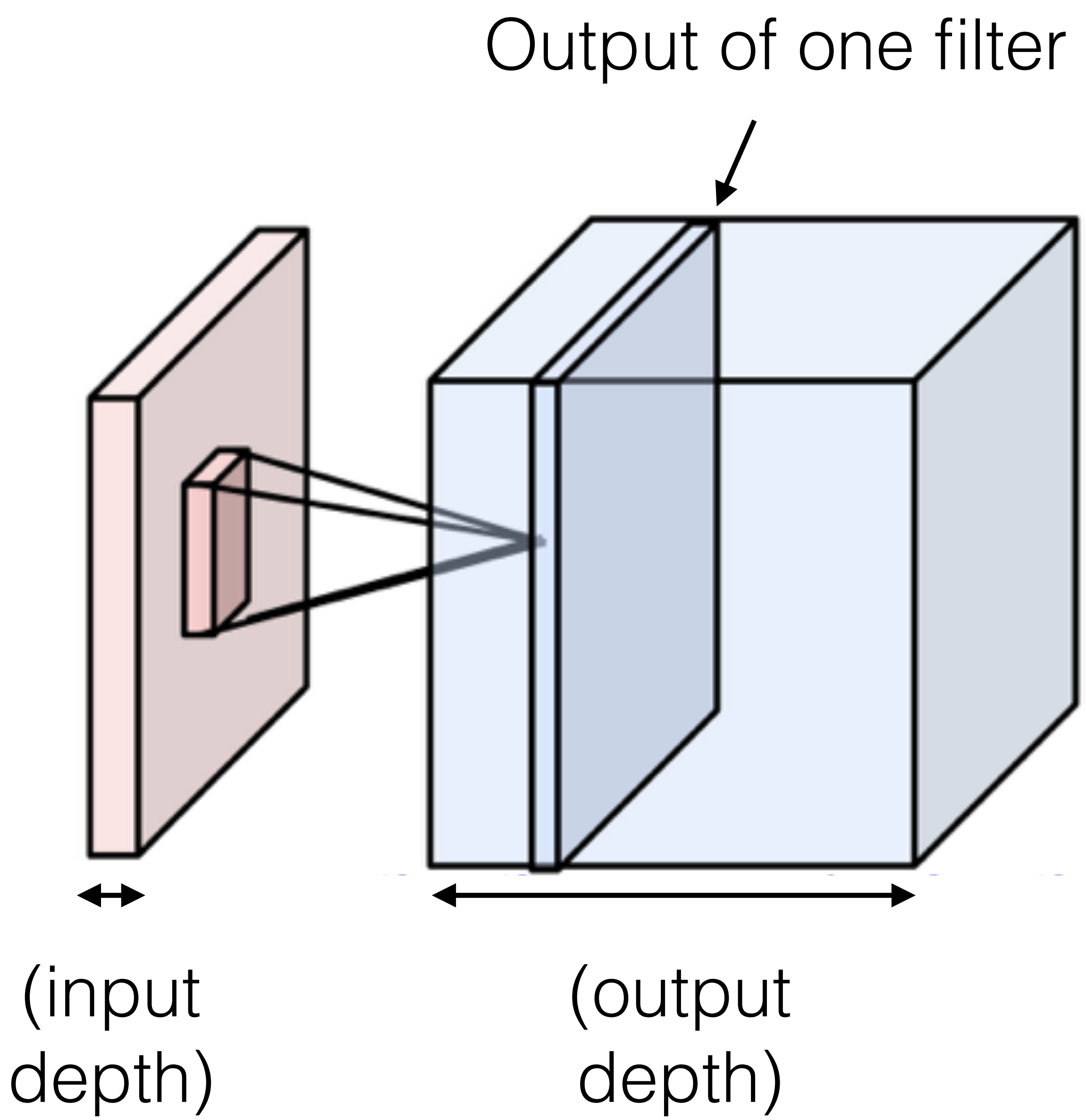
D sets of weights
(also called filters)

With weight sharing, this is called **convolution**

Without weight sharing, this is called a **locally connected layer**

Figure: Andrej Karpathy

3D Activations

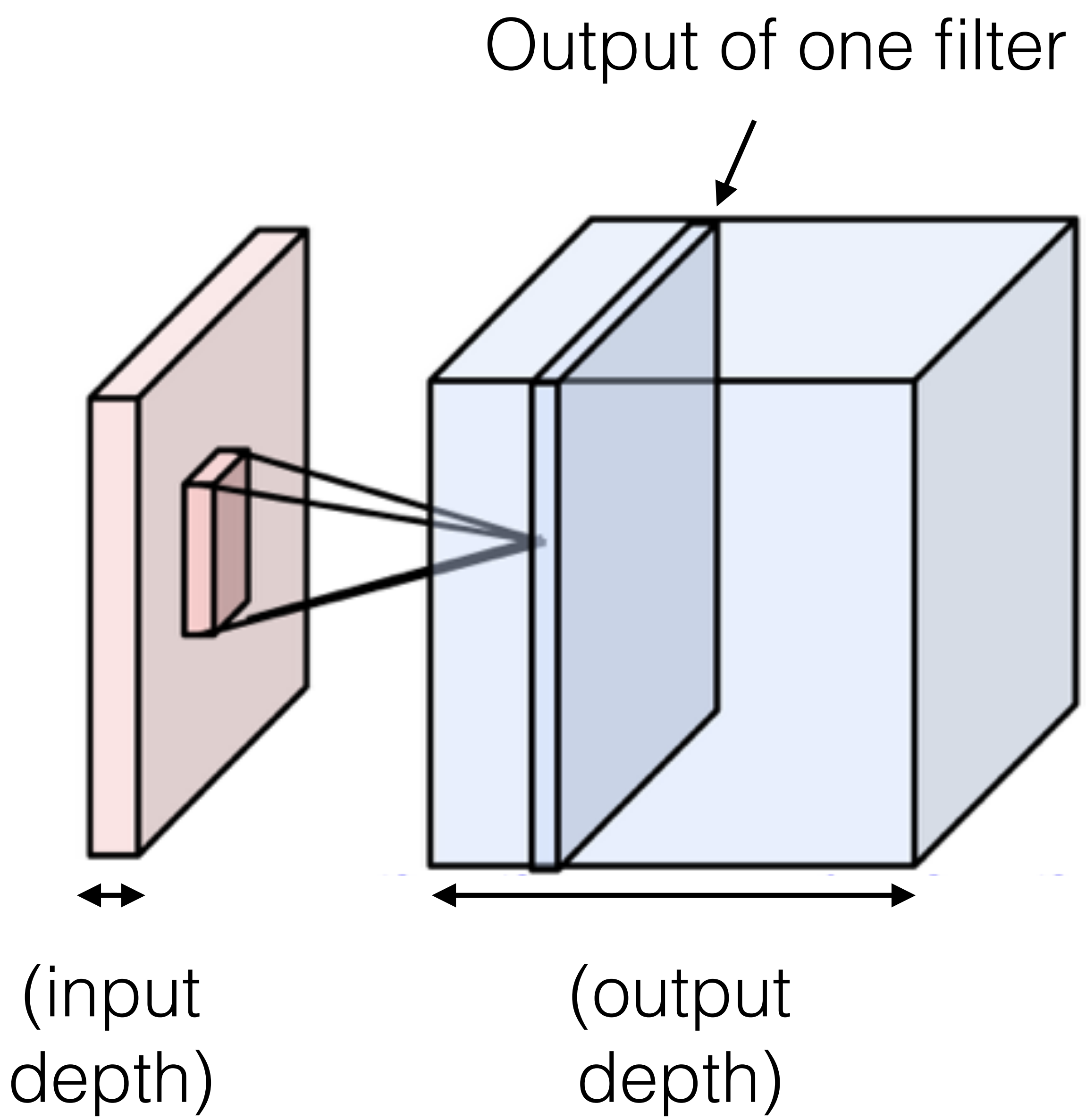


One set of weights gives one slice in the output

To get a 3D output of depth *D*, use *D* different filters

In practice, CNNs use many filters (~64 to 1024)

3D Activations



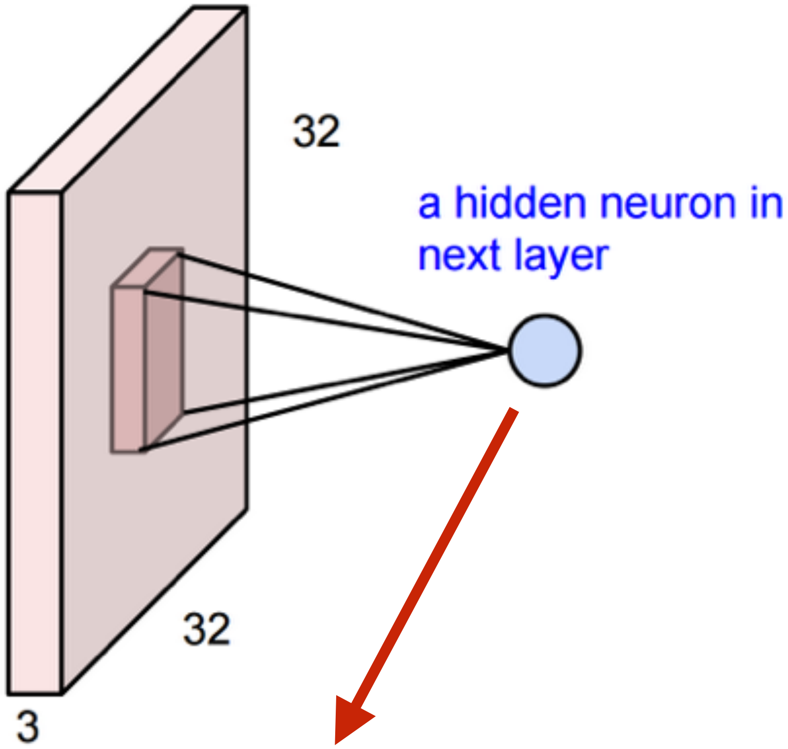
One set of weights gives one slice in the output

To get a 3D output of depth *D*, use *D* different filters

In practice, CNNs use many filters (~64 to 1024)

All together, the weights are **4** dimensional:
(output depth, input depth, kernel height, kernel width)

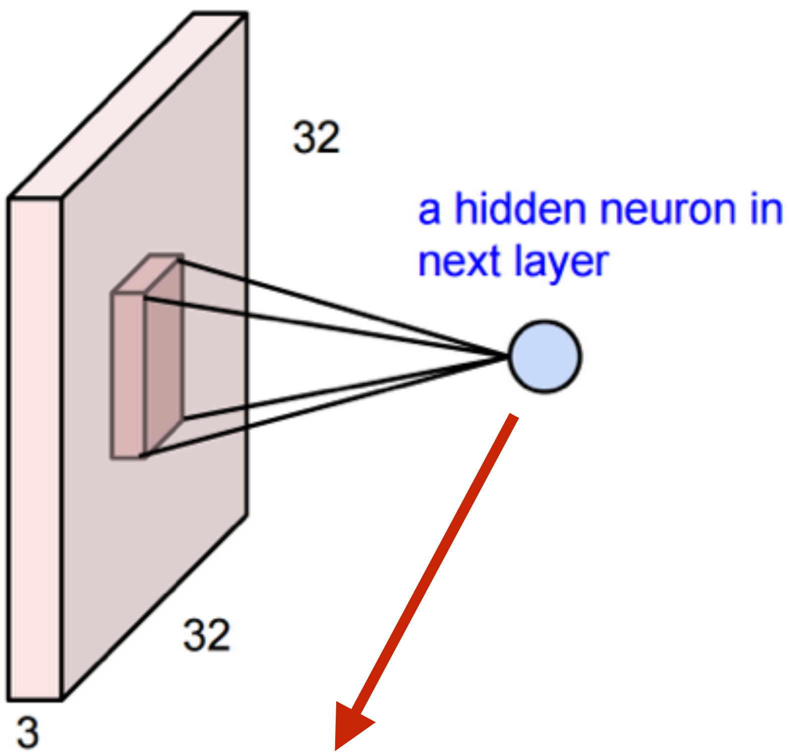
3D Activations




```
out[n, 0, r, c] =
```

Let's code this up in NumPy

3D Activations

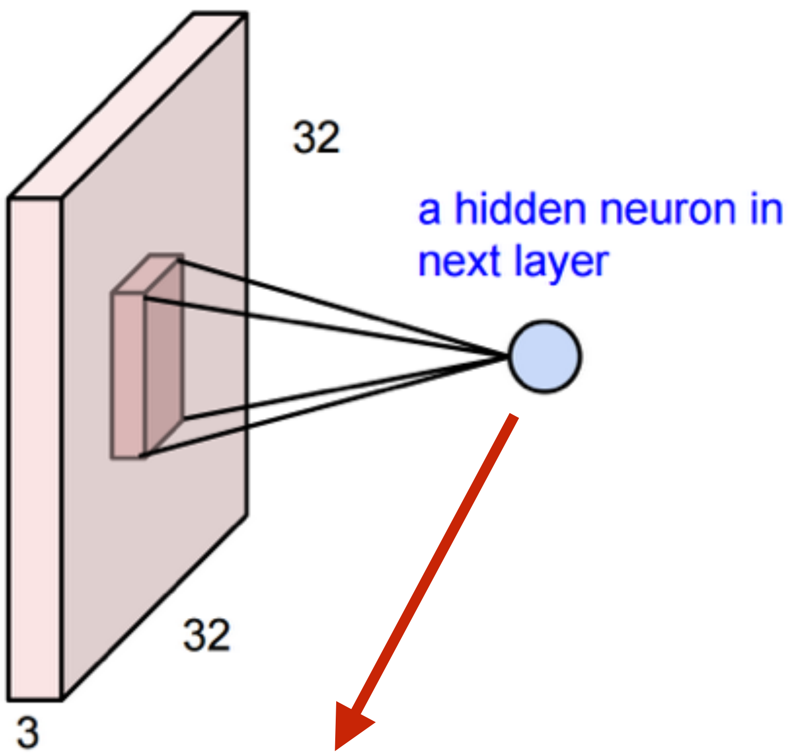


Let's code this up in NumPy

```
out[n, 0, r, c] =
```

n^{th} example

3D Activations



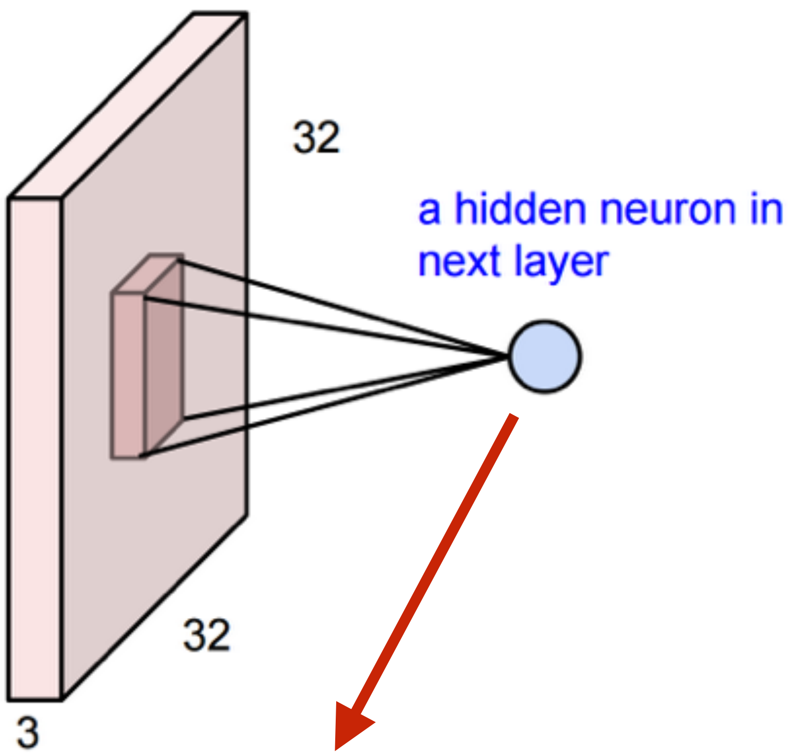
Let's code this up in NumPy

```
out[n, 0, r, c] =
```

first filter

n^{th} example

3D Activations



Let’s code this up in NumPy

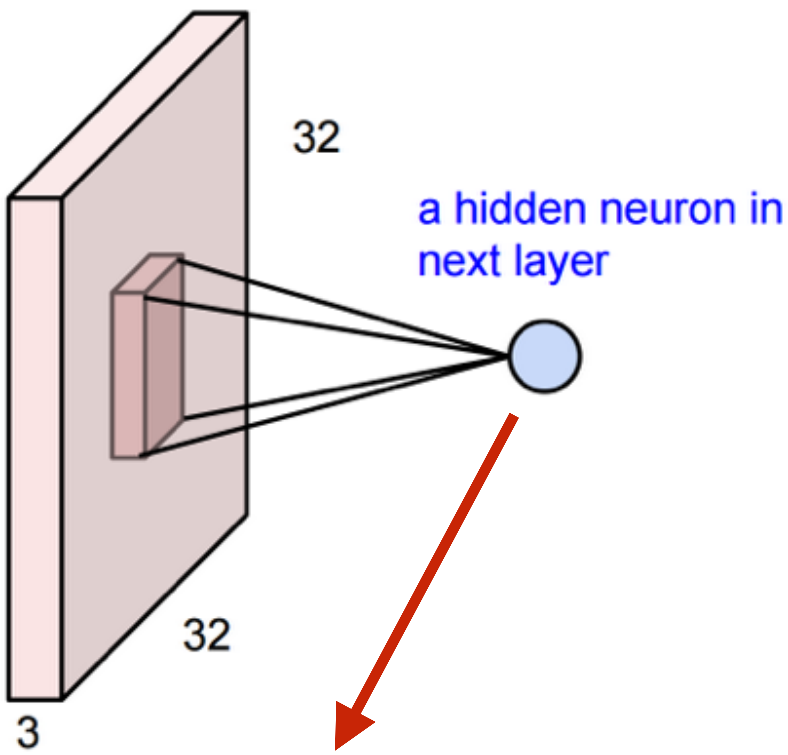
```
out[n, 0, r, c] =
```

output position

first filter

n^{th} example

3D Activations




```
out[n, 0, r, c] = np.sum(
```

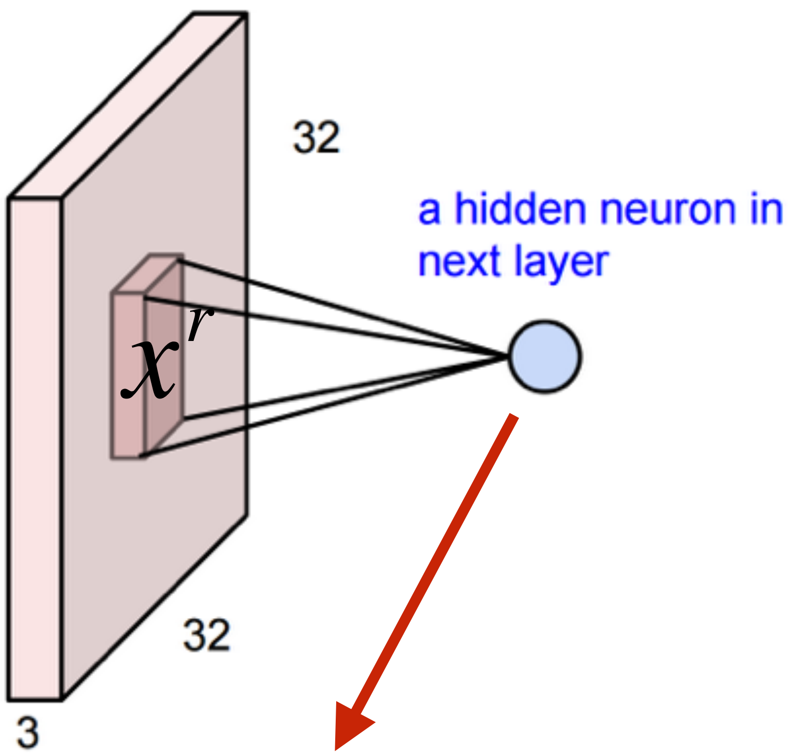
output position

first filter

n^{th} example

Let's code this up in NumPy

3D Activations



Let's code this up in NumPy

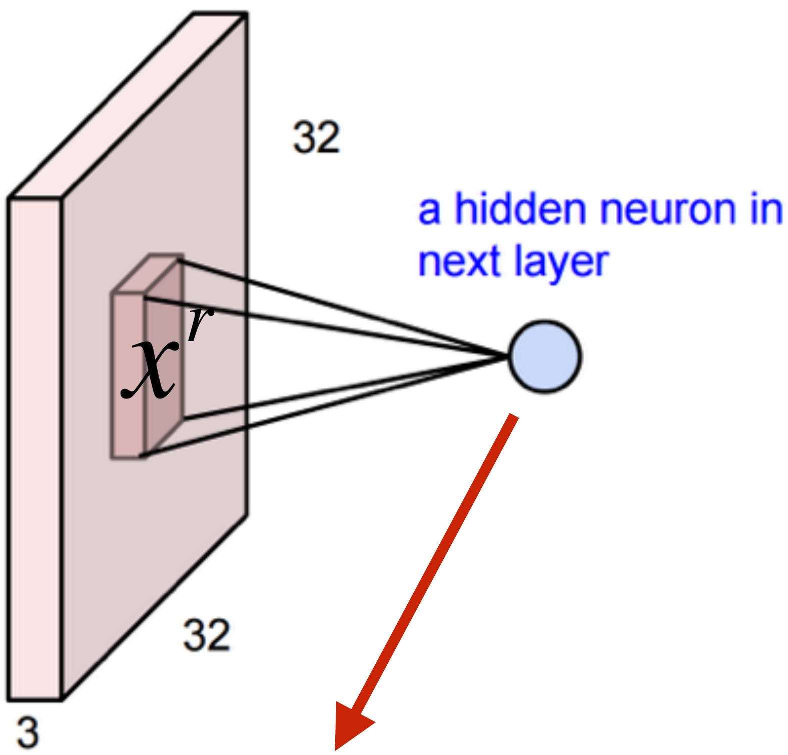
```
out[n, 0, r, c] = np.sum(X[n, :, r0:r1, c0:c1]
```

output position

first filter

nth example

3D Activations



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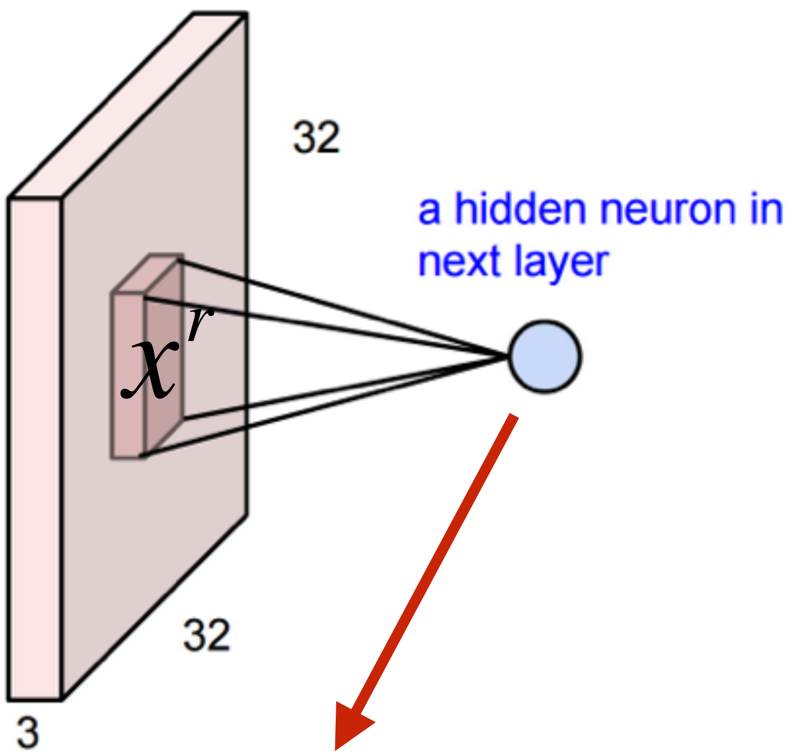
output position

first filter

n^{th} example

n^{th} example

3D Activations



Let's code this up in NumPy

```
out[n, 0, r, c] = np.sum(X[n, :, r0:r1, c0:c1]
```

output position

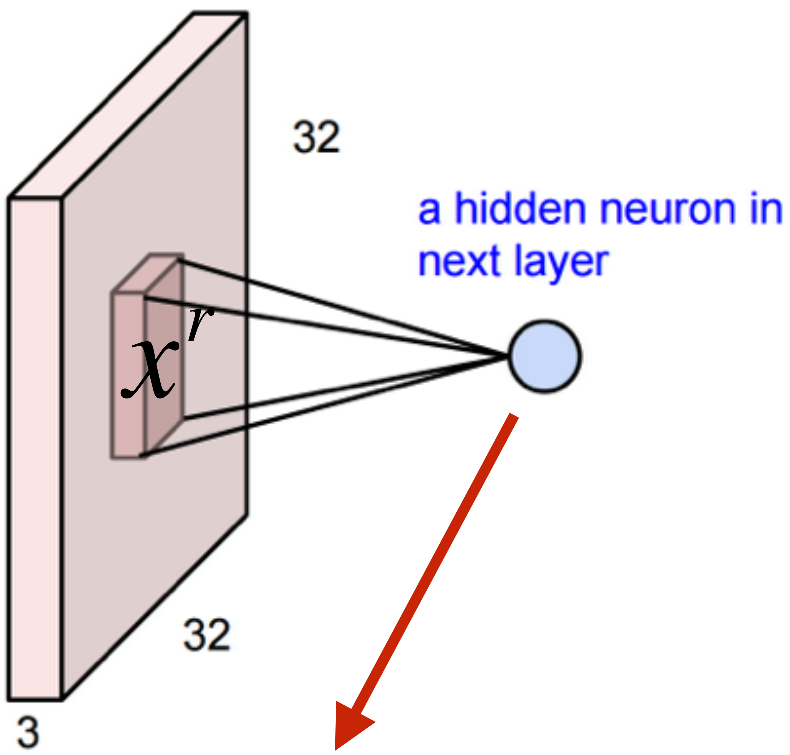
first filter

all input channels

n^{th} example

n^{th} example

3D Activations



Let's code this up in NumPy

```
out[n, 0, r, c] = np.sum(X[n, :, r0:r1, c0:c1]
```

output position

first filter

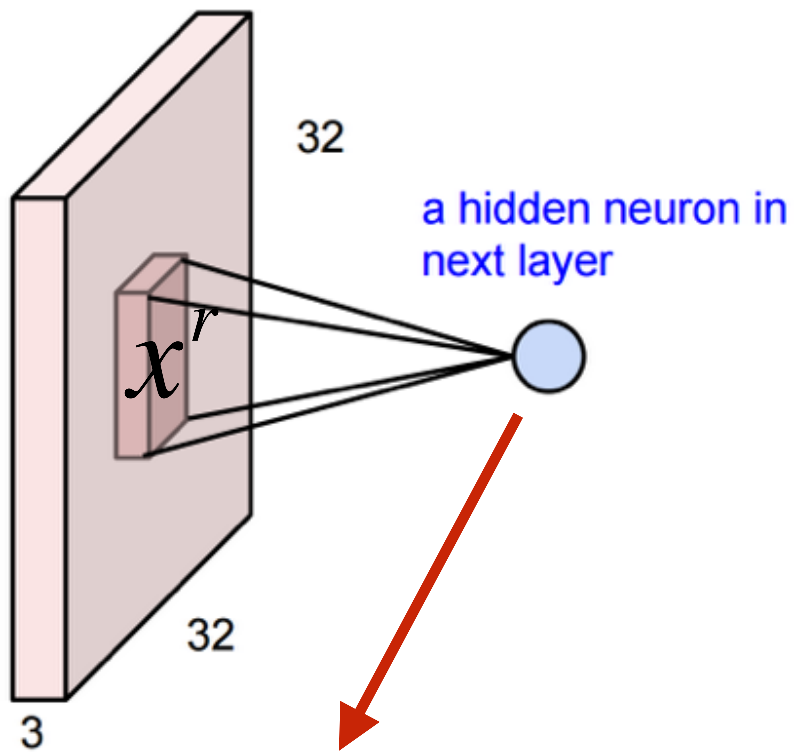
n^{th} example

input region

all input channels

n^{th} example

3D Activations



Let's code this up in NumPy

```
out[n, 0, r, c] = np.sum(X[n, :, r0:r1, c0:c1] * W[0, :, :, :]) + b[0]
```

output position

first filter

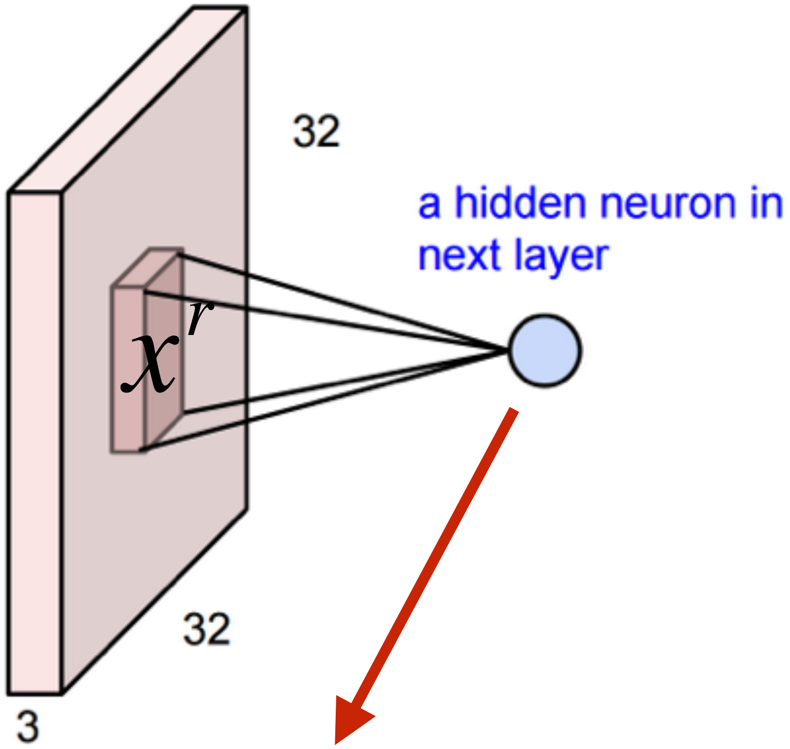
n^{th} example

input region

all input channels

n^{th} example

3D Activations



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out[n, 0, r, c] = np.sum(X[n, :, r0:r1, c0:c1] * W[0, :, :, :]) + b[0]
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output position

first filter

n^{th} example

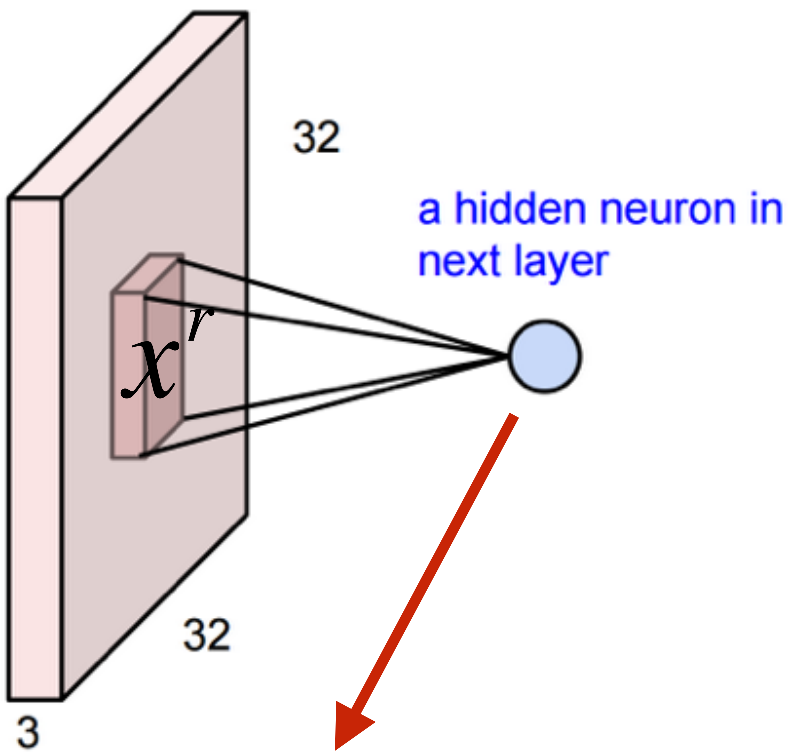
input region

all input channels

n^{th} example

first filter

3D Activations



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```

output position

first filter

nth example

input region

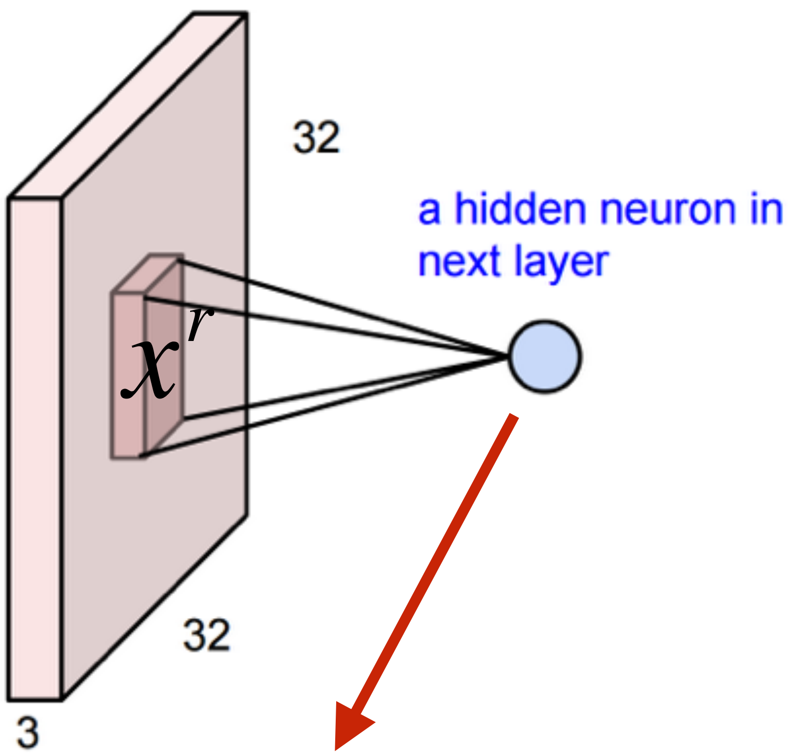
all input channels

nth example

all channels

first filter

3D Activations



Let's code this up in NumPy

```
out[n, 0, r, c] = np.sum(X[n, :, r0:r1, c0:c1] * W[0, :, :, :]) + b[0]
```

output position

first filter

n^th example

input region

all input channels

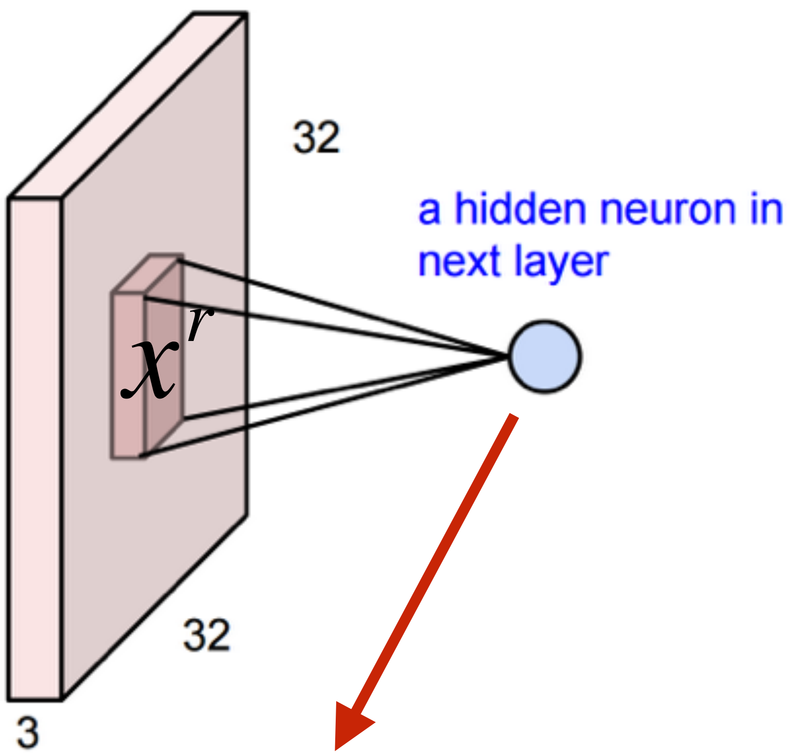
n^th example

all positions

all channels

first filter

3D Activations



Let's code this up in NumPy

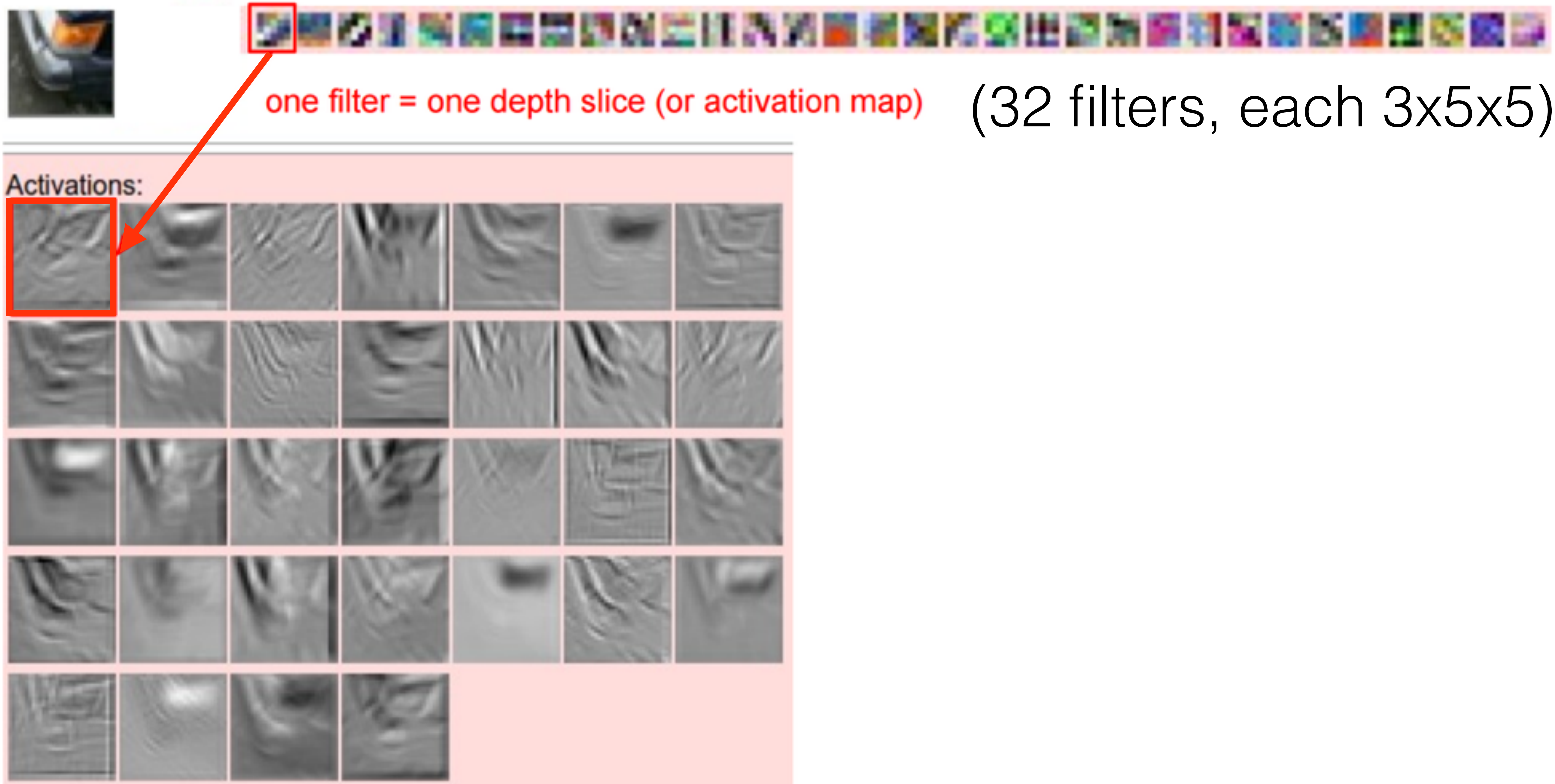
```
out[n, 0, r, c] = np.sum(X[n, :, r0:r1, c0:c1] * W[0, :, :, :]) + b[0]
```

- out[n, ...]: n^{th} example
- out[..., 0, ...]: first filter
- out[..., r, c]: output position
- X[n, ...]: n^{th} example
- X[..., :, ...]: all input channels
- X[..., r0:r1, c0:c1]: input region
- W[0, ...]: first filter
- W[..., :, ...]: all channels
- W[..., :, :]: all positions
- b[0]: bias

3D Activations

We can unravel the 3D cube and show each layer separately:

(Input)



(32 filters, each 3x5x5)

Figure: Andrej Karpathy

3D Activations

We can unravel the 3D cube and show each layer separately:

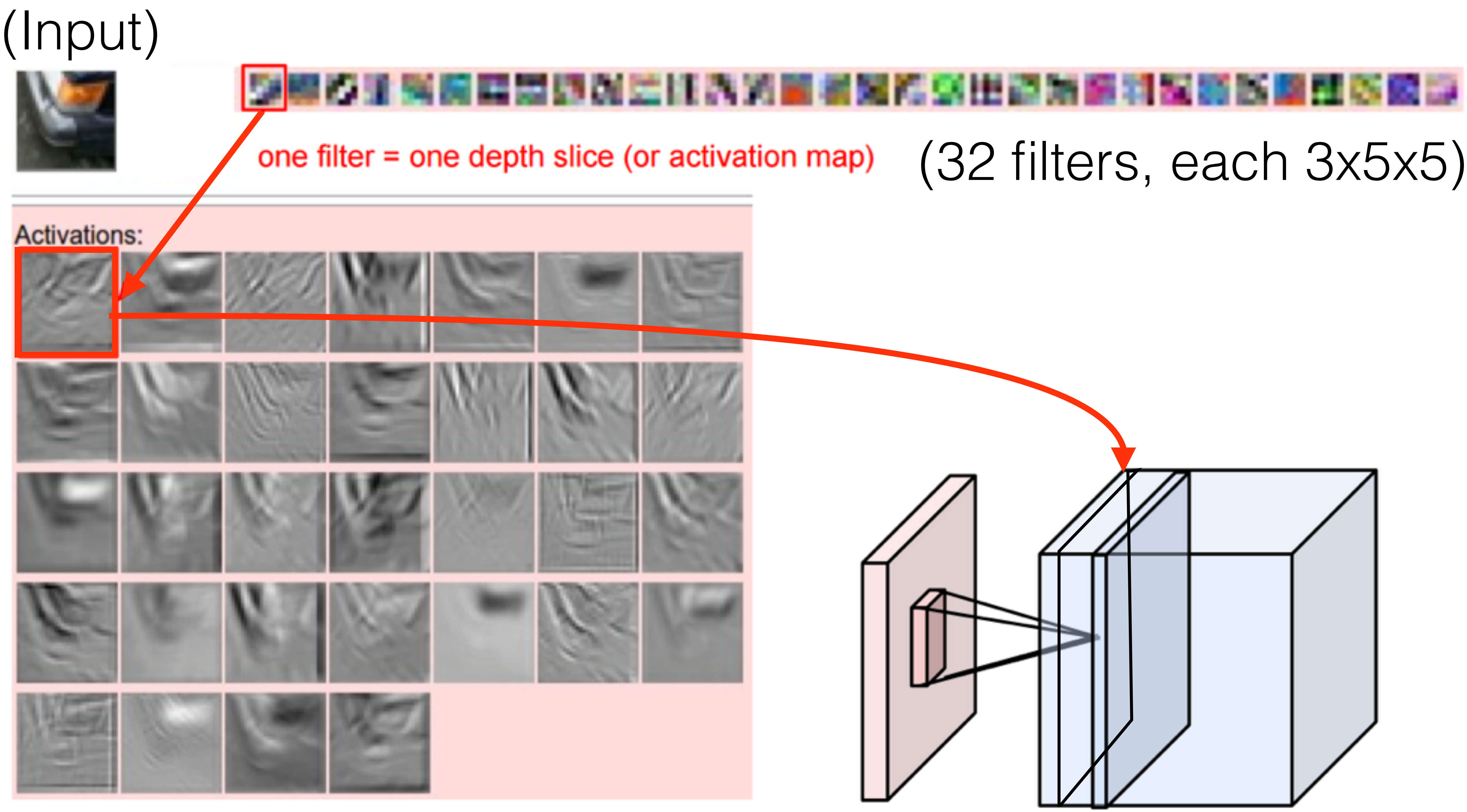


Figure: Andrej Karpathy

3D Activations

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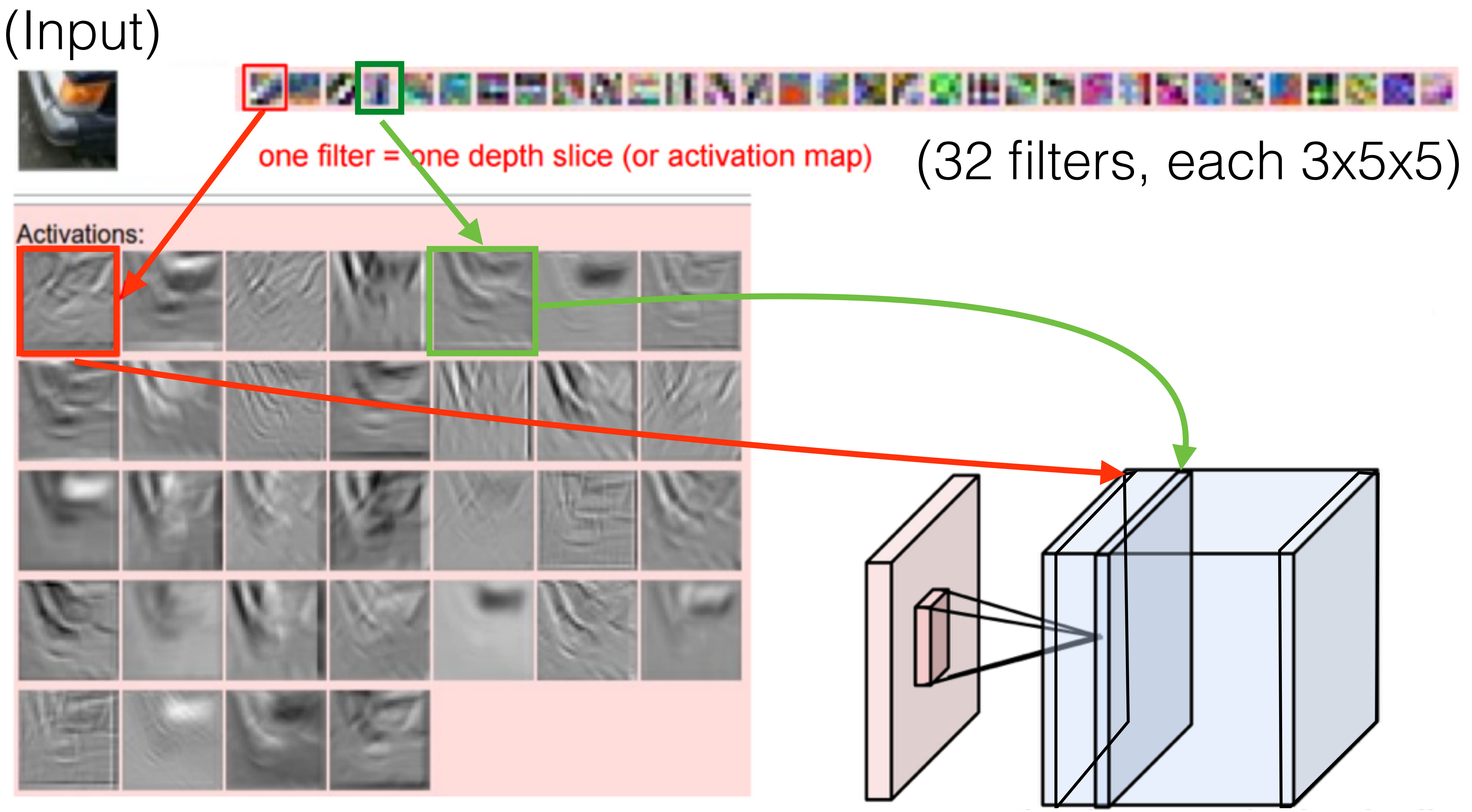


Figure: Andrej Karpathy

3D Activations

We can unravel the 3D cube and show each layer separately:

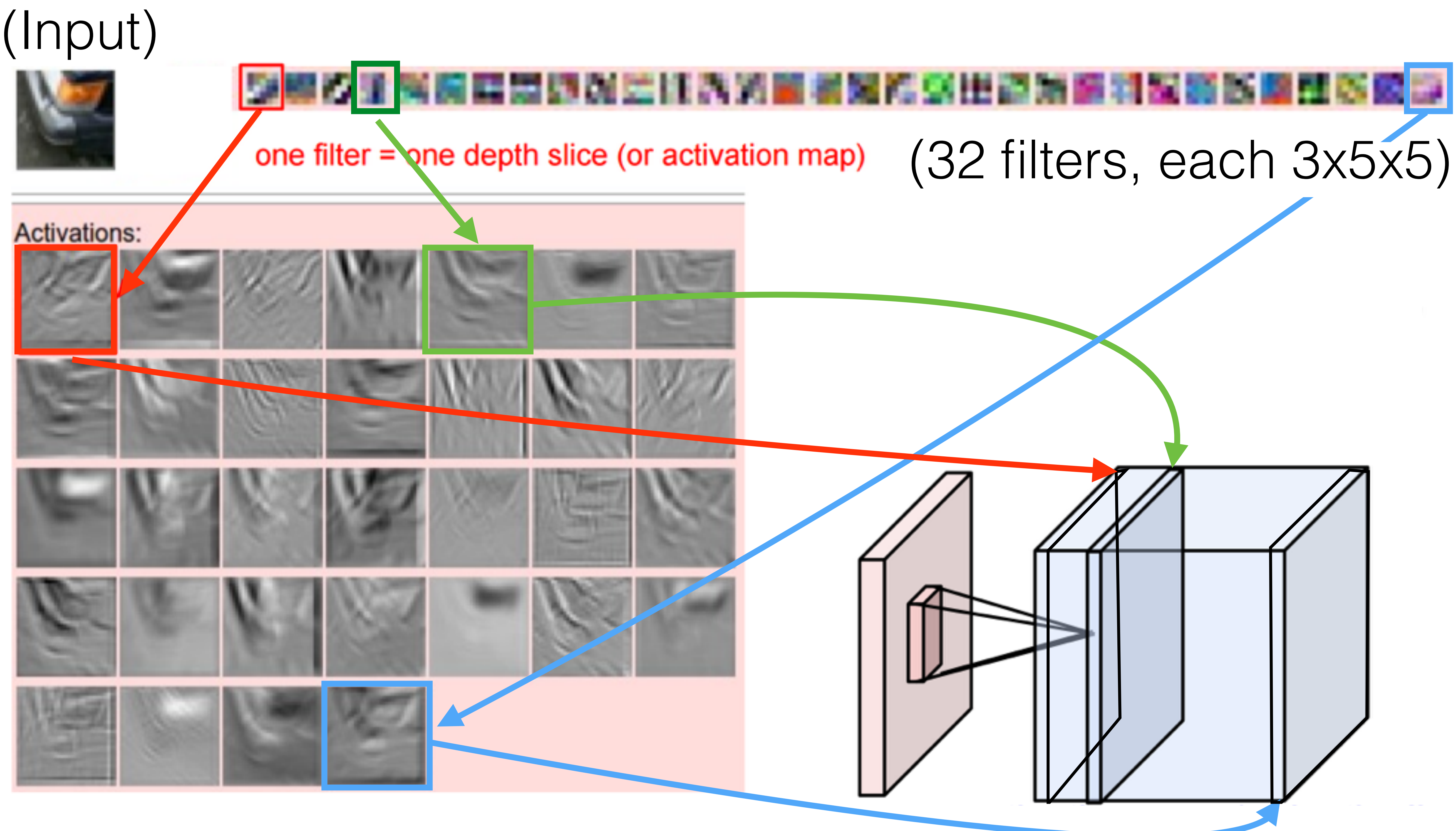
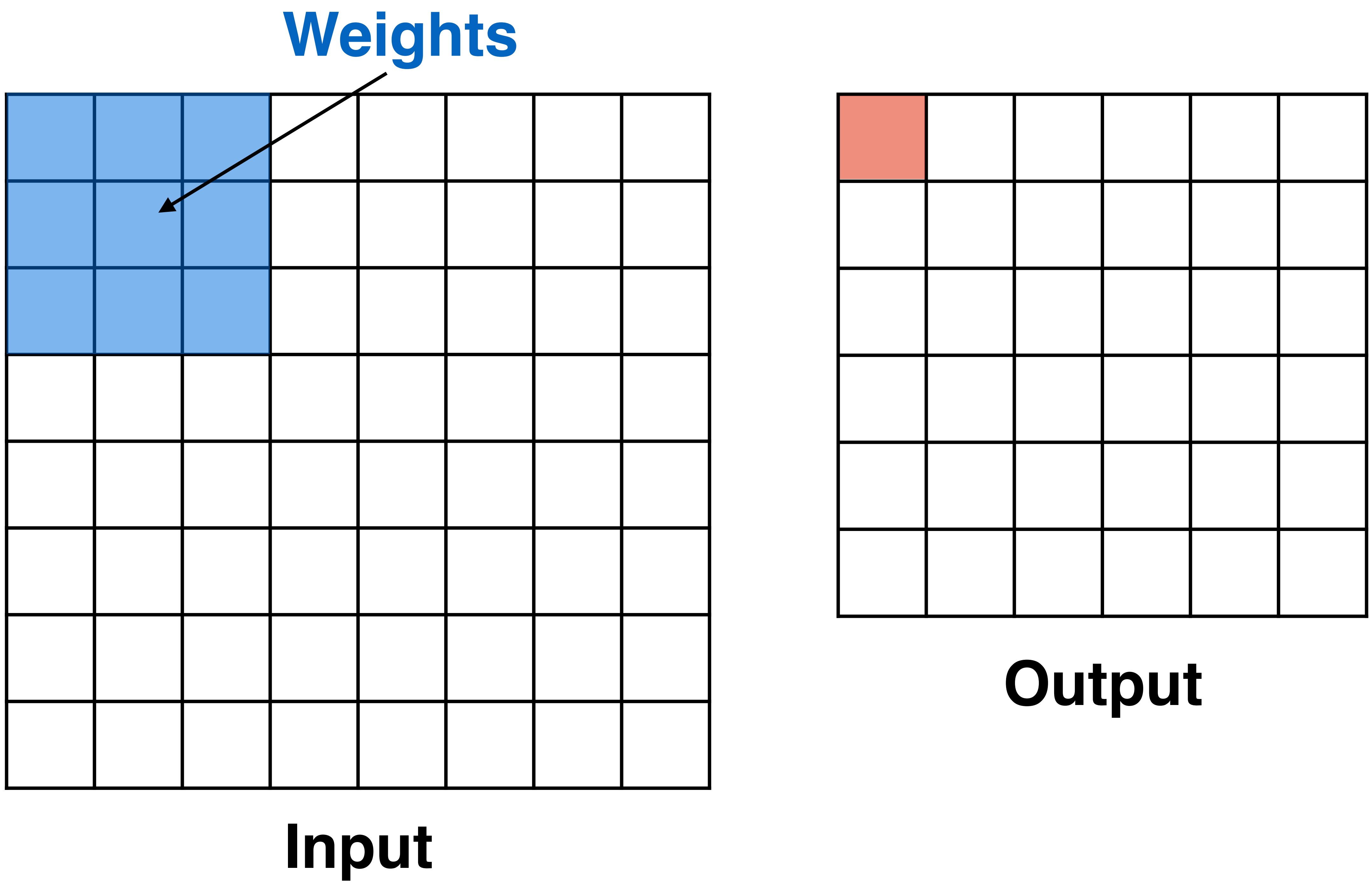


Figure: Andrej Karpathy

Questions?

Convolution: Stride

During convolution, the weights "slide" along the input to generate each output



Convolution: Stride

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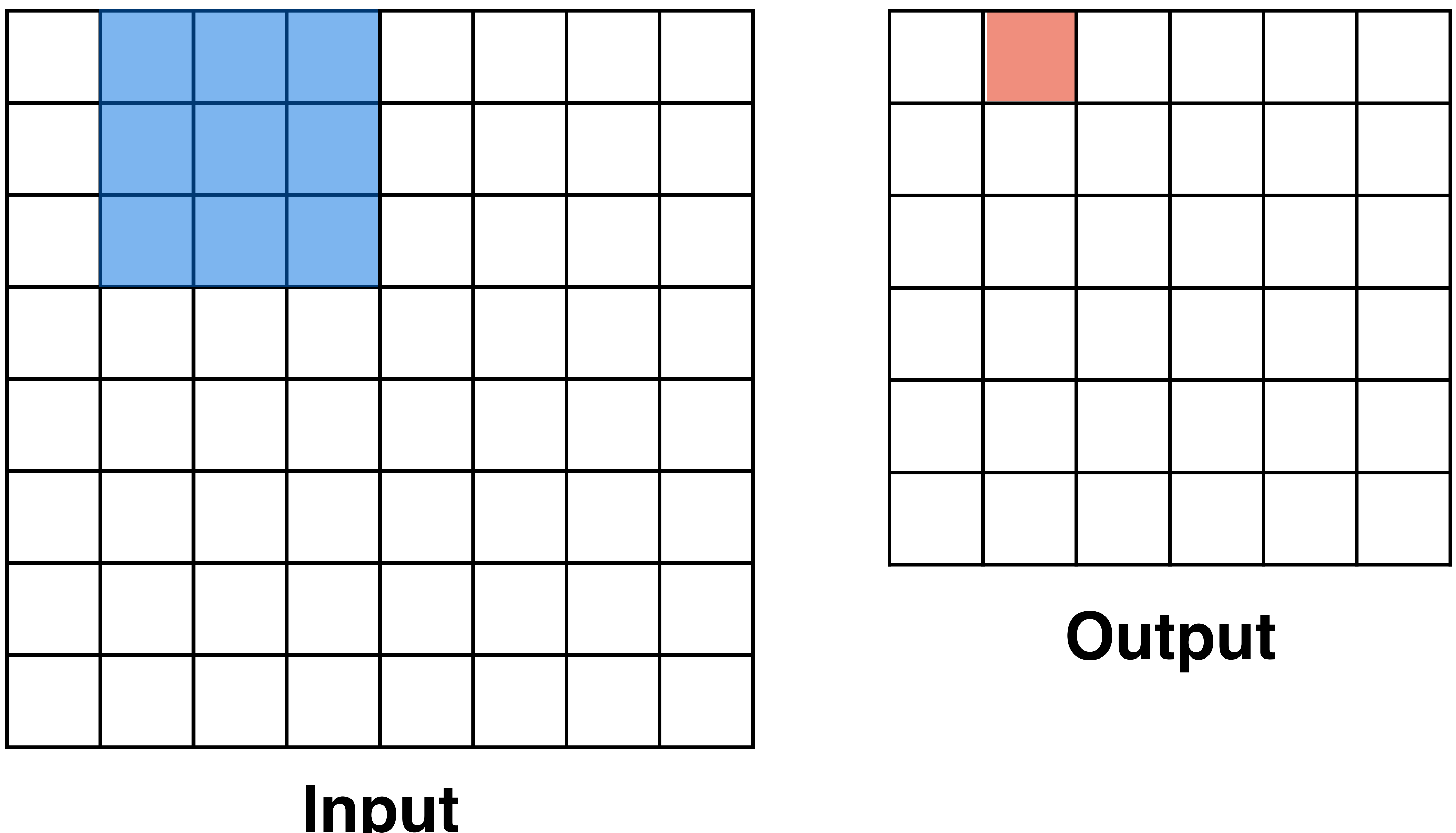

Convolution: Stride

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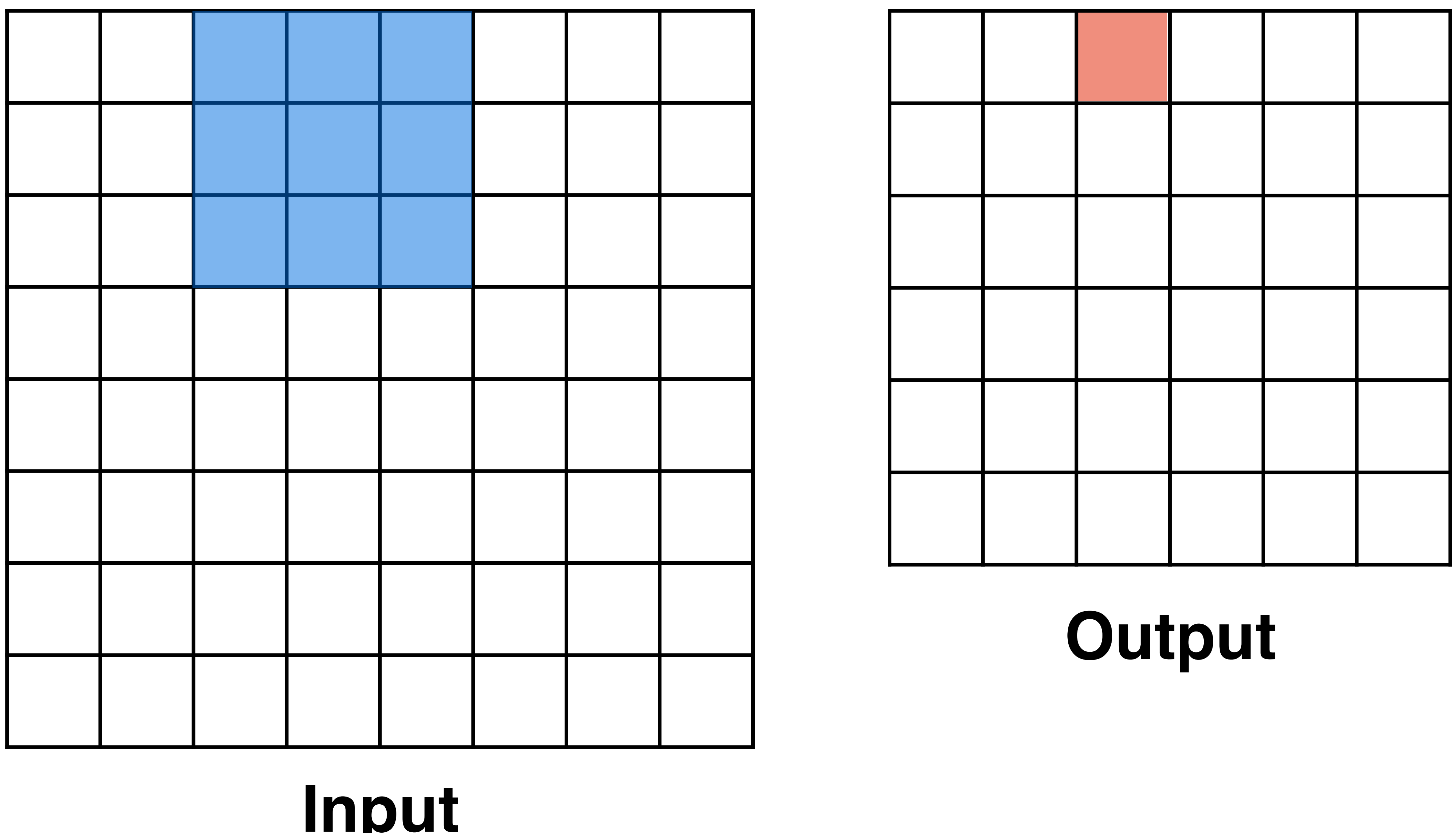

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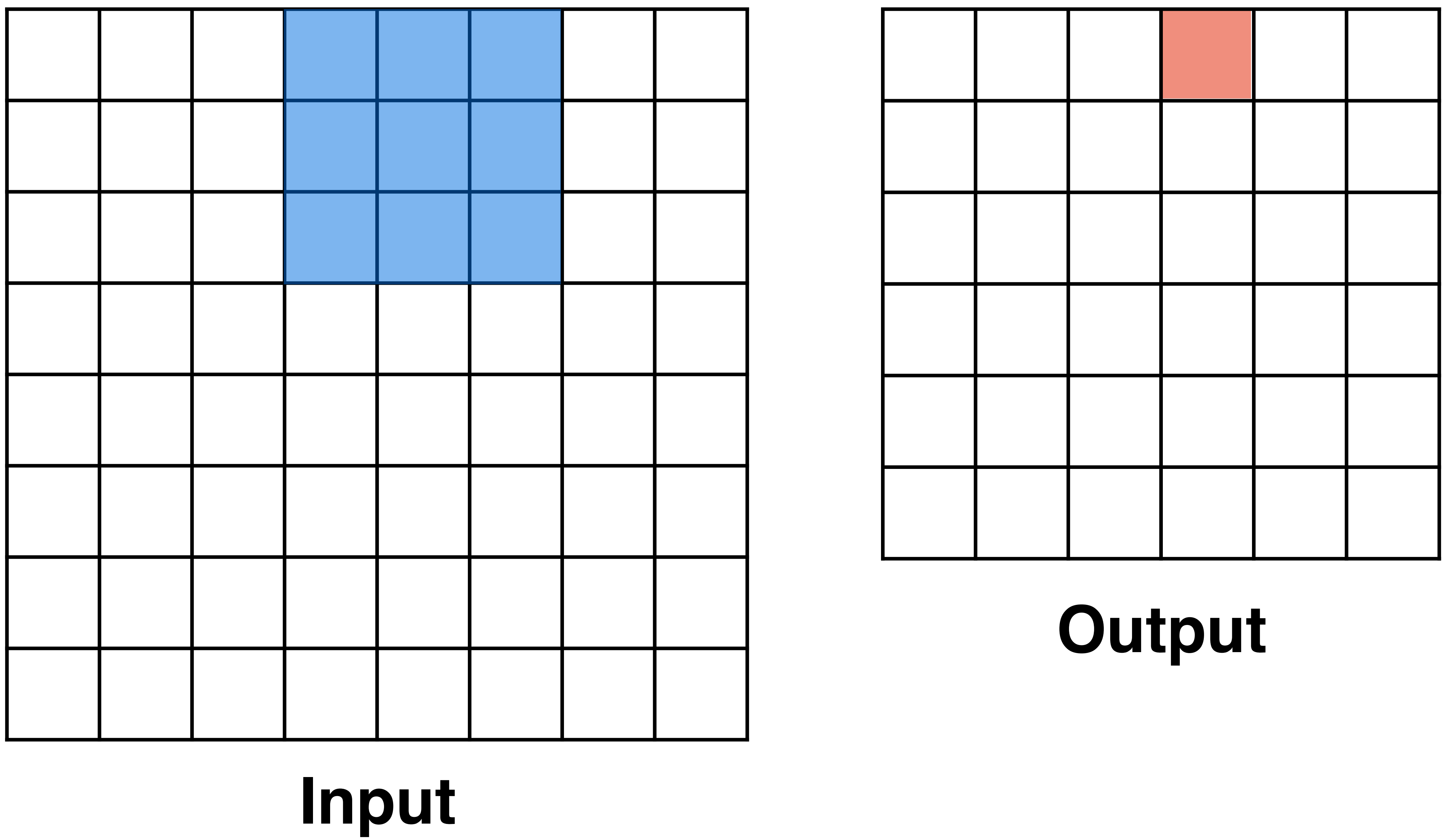

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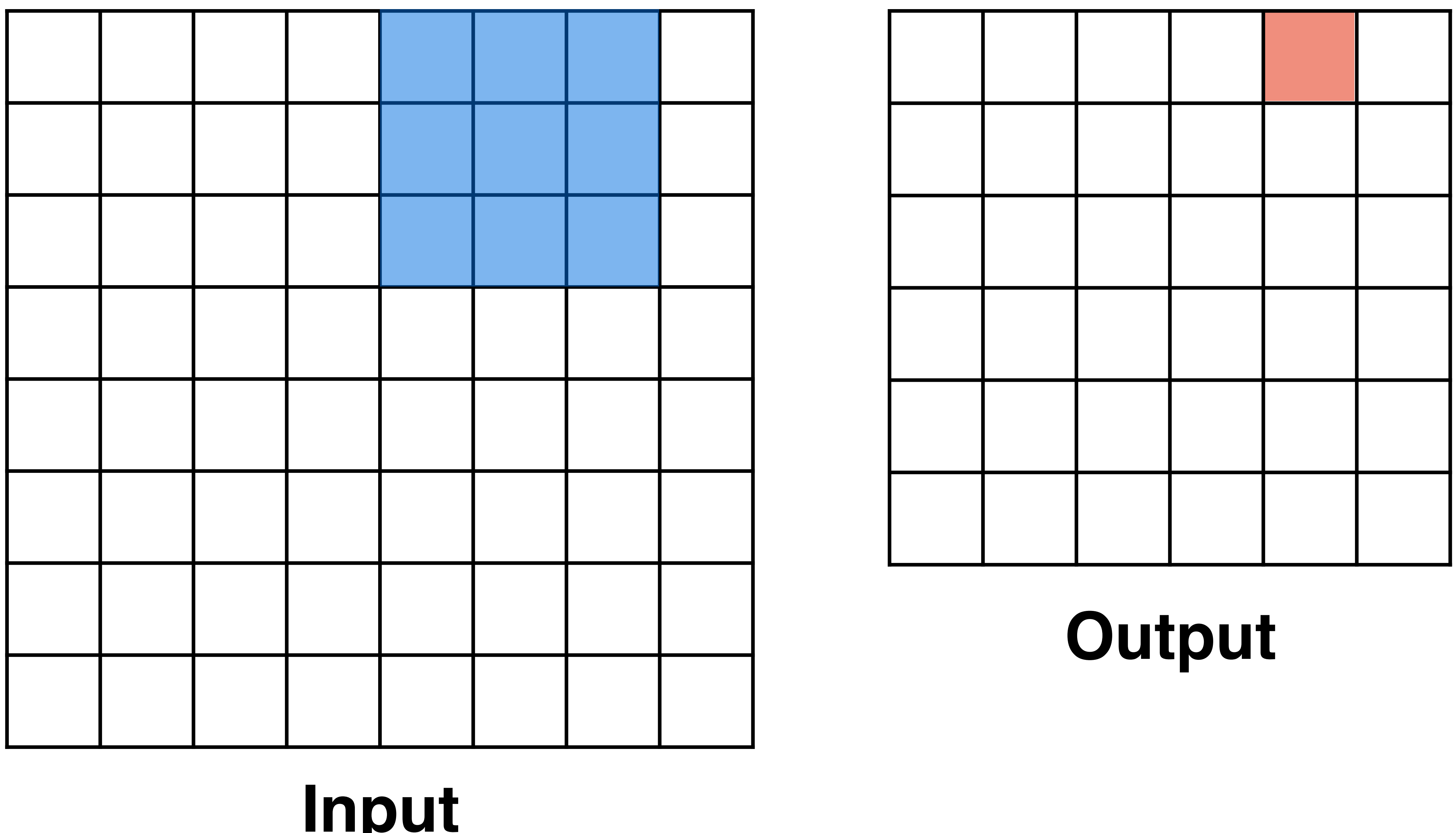

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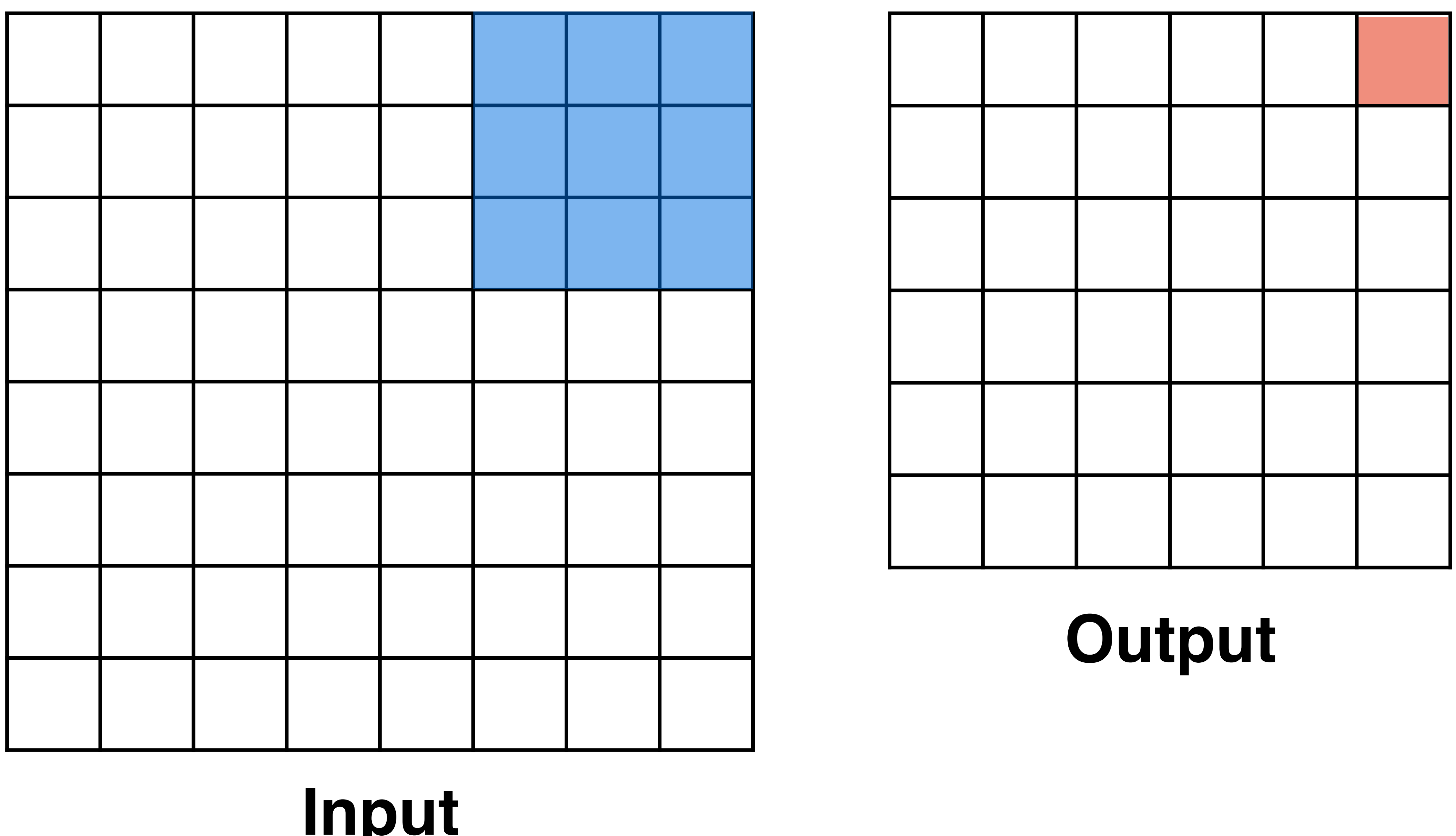

Convolution: Stride

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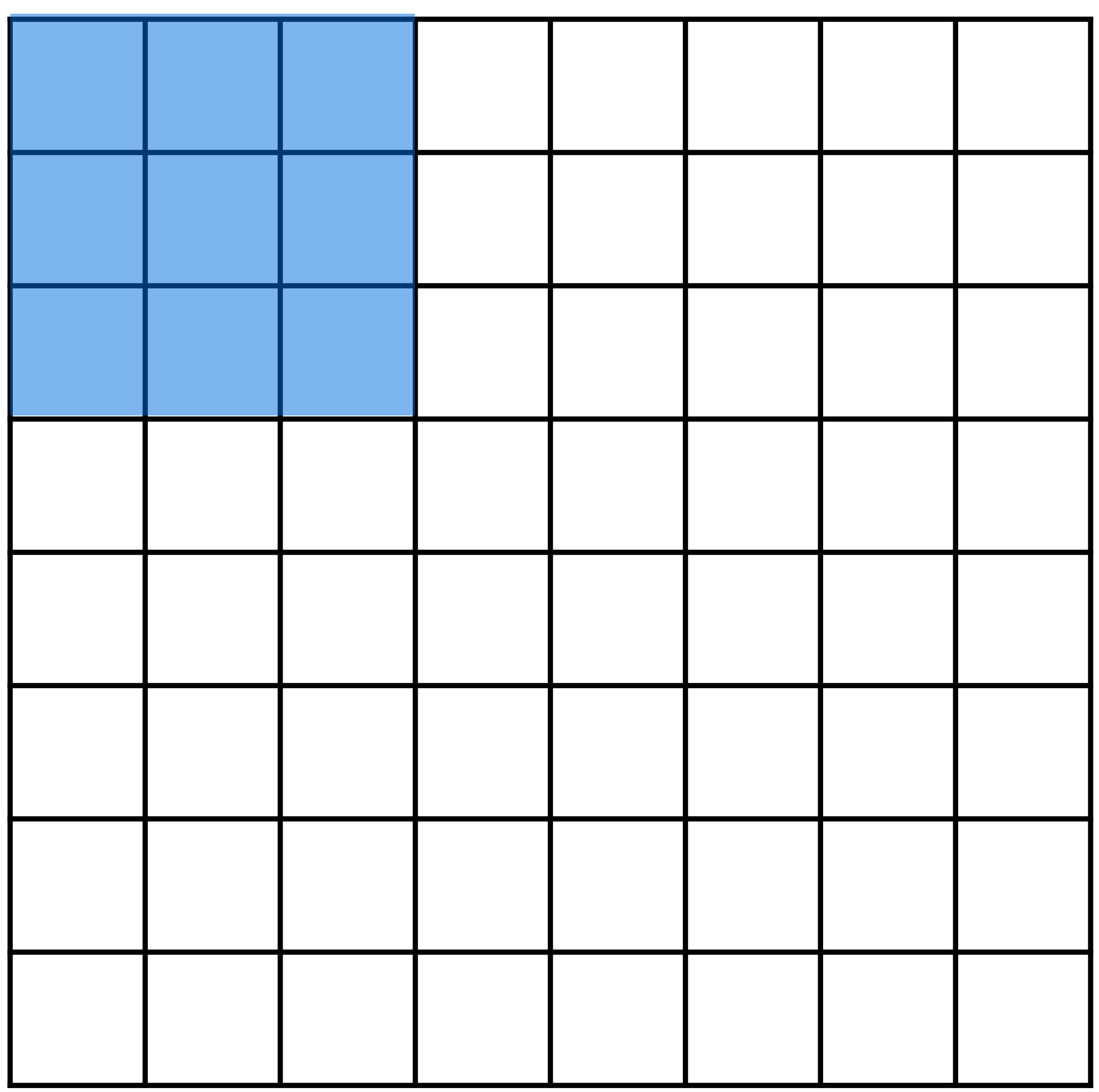


Input

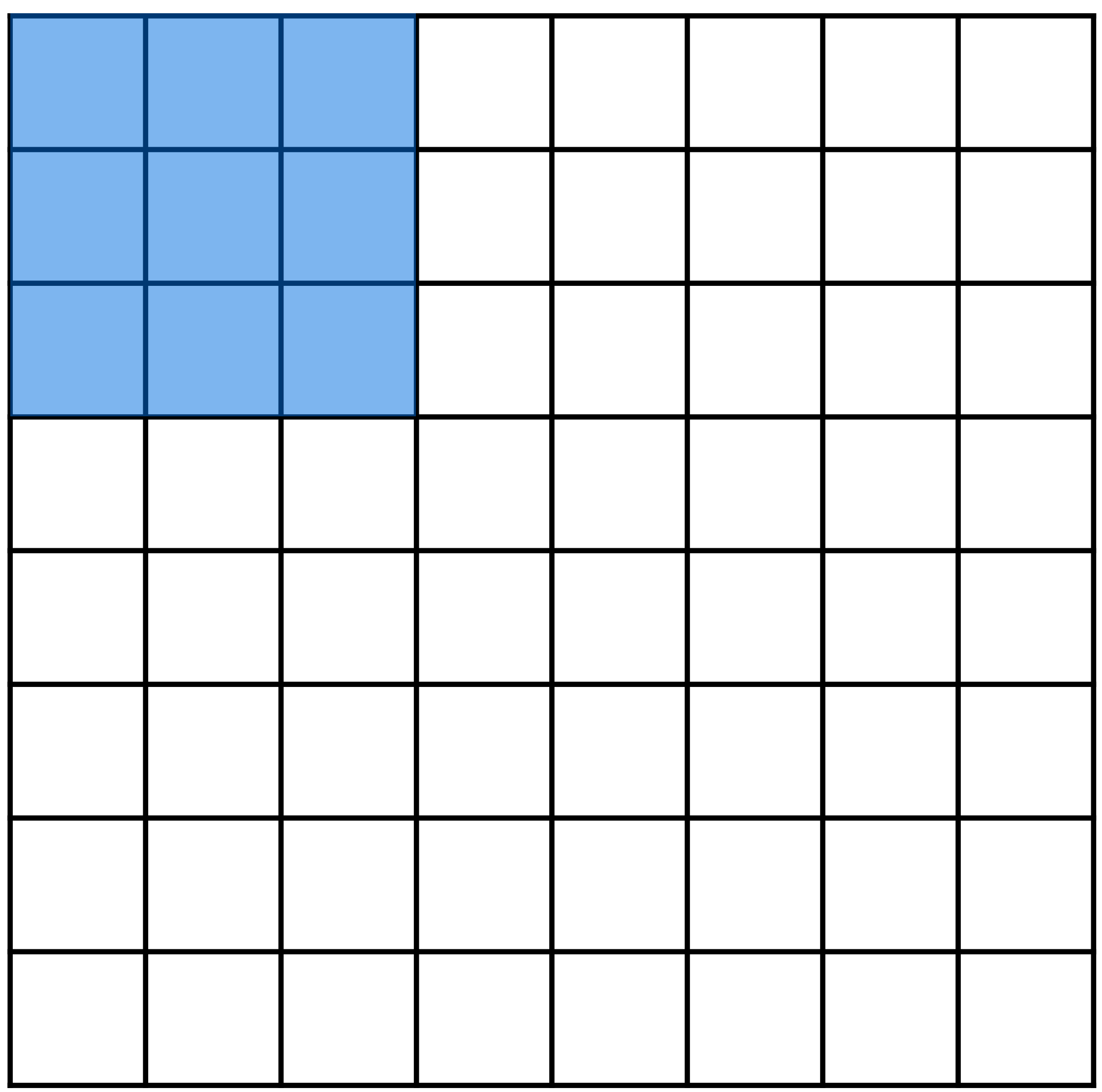
Recall that at each position, we are doing a **3D** sum:

$$h^r = \sum_{ijk} x^r{}_{ijk} W_{ijk} + b$$

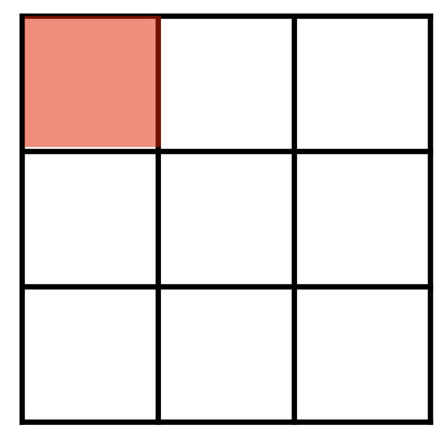
(channel, row, column)

Convolution: Stride

But we can also convolve with a **stride**, e.g. stride = 2



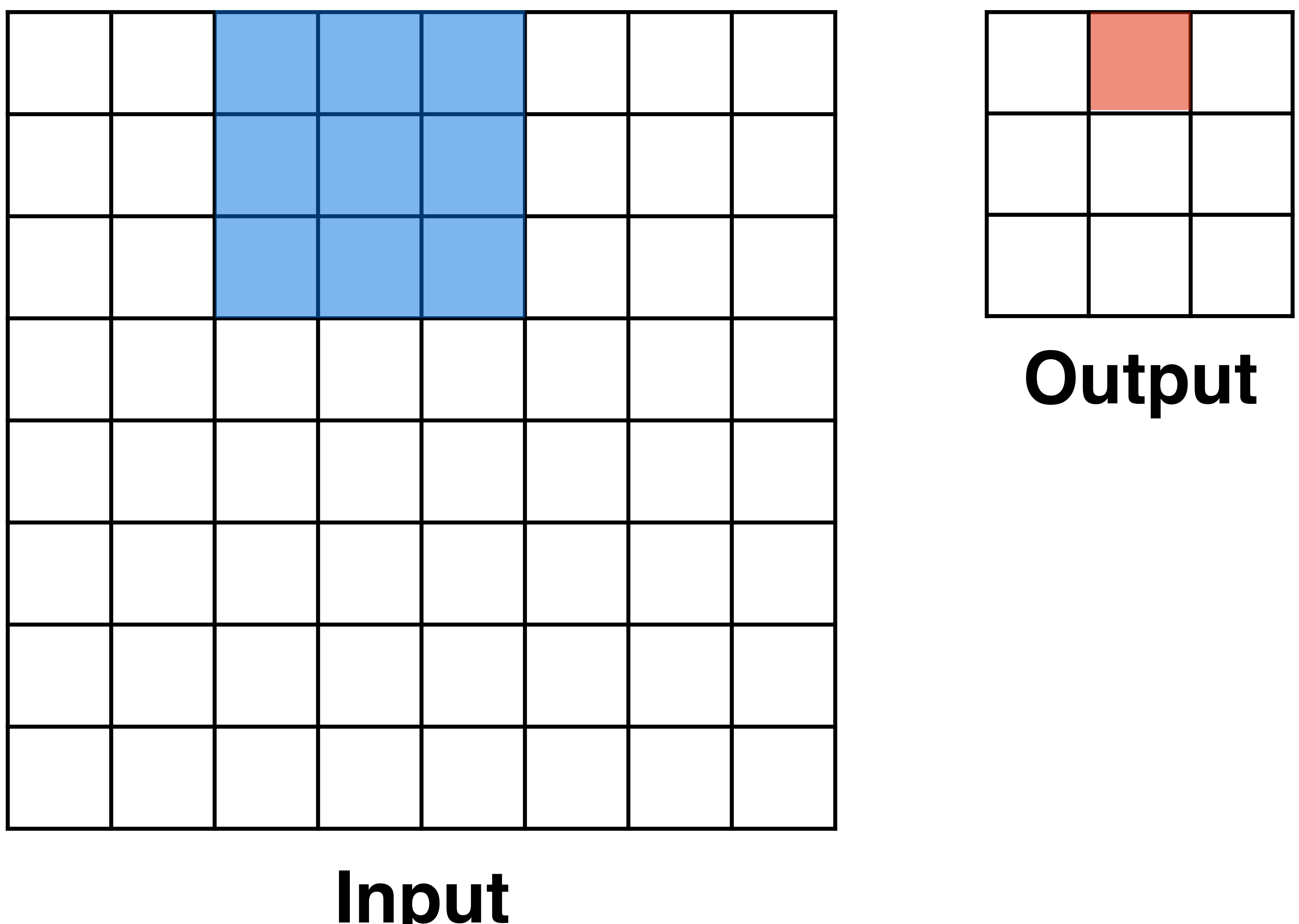
Input



Output

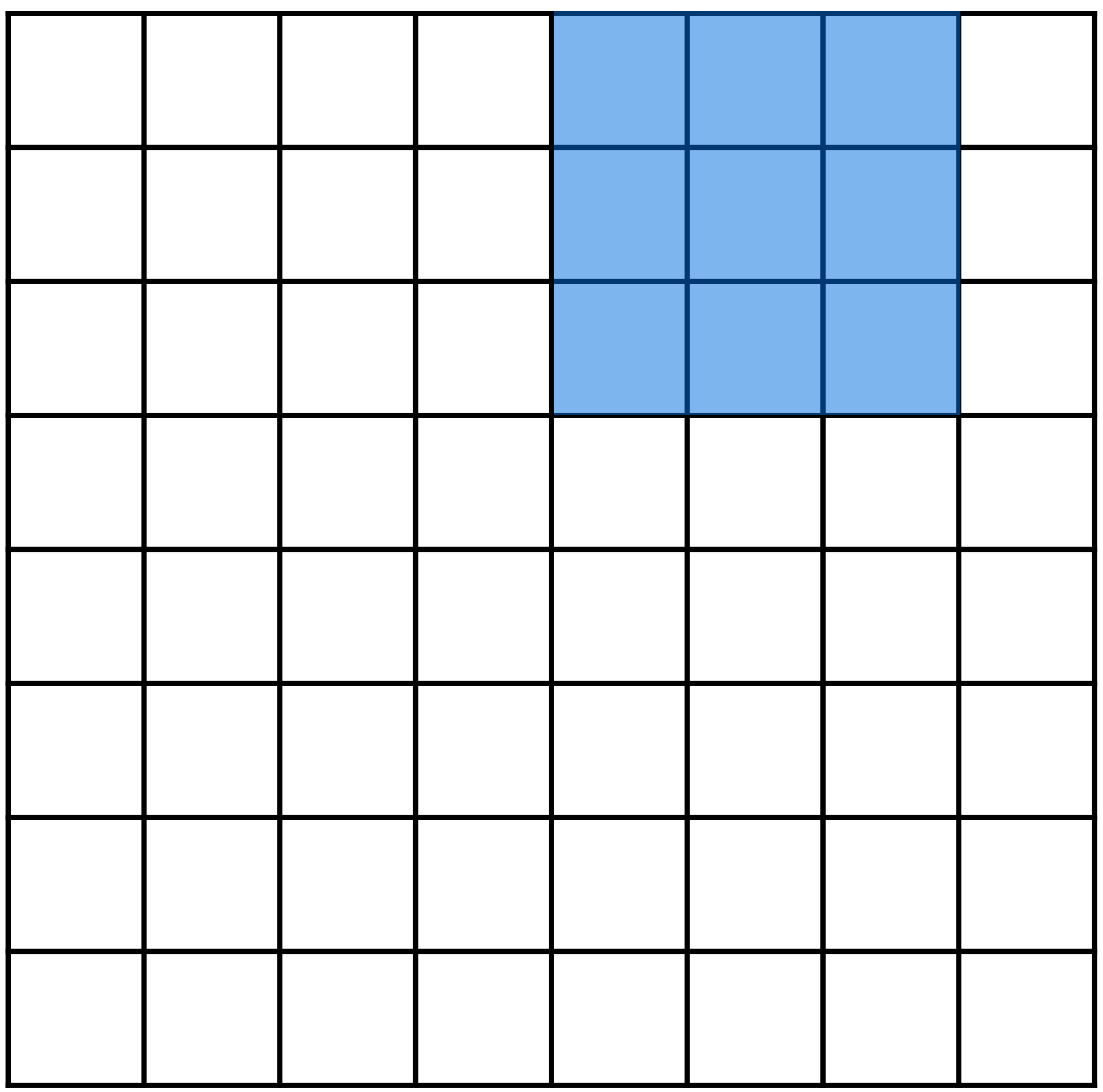
Convolution: Stride

But we can also convolve with a **stride**, e.g. stride = 2

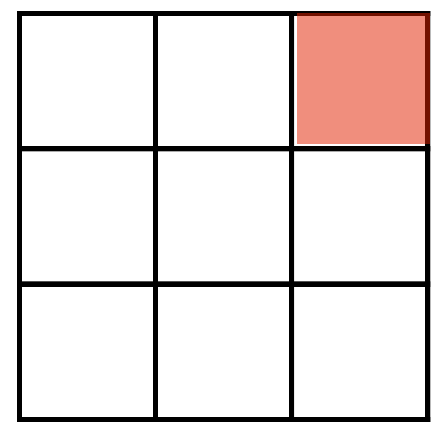


Convolution: Stride

But we can also convolve with a **stride**, e.g. stride = 2



Input



Output

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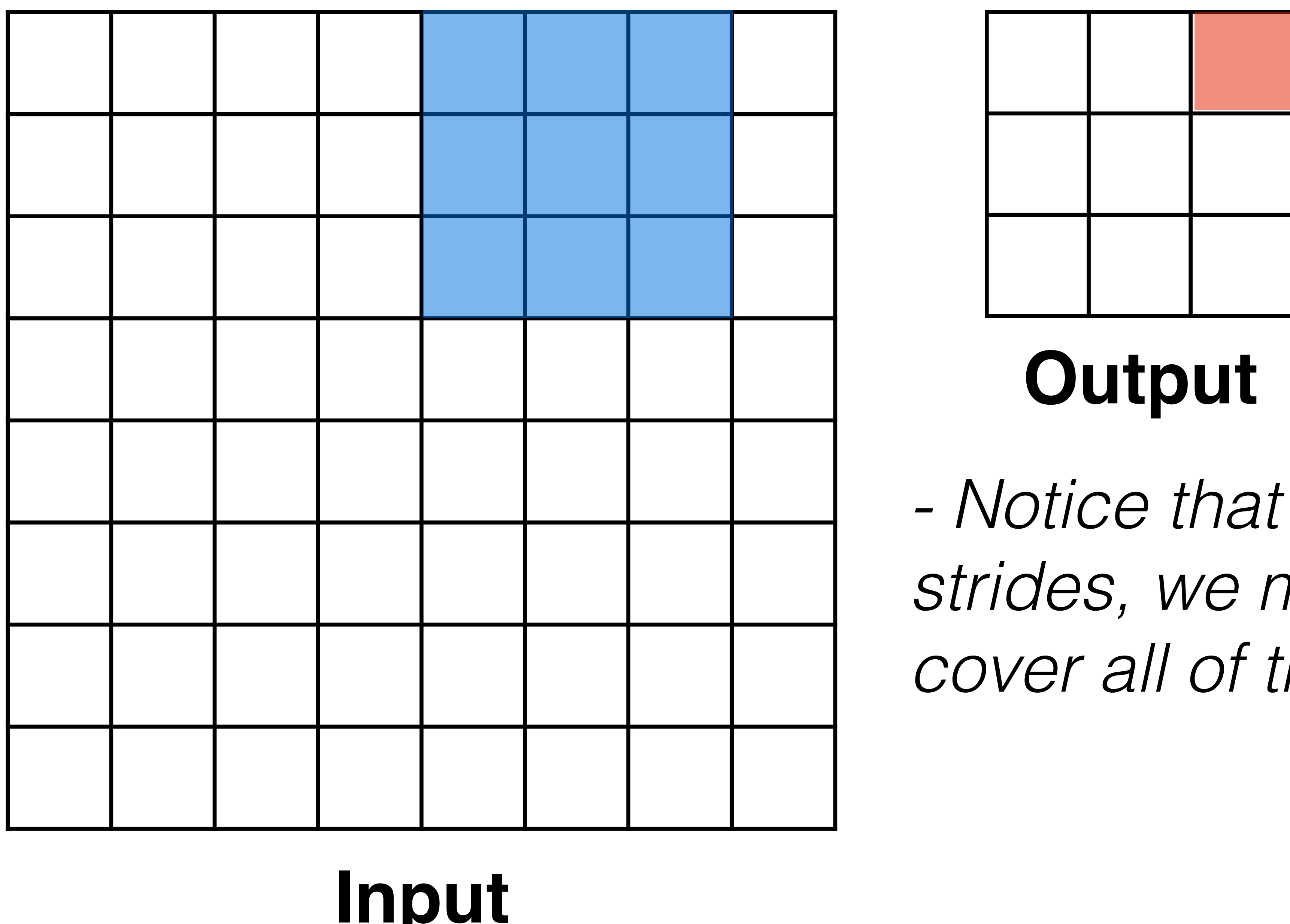

- Notice that with certain strides, we may not be able to cover all of the input

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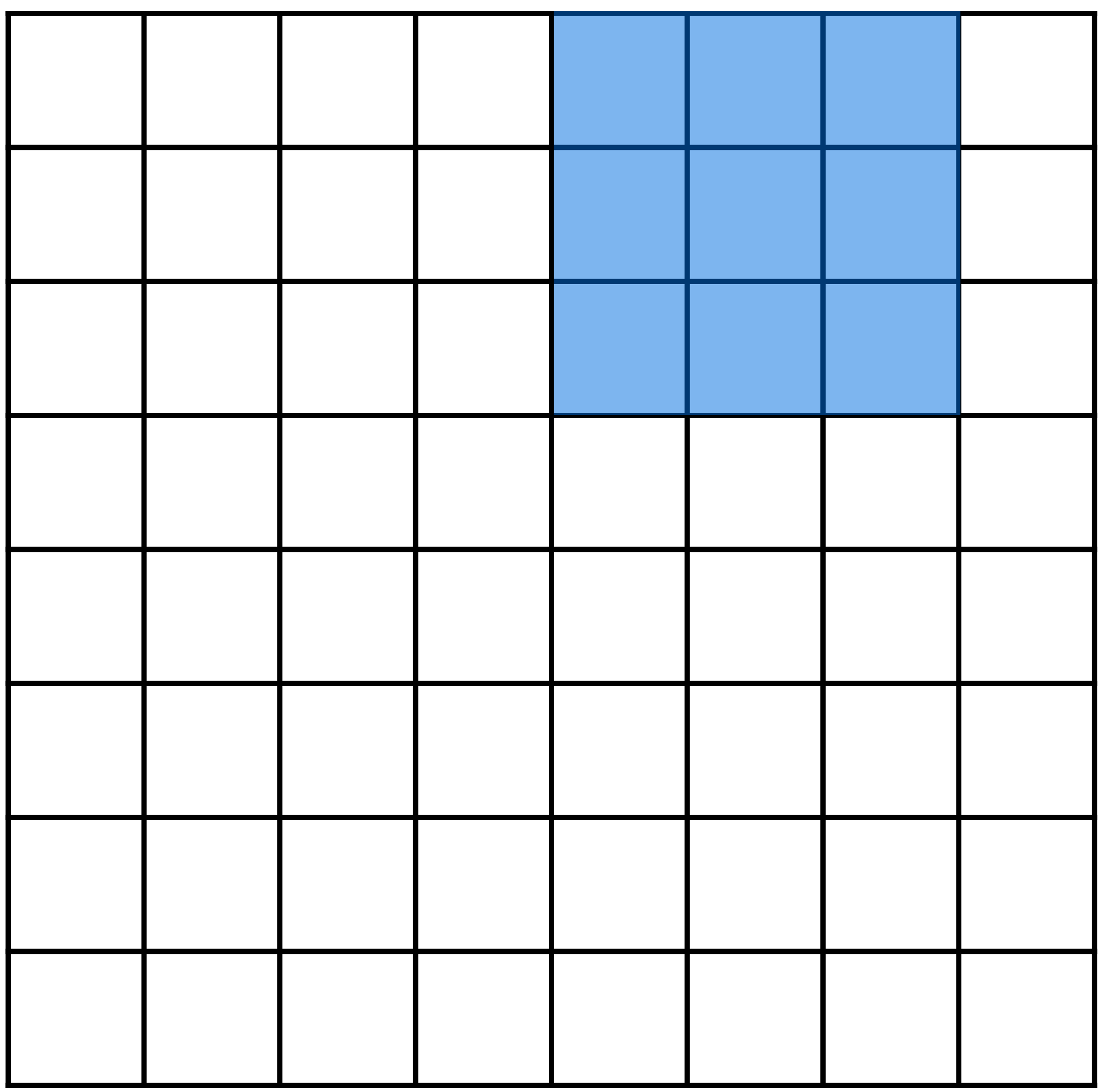
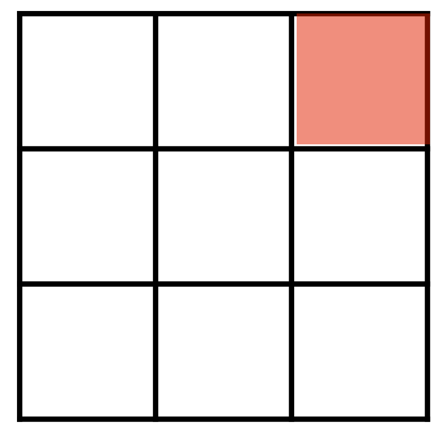


Input



Output

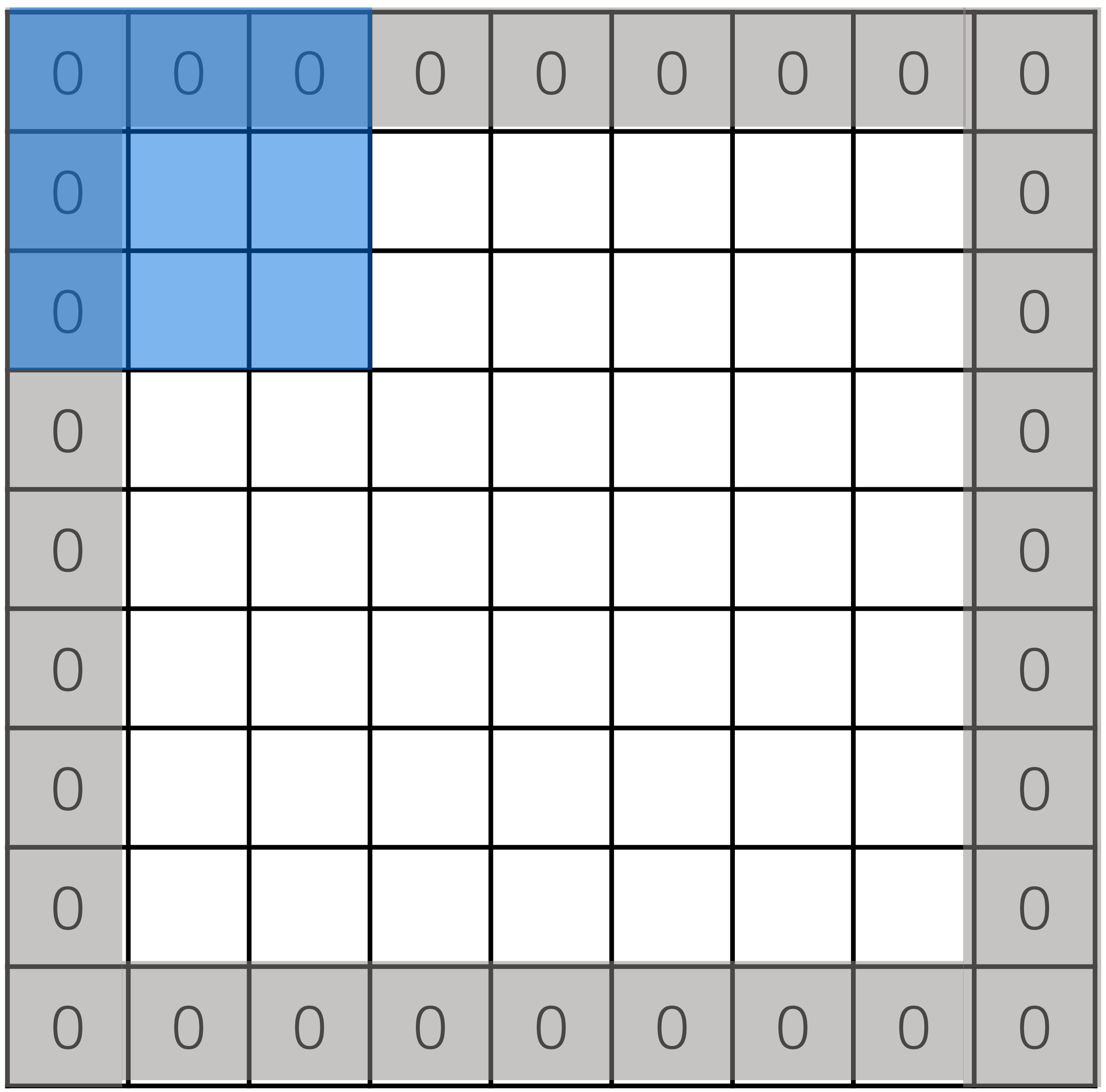
- *Notice that with certain strides, we may not be able to cover all of the input*

- *The output is also half the size of the input*

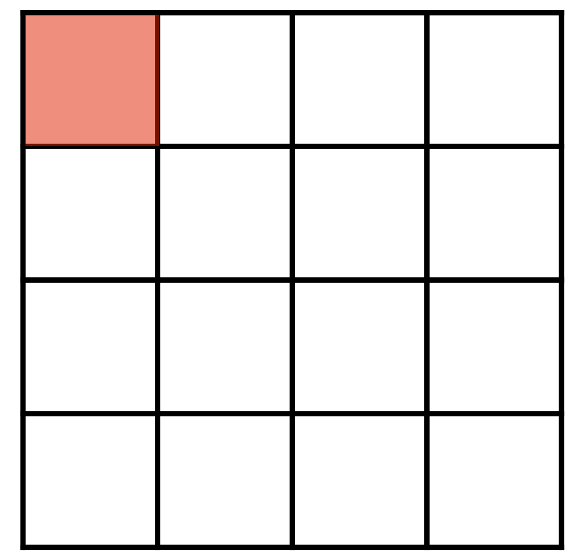
Convolution: Padding

We can also pad the input with zeros.

Here, **pad = 1, stride = 2**



Input



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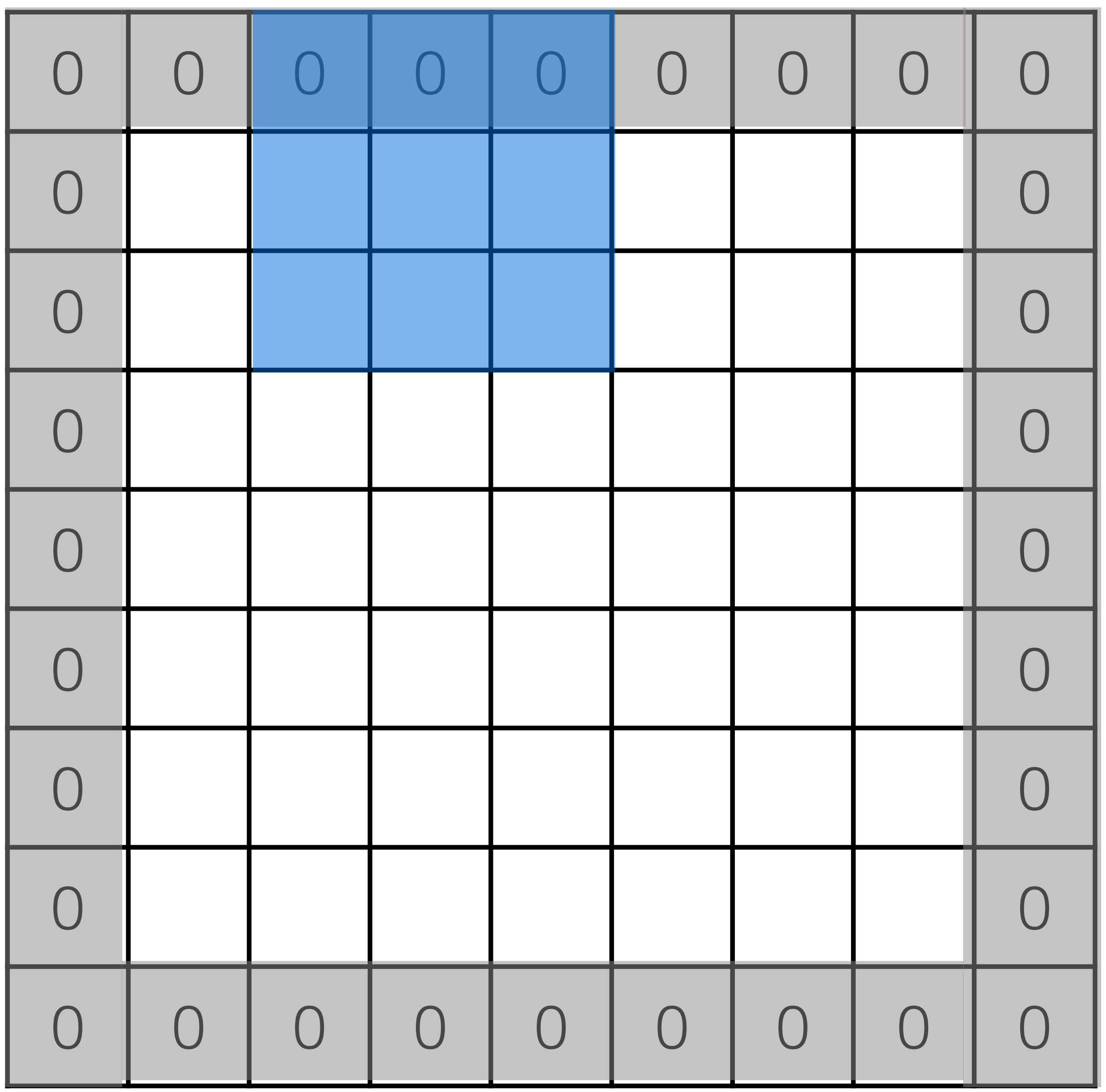
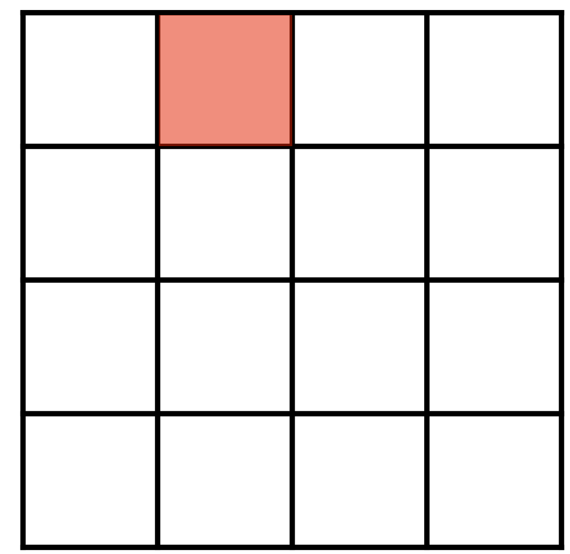


Input



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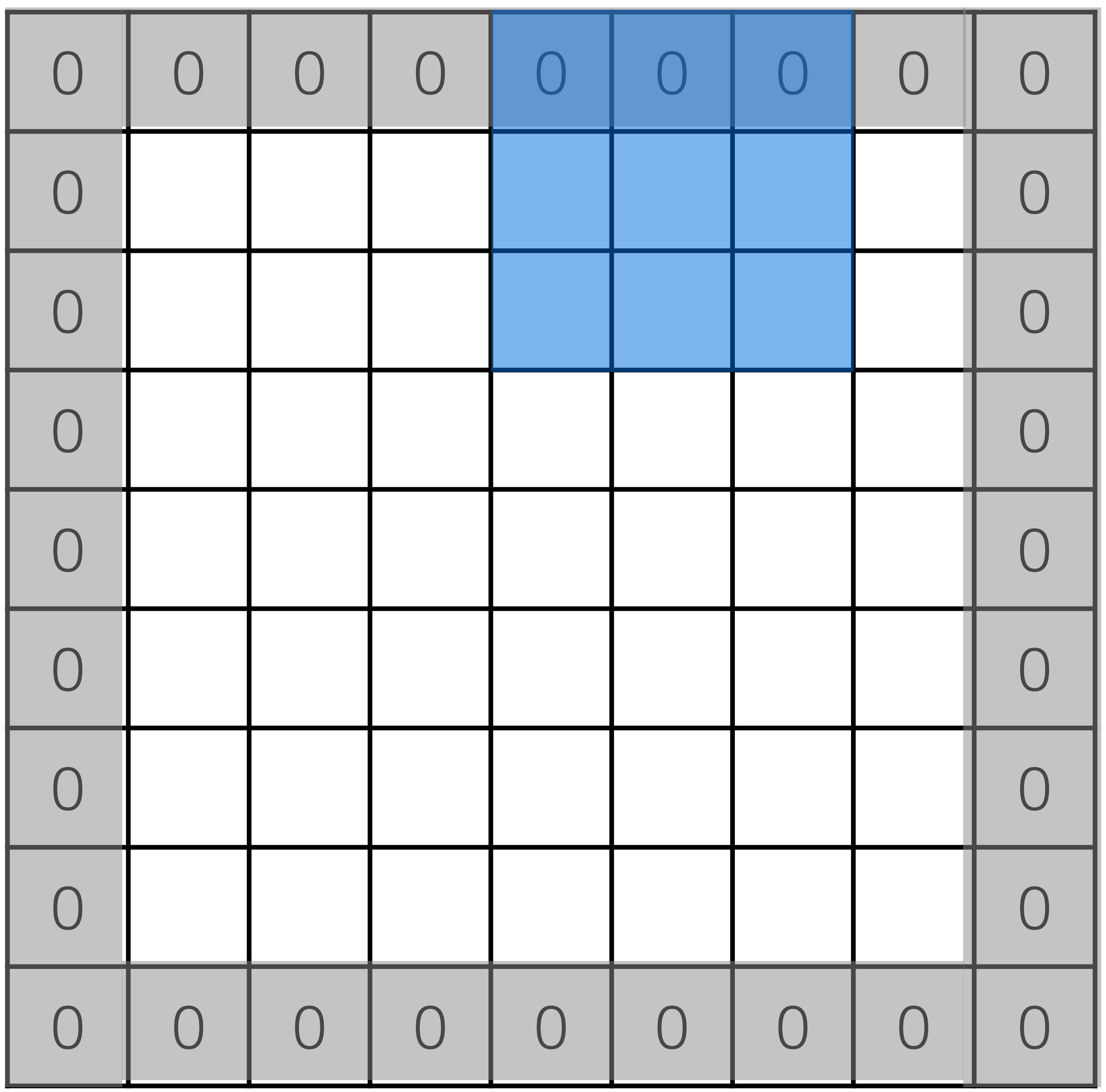

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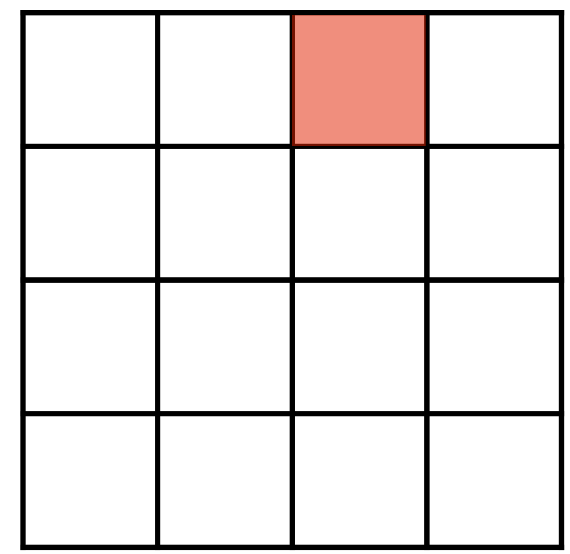

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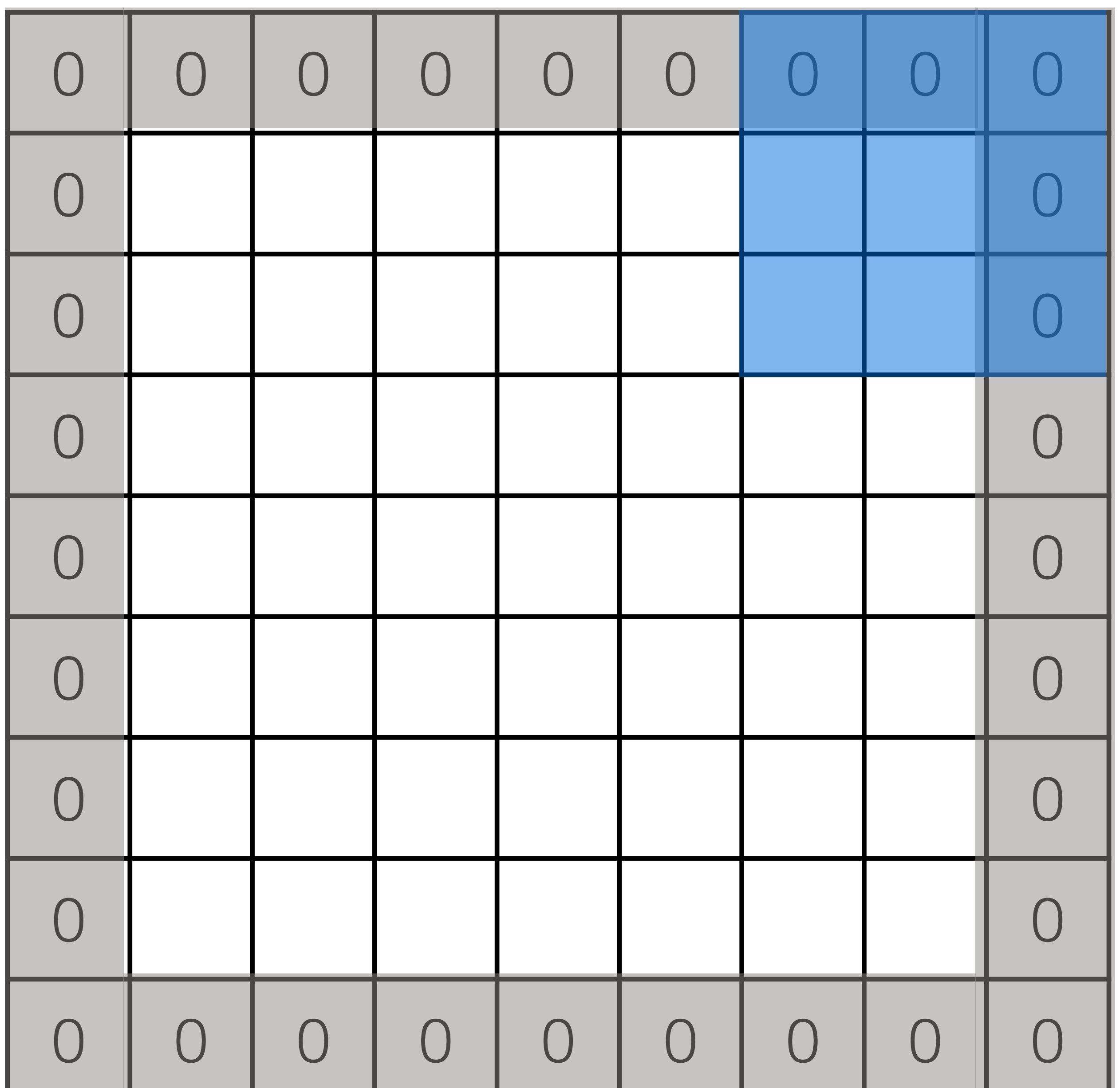
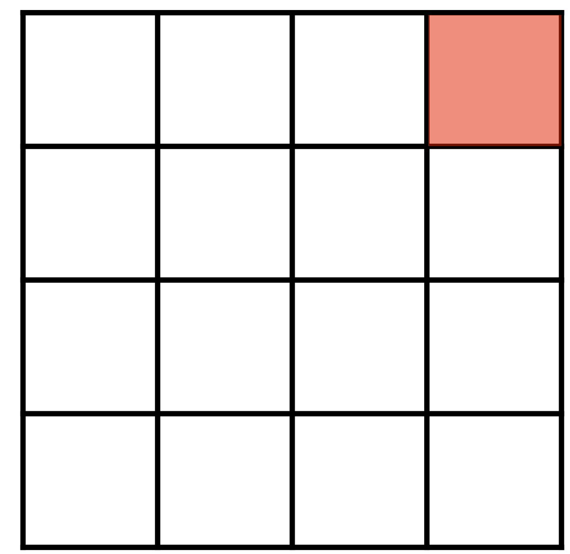


Input



Output

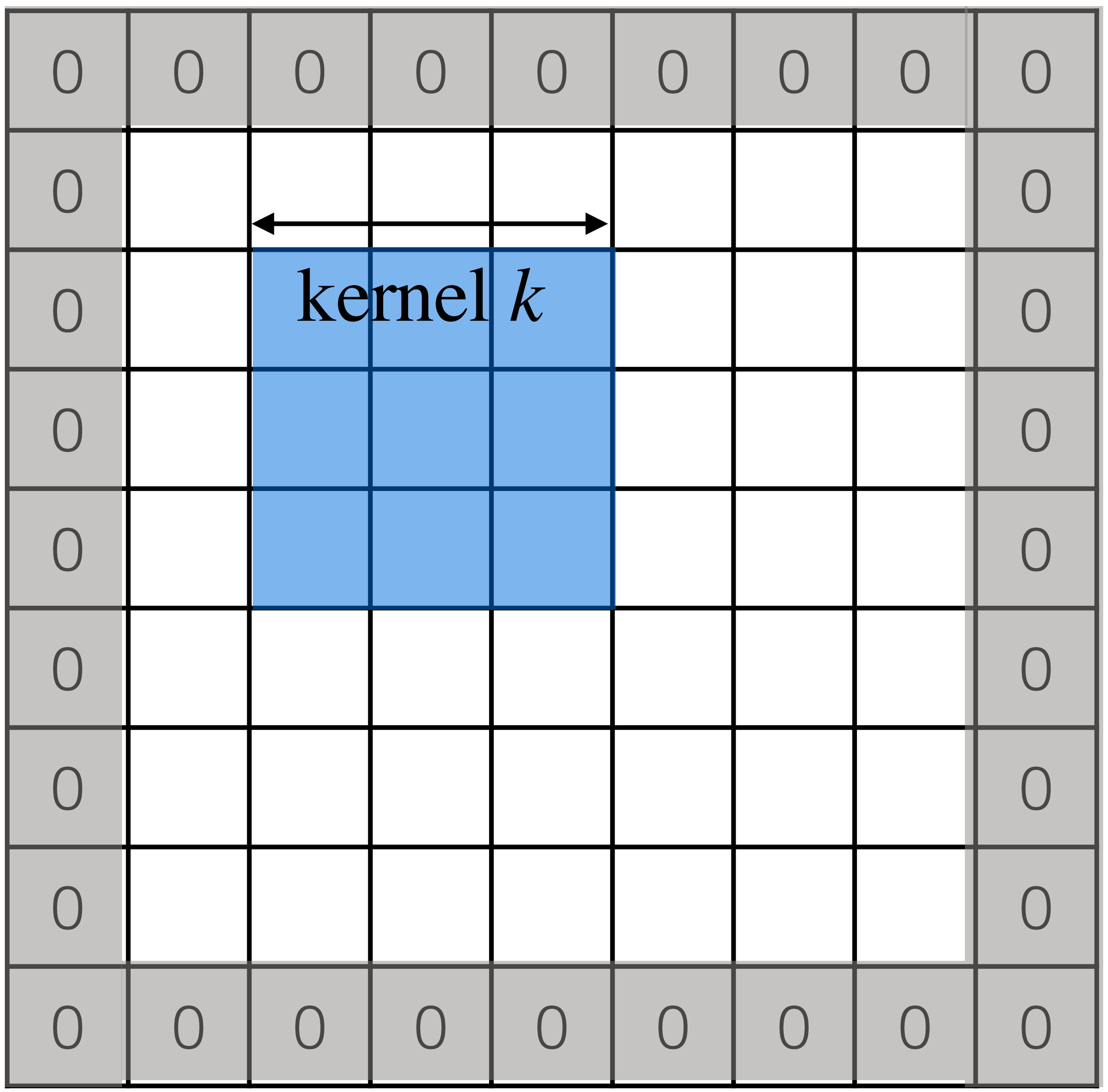
Convolution:

How big is the output?

0	0	0	0	0	0	0	0	0
0								0
0								0
0								0
0								0
0								0
0								0
0								0
0	0	0	0	0	0	0	0	0

Convolution:

How big is the output?



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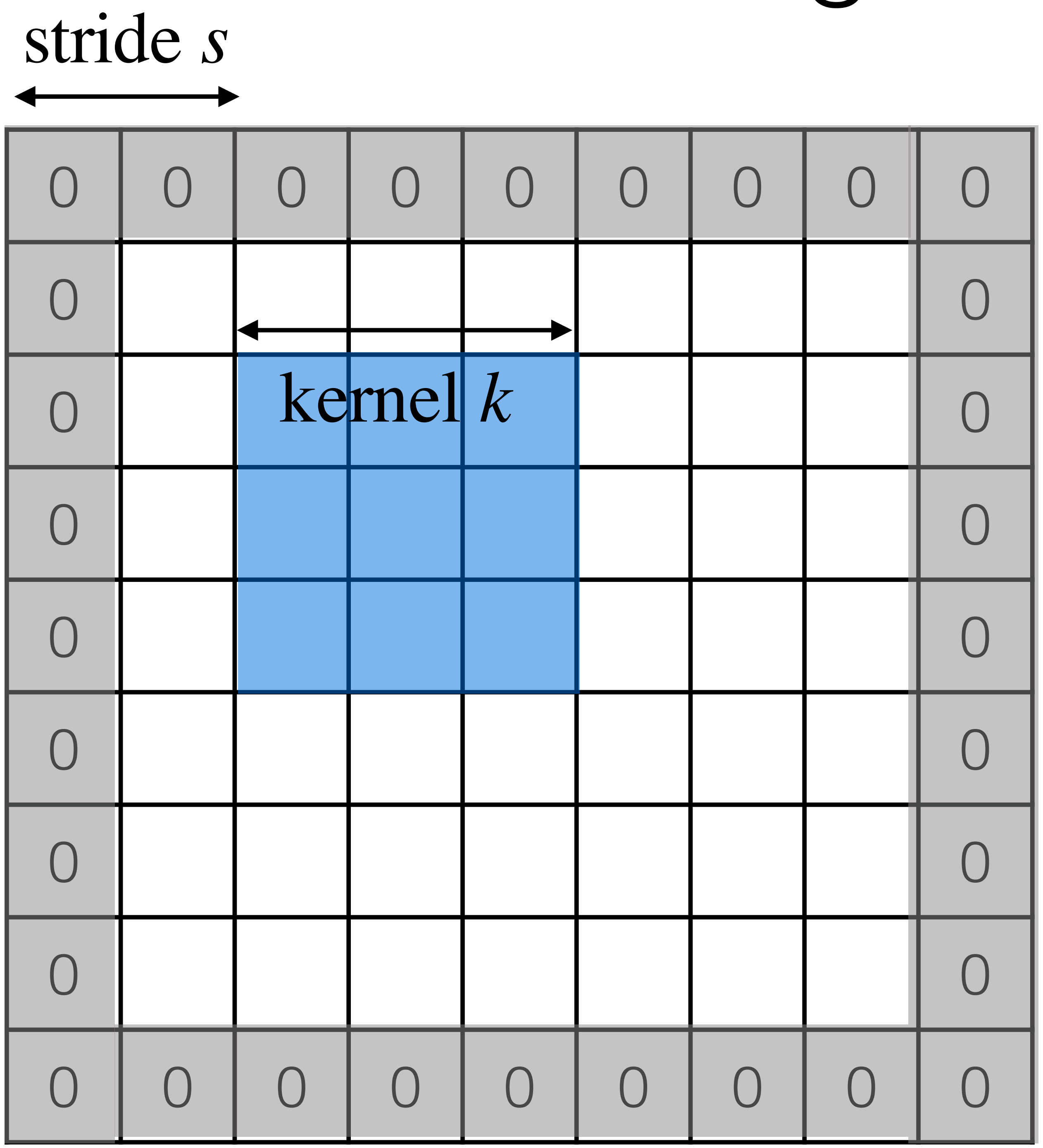

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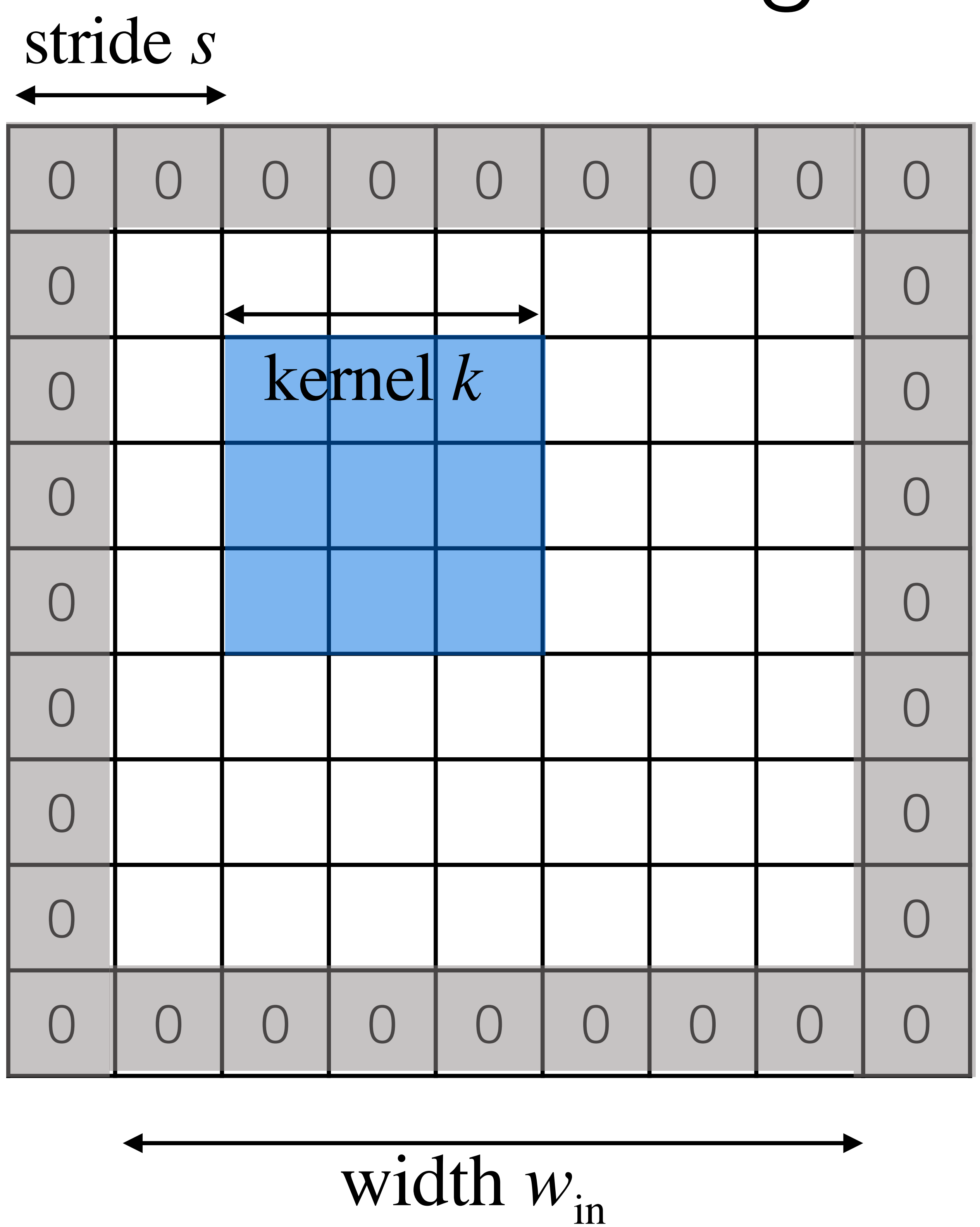

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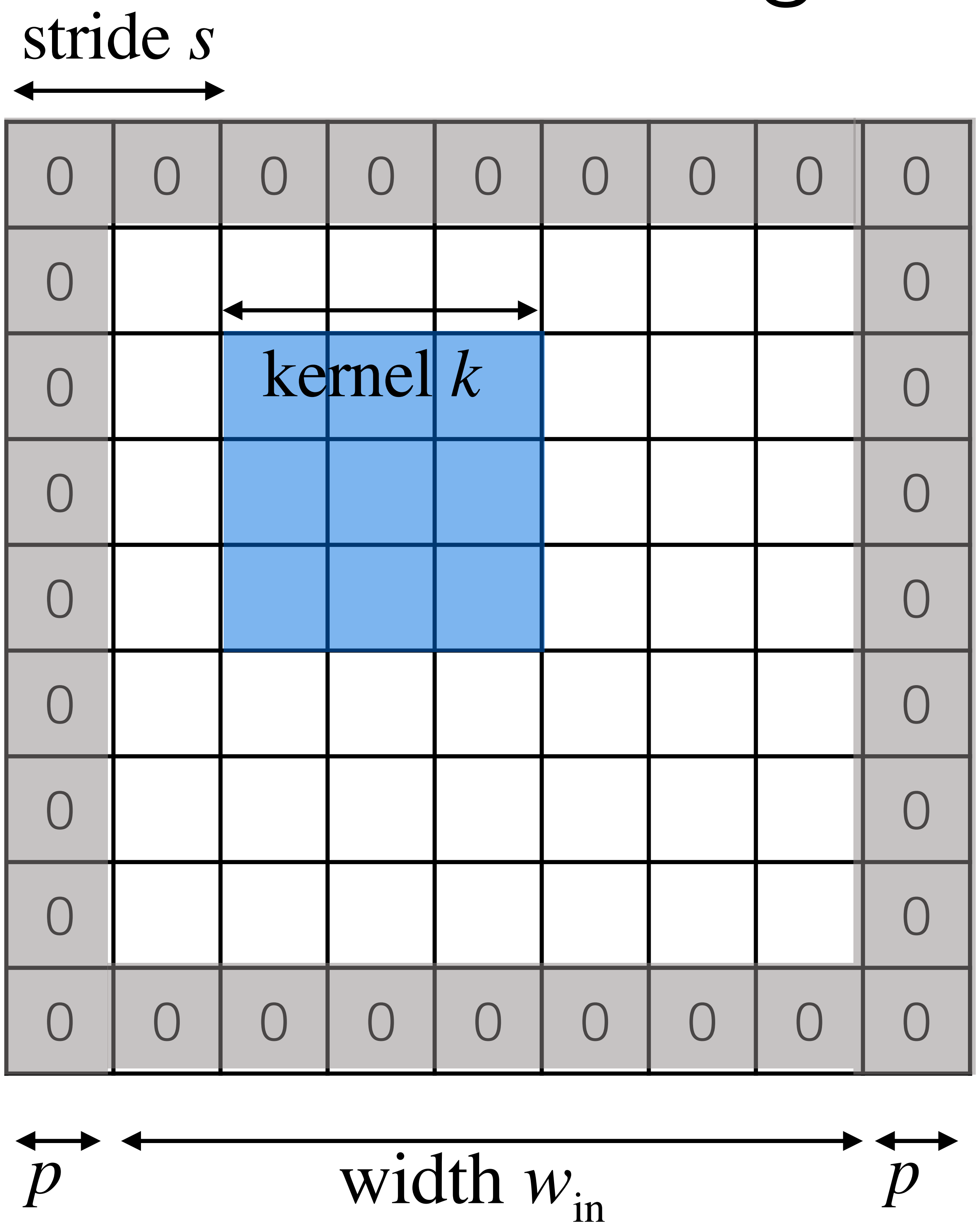

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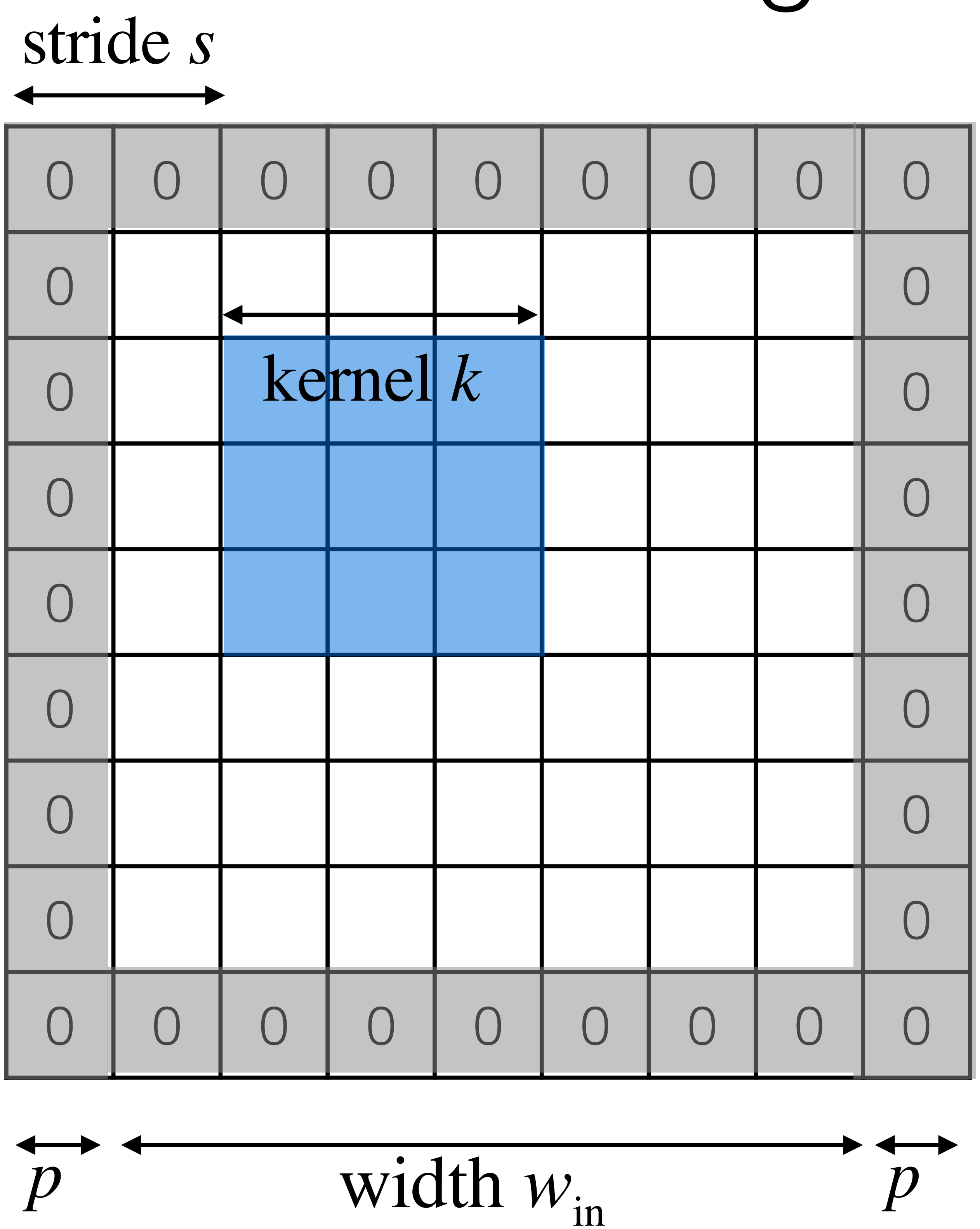
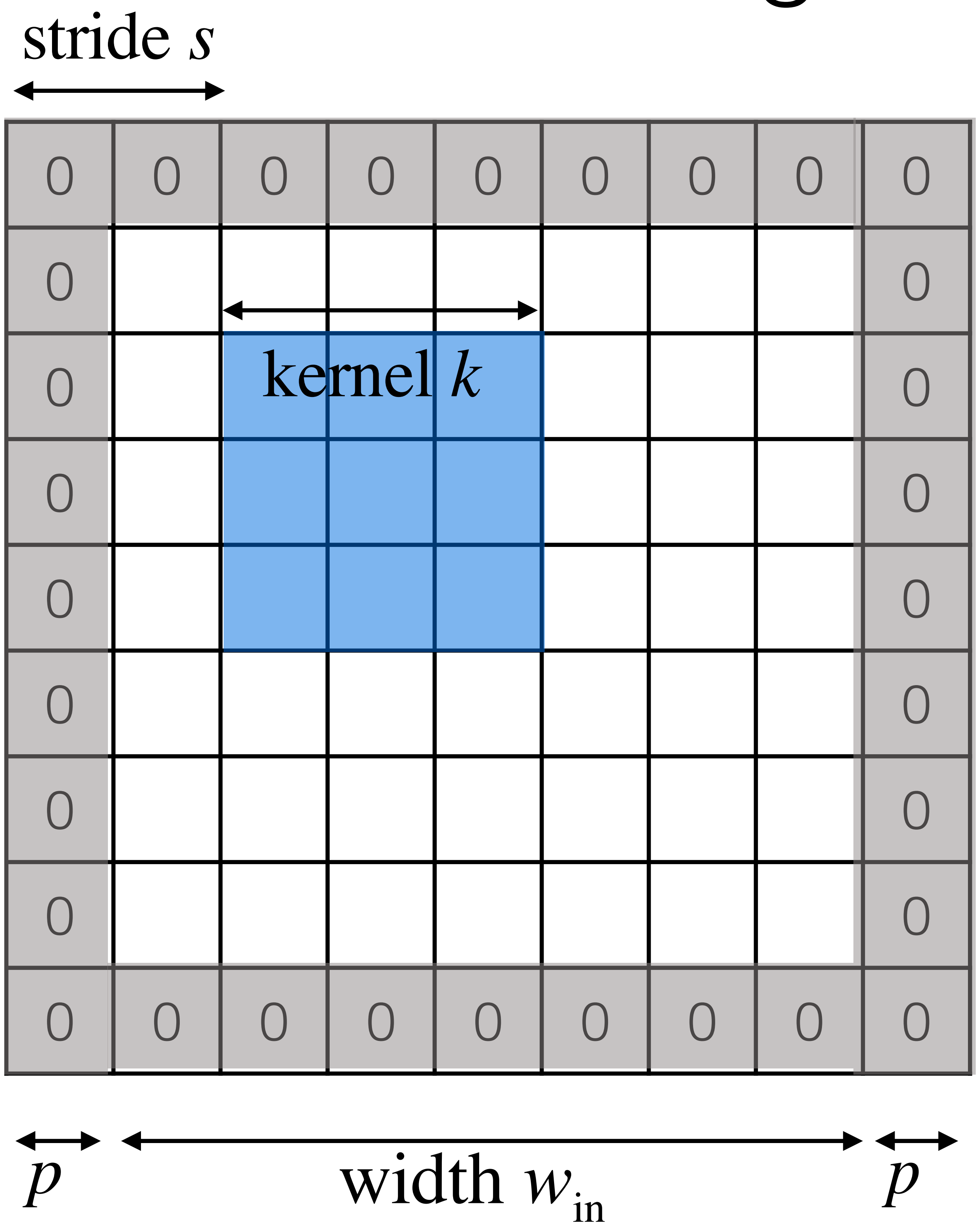


In general, the output has size:

$$w_{\text{out}} = \left\lfloor \frac{w_{\text{in}} + 2p - k}{s} \right\rfloor + 1$$

Convolution:

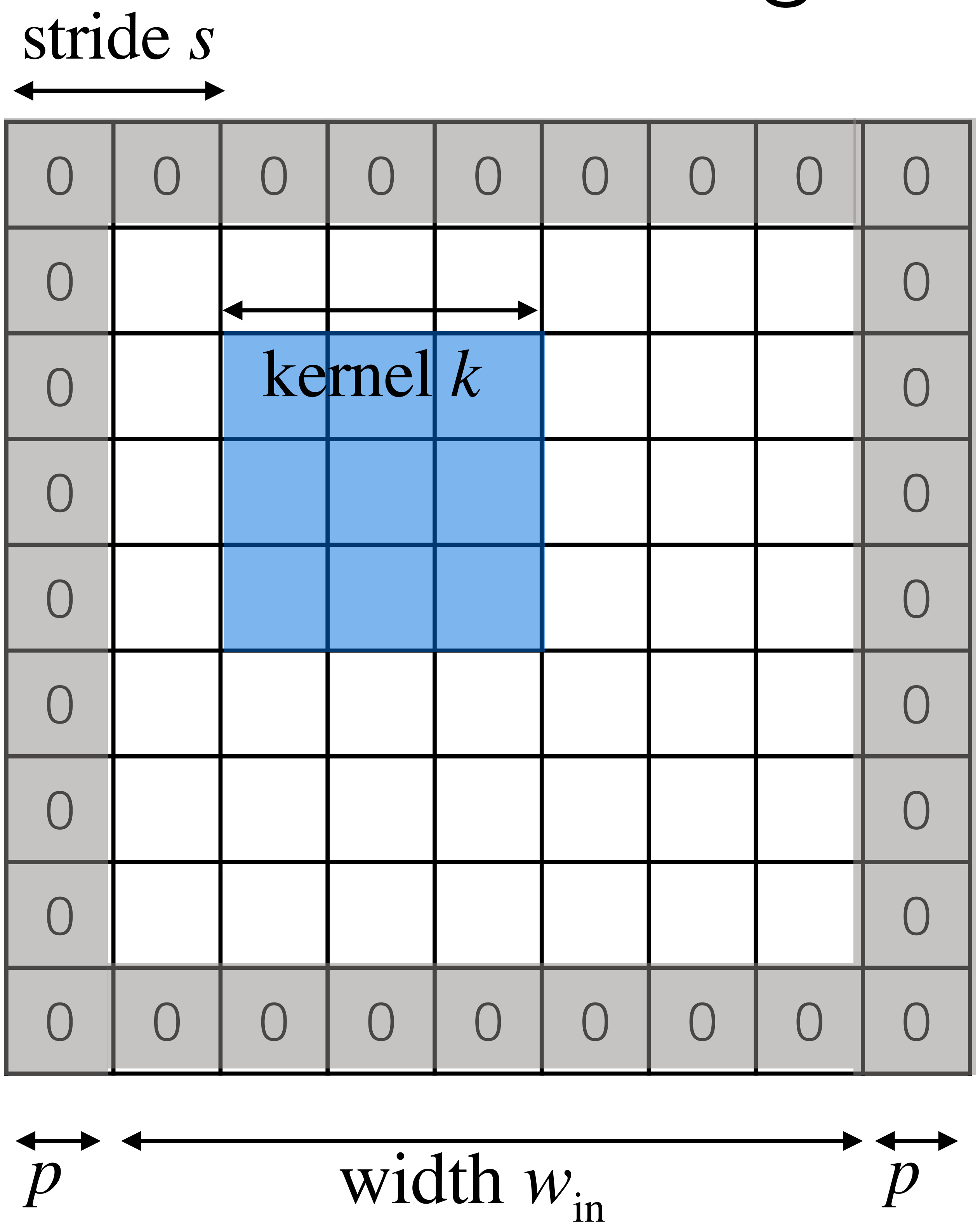
How big is the output?



Example: k=3, s=1, p=1

Convolution:

How big is the output?



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$$w_{\text{out}} = \left\lfloor \frac{w_{\text{in}} + 2p - k}{s} \right\rfloor + 1$$

$$= \left\lfloor \frac{w_{\text{in}} + 2 - 3}{1} \right\rfloor + 1$$

$$= w_{\text{in}}$$

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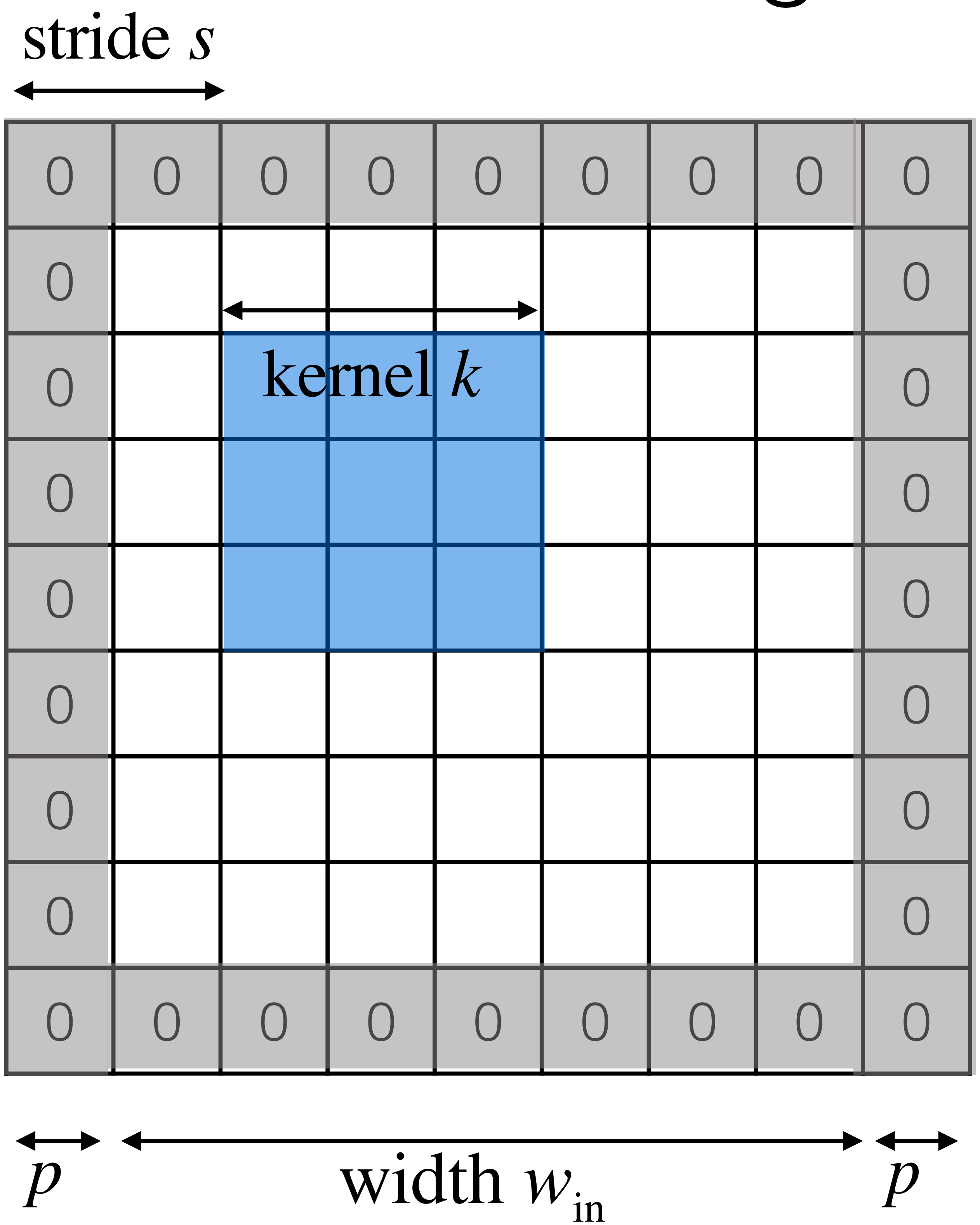

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VGGNet [Simonyan 2014] uses filters of this shape

Max Pooling

For most CNNs, **convolution** is often followed by **pooling**:

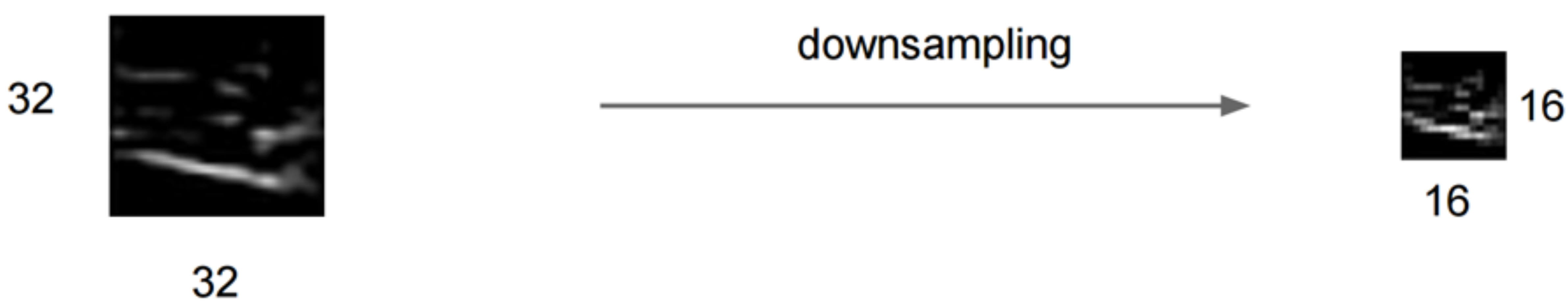
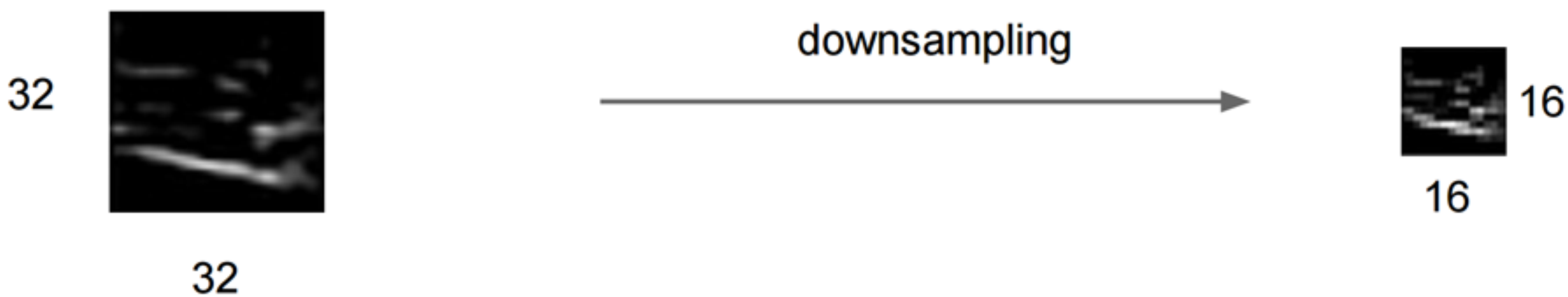


Figure: Andrej Karpathy

Max Pooling

For most CNNs, **convolution** is often followed by **pooling**:

- Creates a smaller representation while retaining the most important information



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For most CNNs, **convolution** is often followed by **pooling**:

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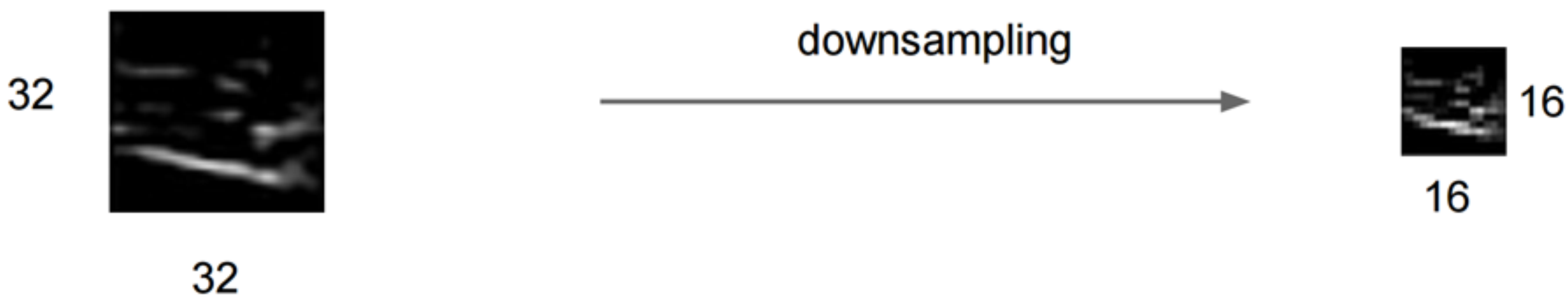
- The "max" operation is the most common



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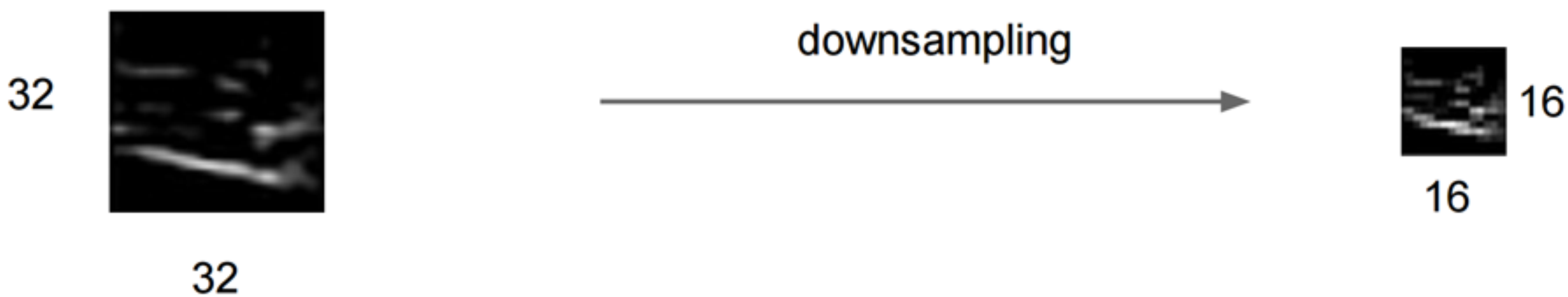
Max Pooling

For most CNNs, **convolution** is often followed by **pooling**:

- Creates a smaller representation while retaining the most important information
- The “max” operation is the most common
- Why might “avg” be a poor choice?



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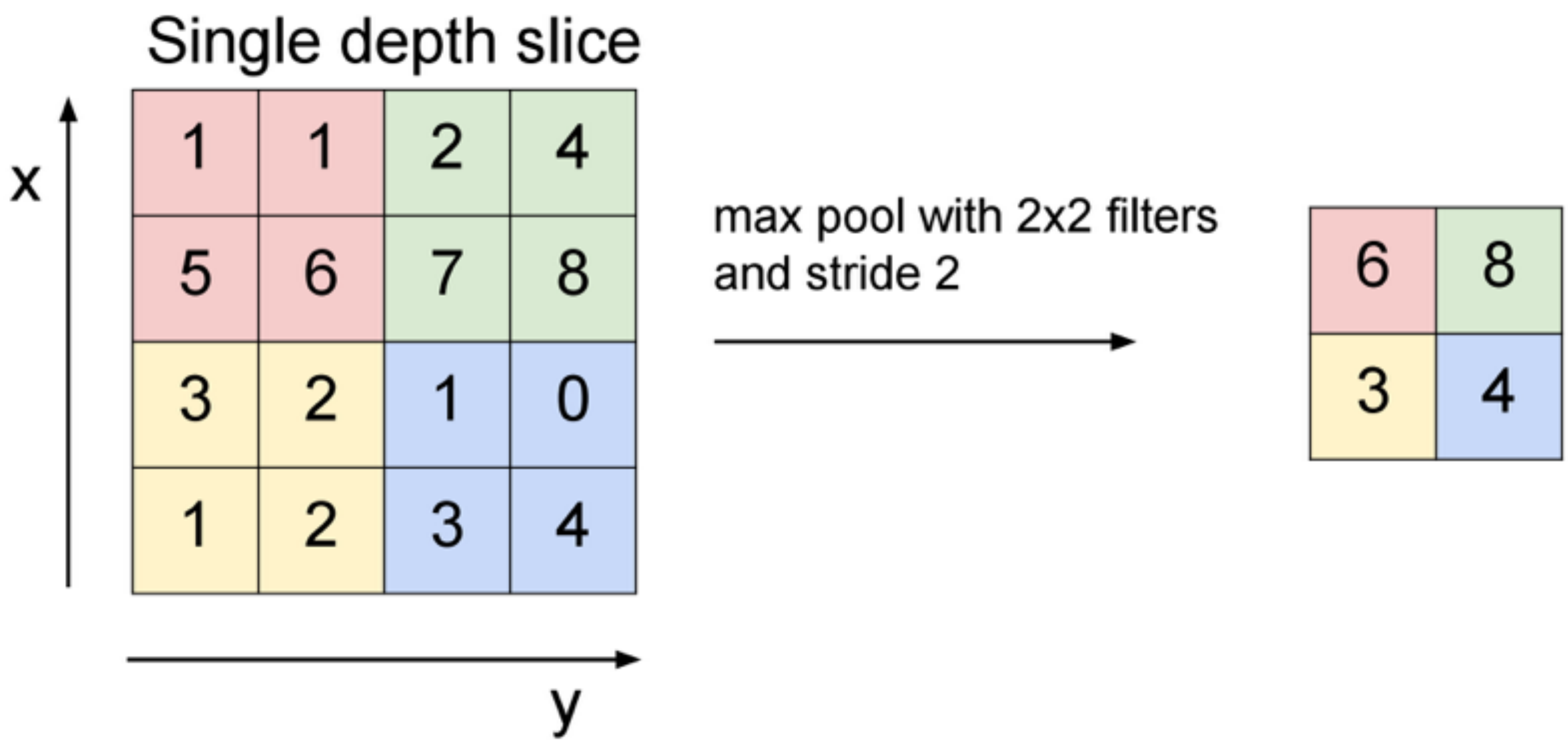
Max Pooling



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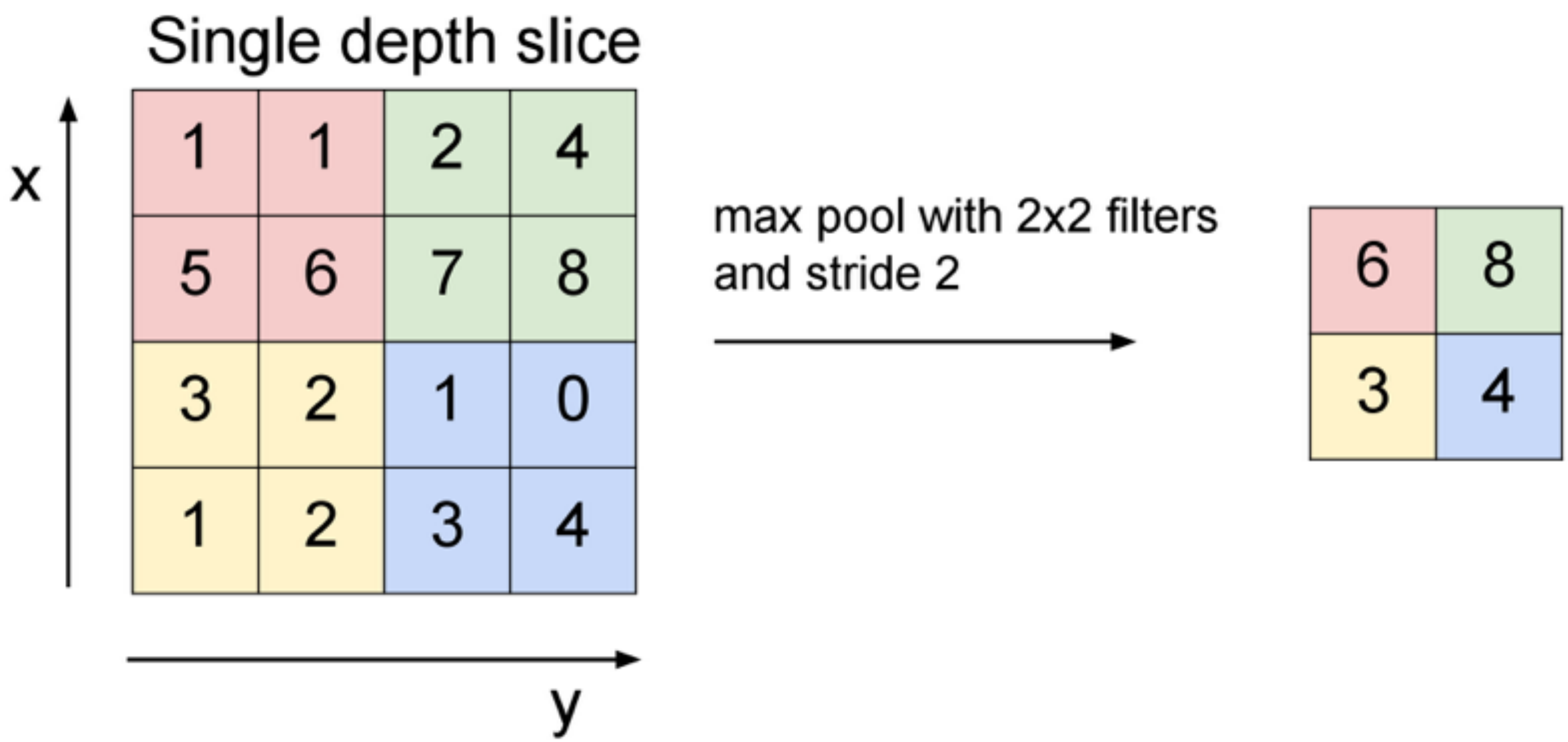
Max Pooling



What's the backprop rule for max pooling?

Figure: Andrej Karpathy

Max Pooling

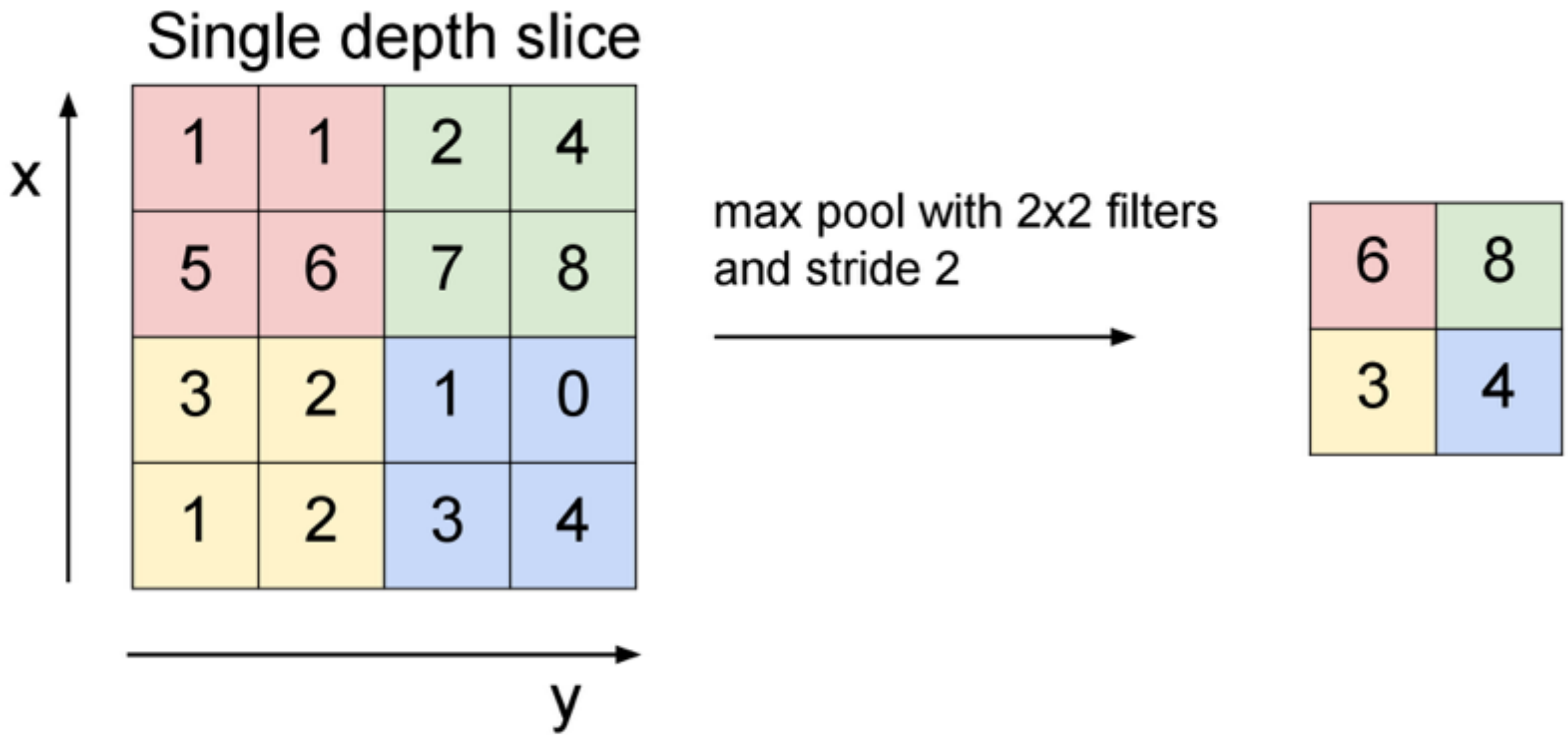


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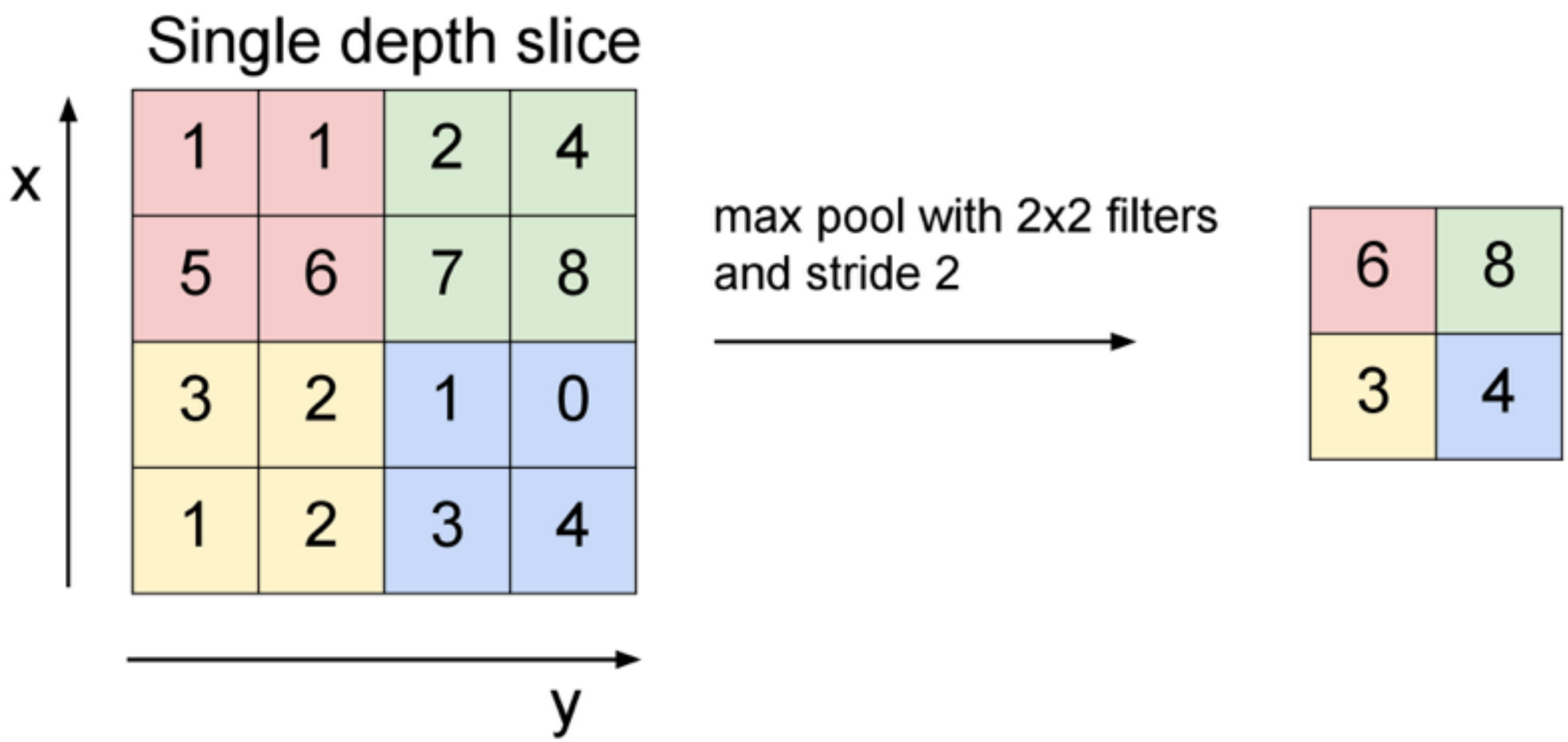
Figure: Andrej Karpathy

Max Pooling



What's the backprop rule for max pooling?

- In the forward pass, store the index that took the max
- The backprop gradient is the input gradient at that index

Figure: Andrej Karpathy

Example CNN

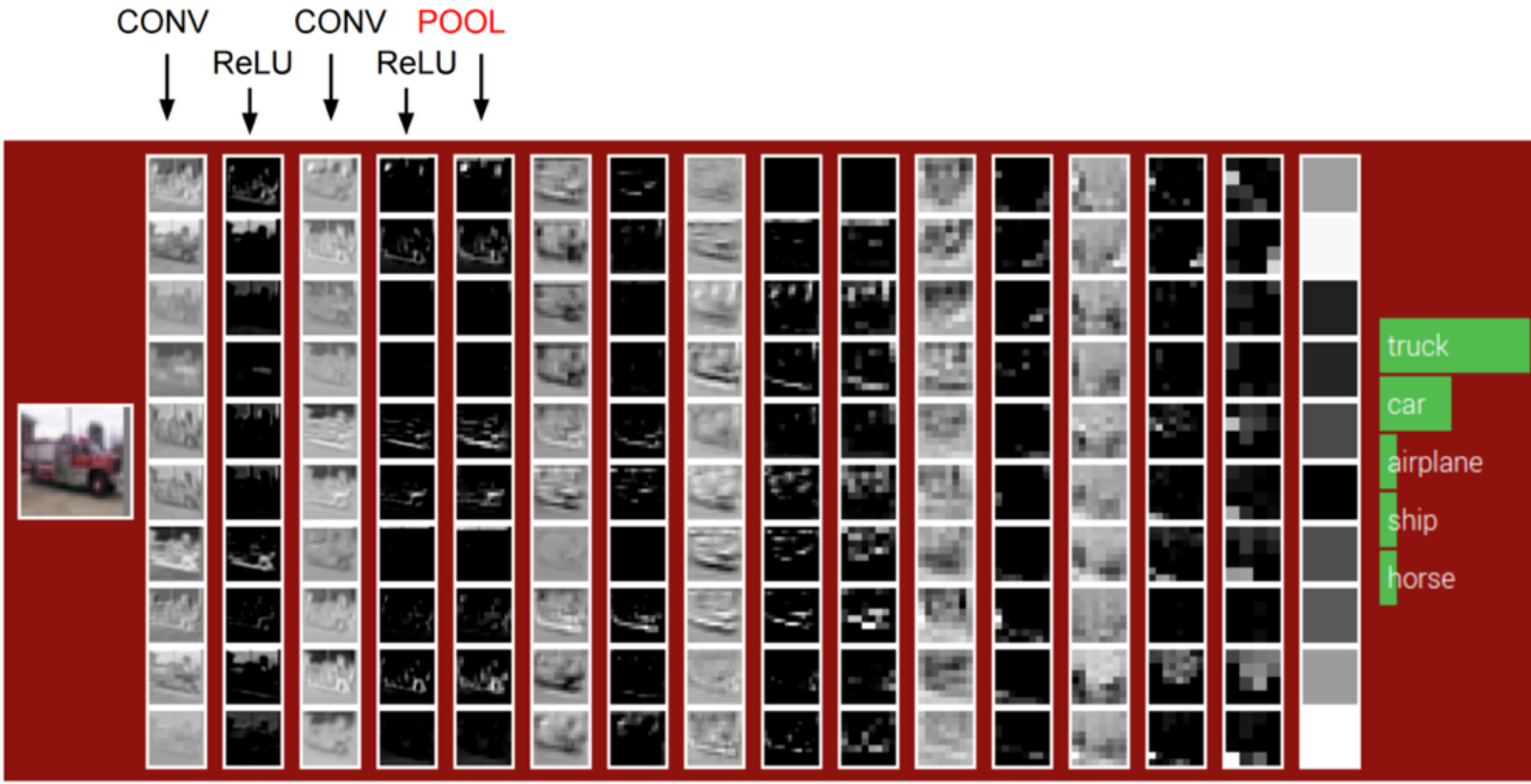


Figure: Andrej Karpathy

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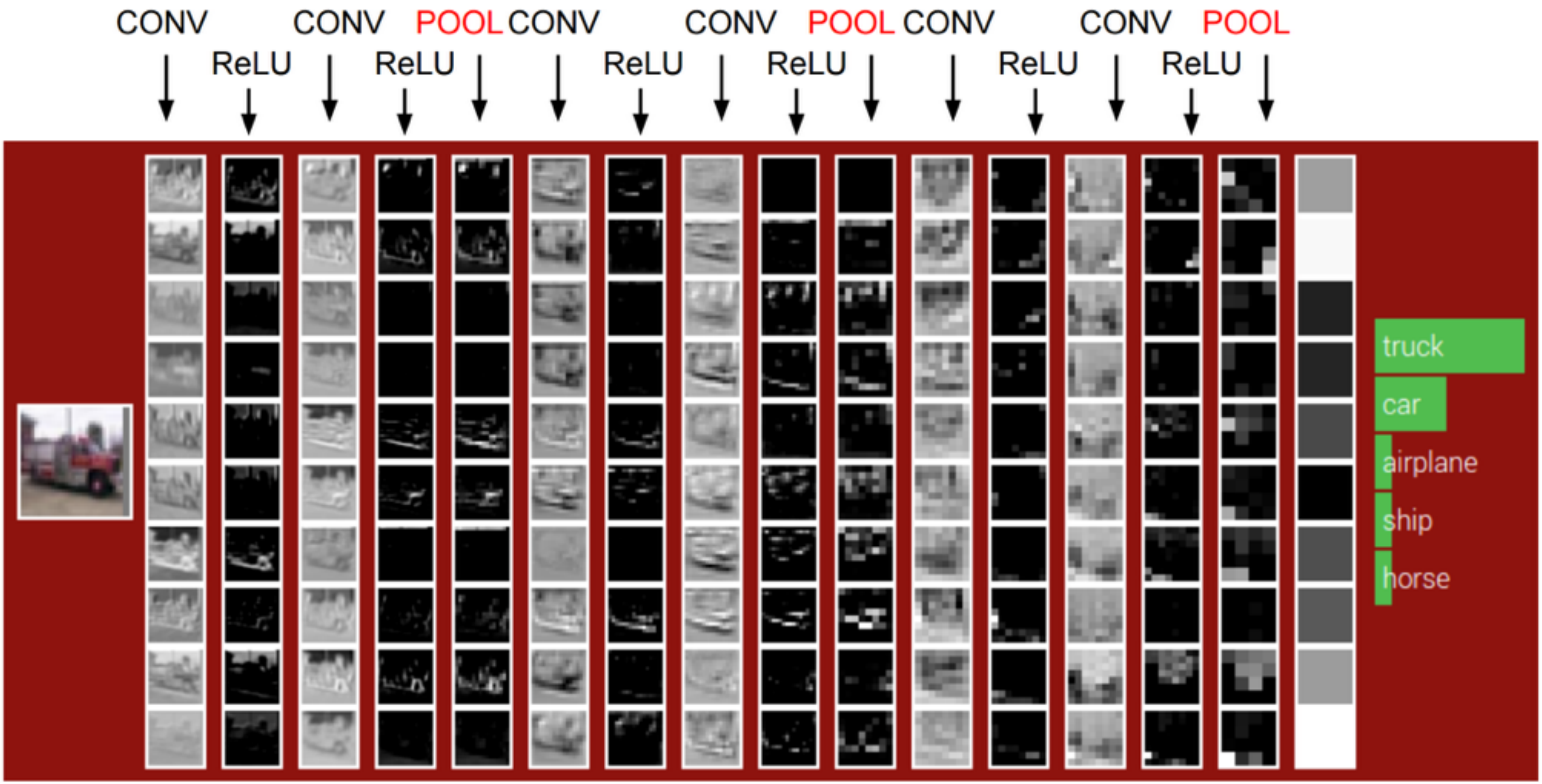

Figure: Andrej Karpathy

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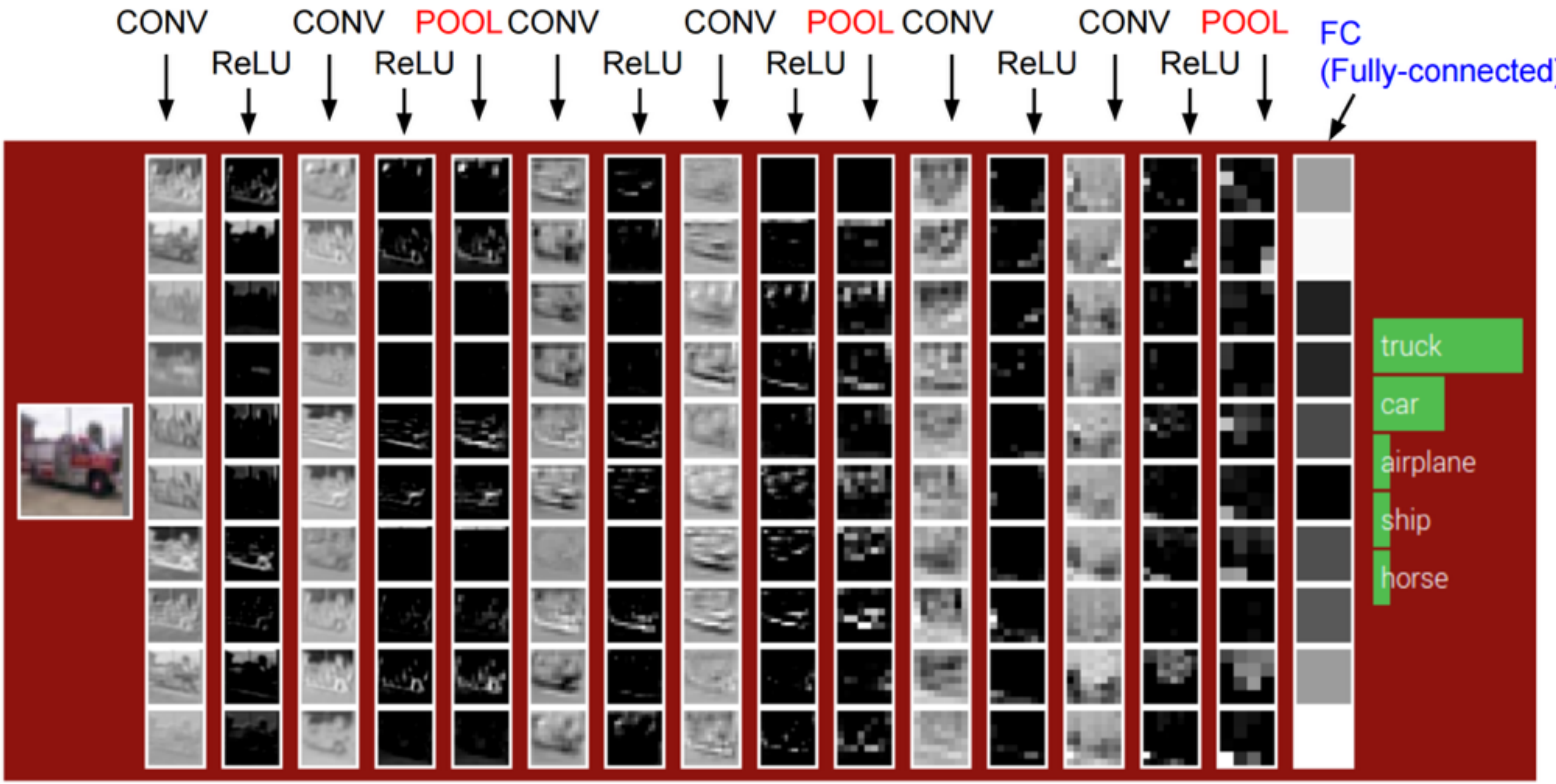

Figure: Andrej Karpathy

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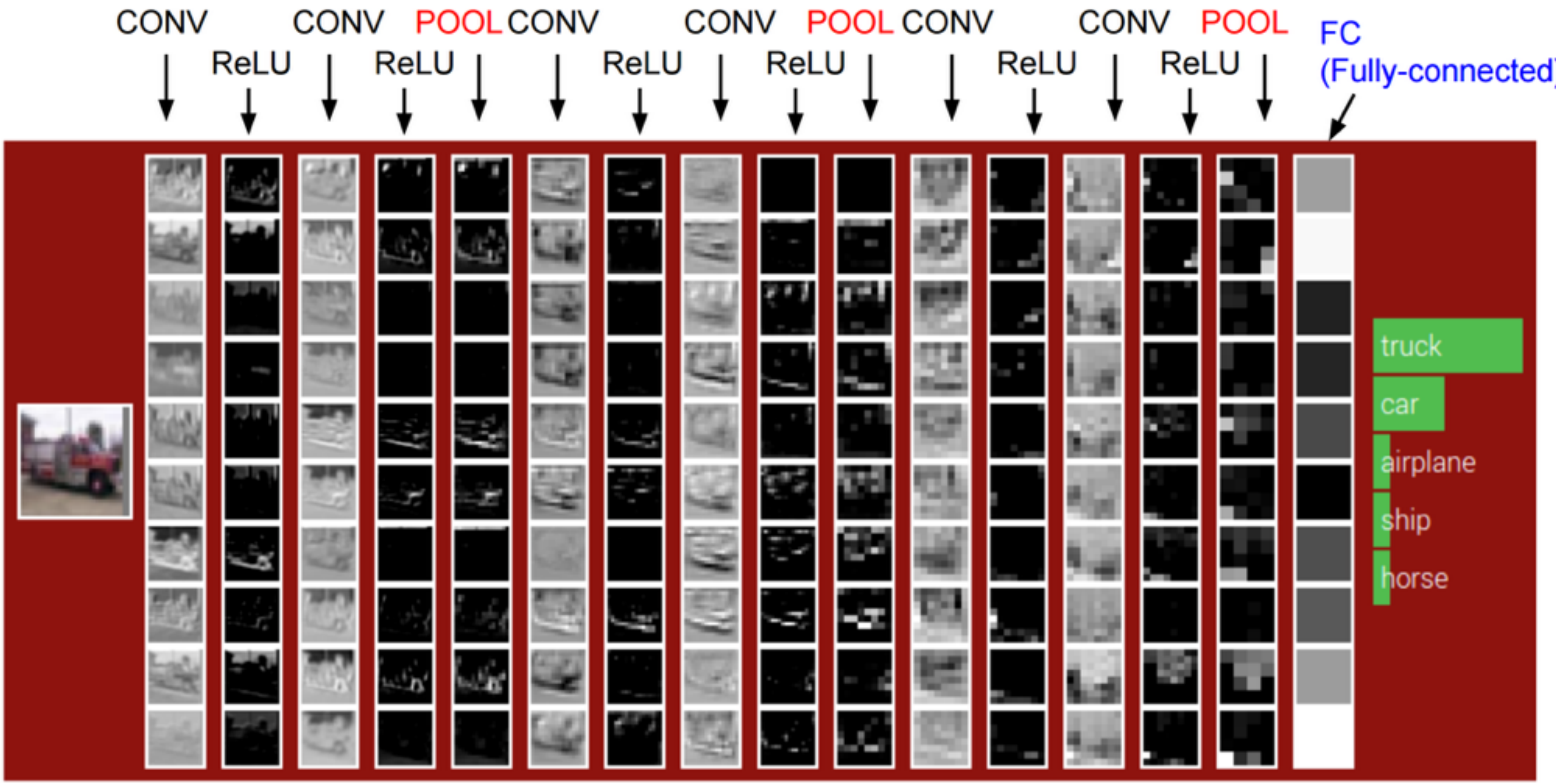

10x3x3 conv filters, stride 1, pad 1

2x2 pool filters, stride 2

Figure: Andrej Karpathy

Questions?