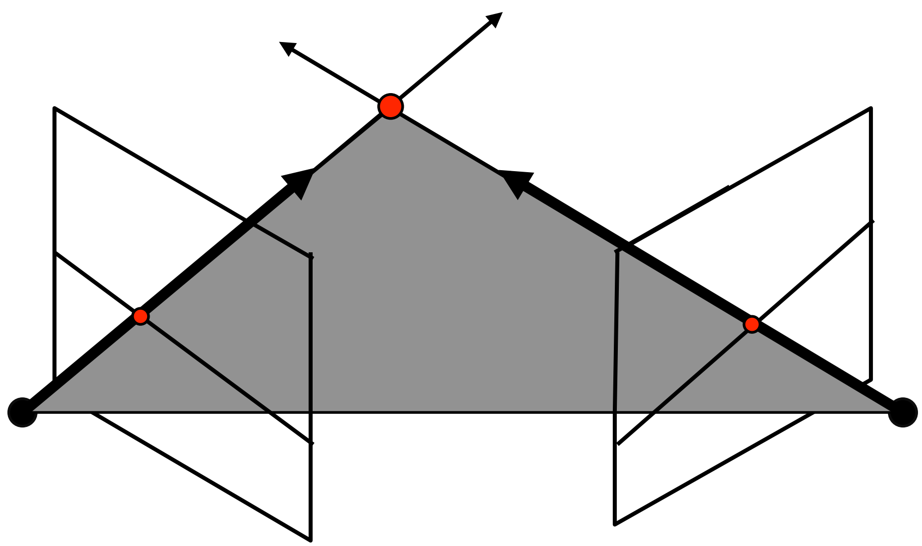
CS4670 / 5670: Computer Vision

Kavita Bala

Lec 21: Fundamental Matrix

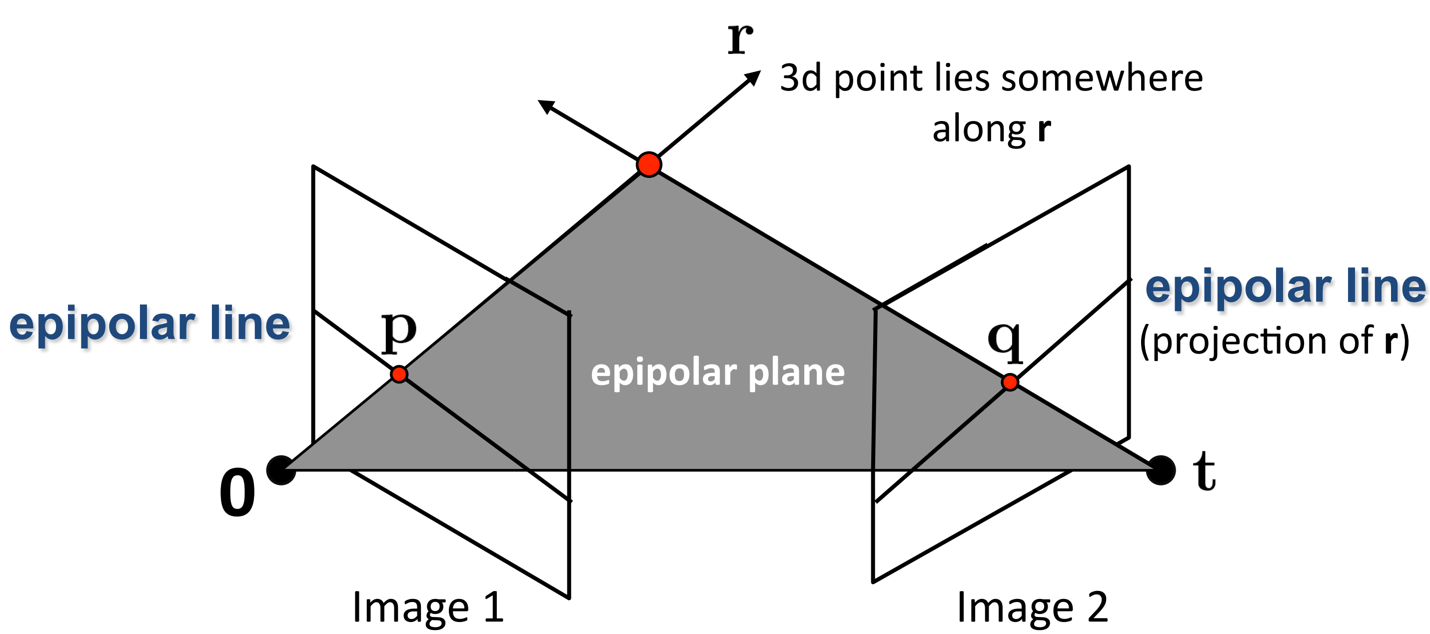


Readings

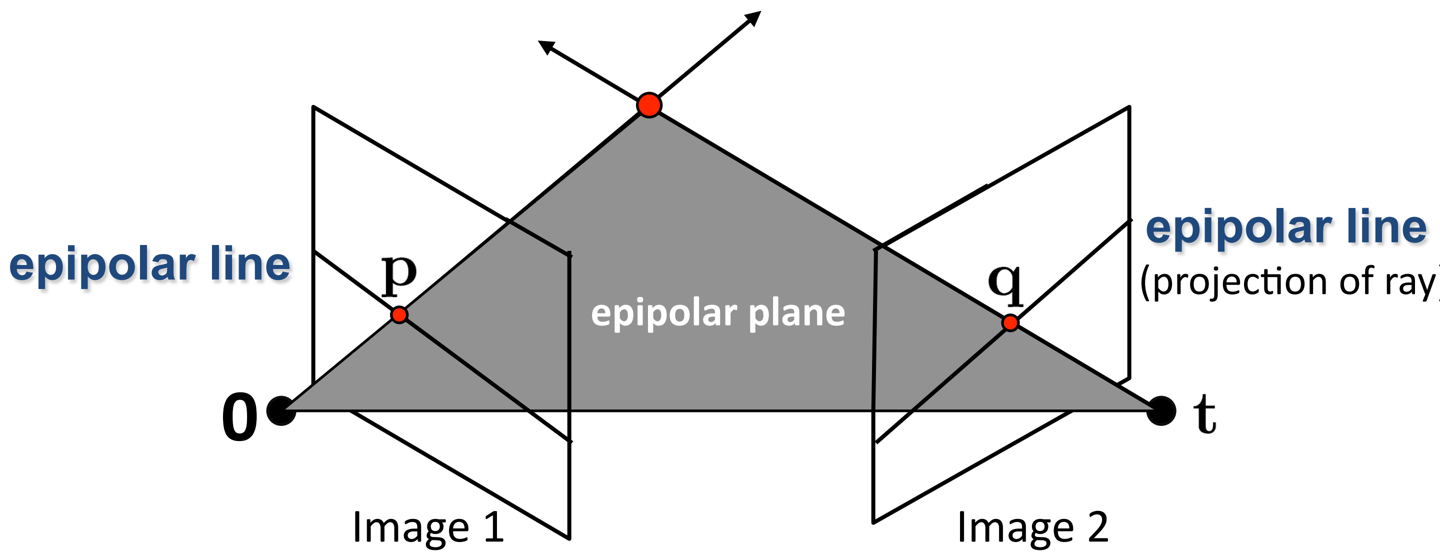
- Szeliski, Chapter 7.2
- “Fundamental matrix song”

Two-view geometry

- Where do epipolar lines come from?

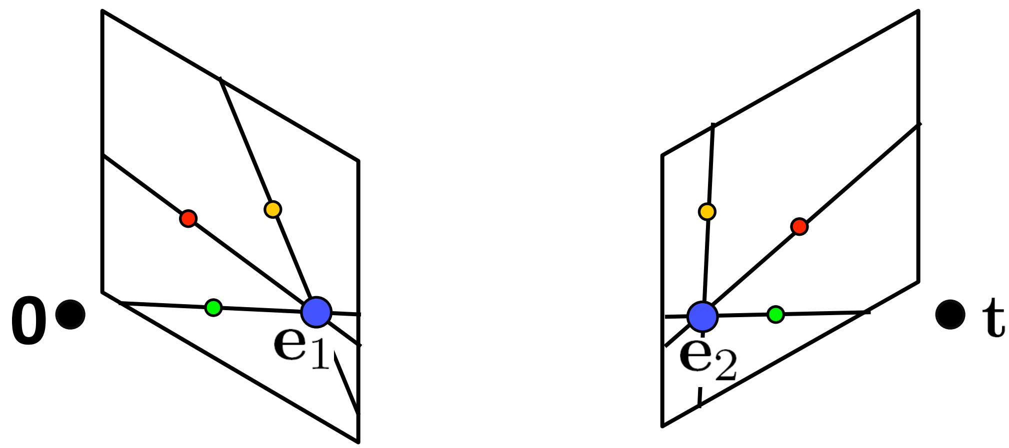


Fundamental matrix



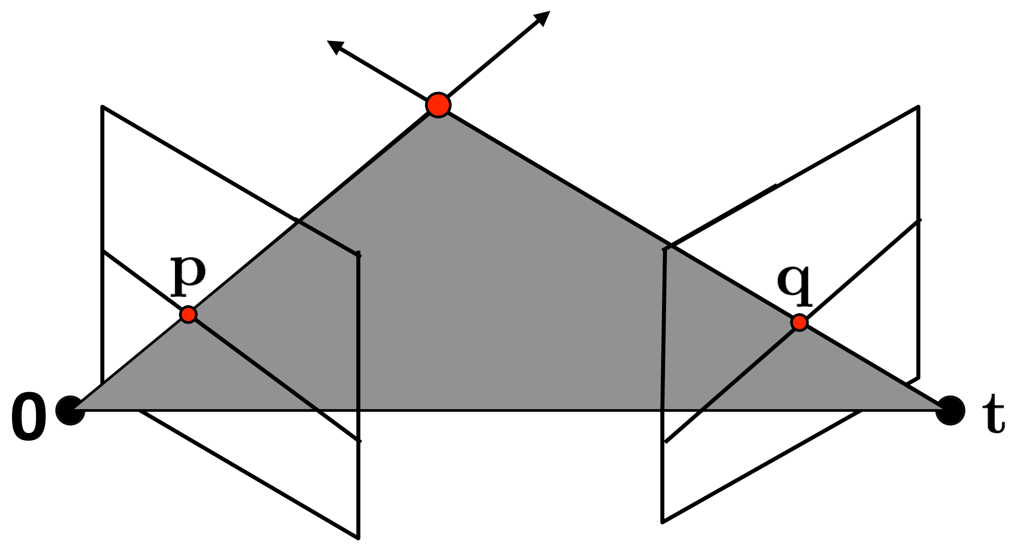
- This *epipolar geometry* of two views is described by a Very Special 3x3 matrix $\mathbf{F}$, called the *fundamental matrix*
- $\mathbf{F}$ maps (homogeneous) *points* in image 1 to *lines* in image 2!
- The epipolar line (in image 2) of point **p** is: $\mathbf{F}\mathbf{p}$
- *Epipolar constraint* on corresponding points: $\mathbf{q}^T\mathbf{F}\mathbf{p} = 0$

Fundamental matrix



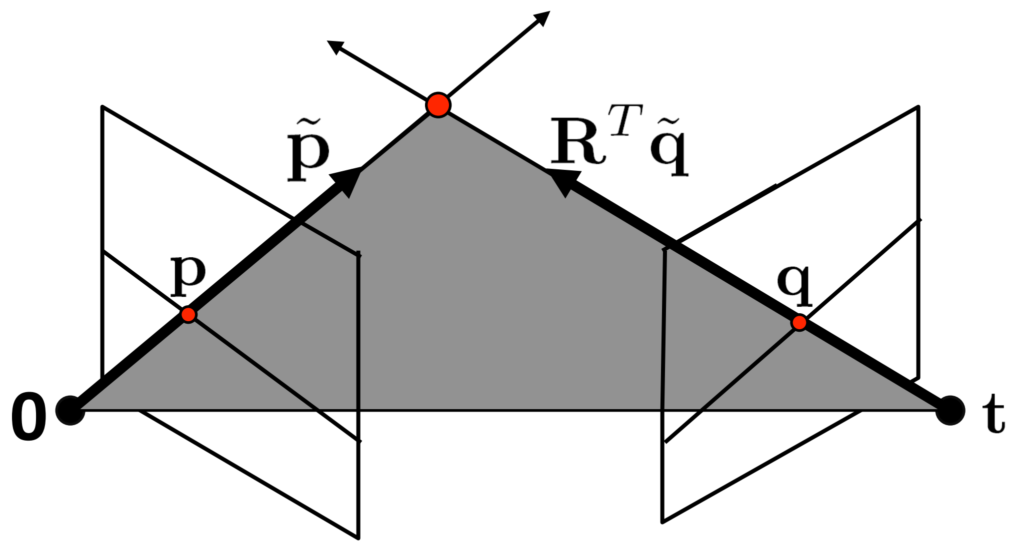
- Two special points: $\mathbf{e}_1$ and $\mathbf{e}_2$ (the *epipoles*): projection of one camera into the other
- All of the epipolar lines in an image pass through the epipole

Fundamental matrix



- Why does **F** exist?
- Let's derive it…

Fundamental matrix



$\mathbf{K}_1$: intrinsics of camera 1

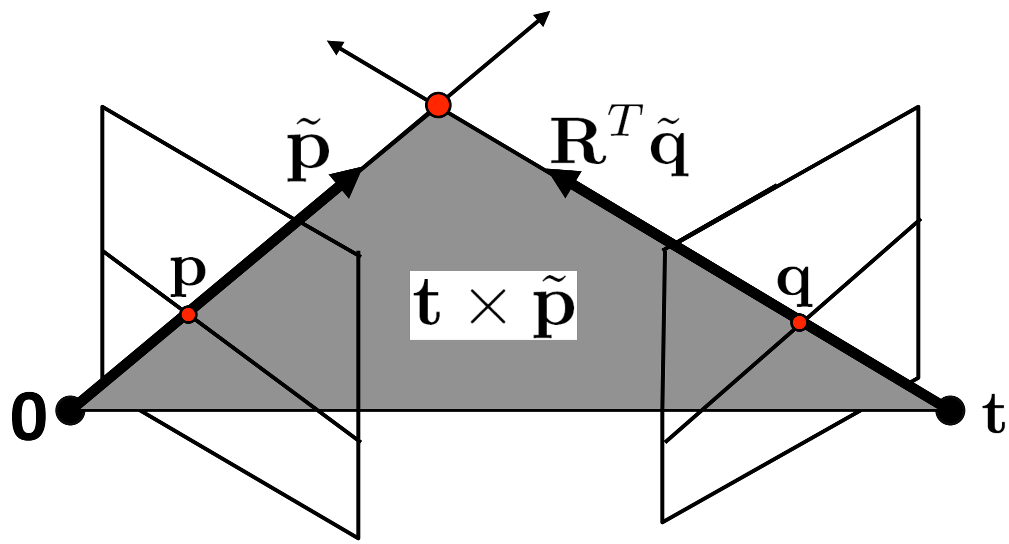
$\mathbf{K}_2$: intrinsics of camera 2

$\mathbf{R}$: rotation of image 2 w.r.t. camera 1

$\tilde{\mathbf{p}} = \mathbf{K}_1^{-1}\mathbf{p}$: ray through **p** in camera 1's (and world) coordinate system

$\tilde{\mathbf{q}} = \mathbf{K}_2^{-1}\mathbf{q}$: ray through **q** in camera 2's coordinate system

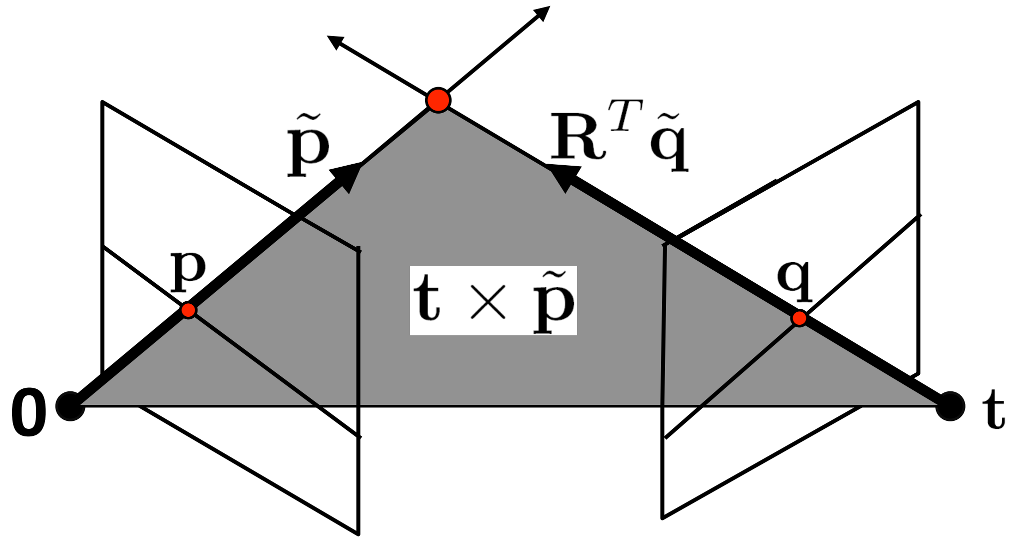
Fundamental matrix



- $\tilde{\mathbf{p}}$, $\mathbf{R}^T\tilde{\mathbf{q}}$, and $\mathbf{t}$ are coplanar
- epipolar plane can be represented as $\mathbf{t} \times \tilde{\mathbf{p}}$

$$(\mathbf{R}^T\tilde{\mathbf{q}})^T(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

Fundamental matrix



$$(\mathbf{R}^T\tilde{\mathbf{q}})^T(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

$$\Downarrow$$

$$\tilde{\mathbf{q}}^T\mathbf{R}(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

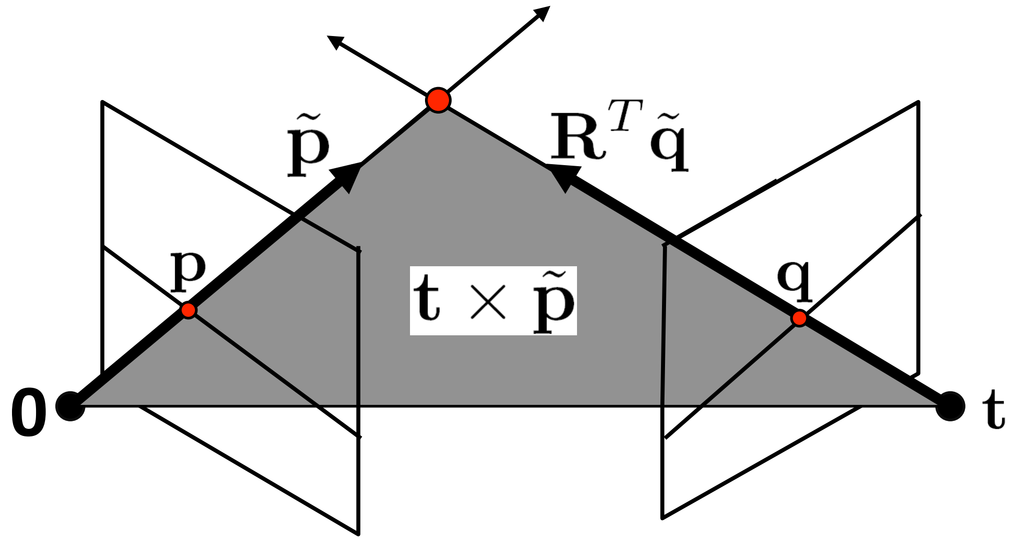
Cross-product as linear operator

Useful fact: Cross product with a vector **t** can be represented as multiplication with a (*skew-symmetric*) 3x3 matrix

$$[\mathbf{t}]_\times = \begin{bmatrix} 0 & -t_z & t_y \\ t_z & 0 & -t_x \\ -t_y & t_x & 0 \end{bmatrix}$$

$$\mathbf{t} \times \tilde{\mathbf{p}} = [\mathbf{t}]_\times \tilde{\mathbf{p}}$$

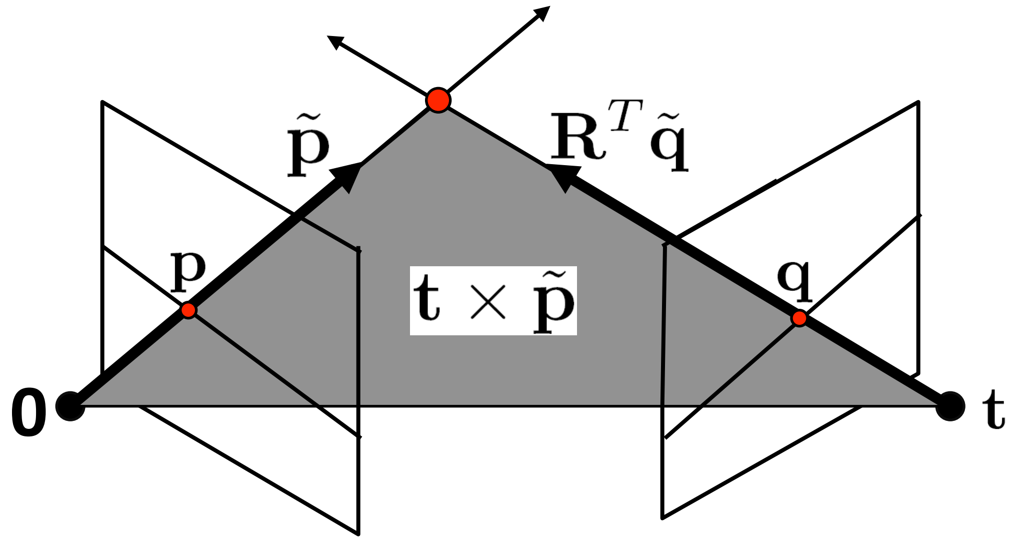
Fundamental matrix



$$\tilde{\mathbf{q}}^T\mathbf{R}(\mathbf{t} \times \tilde{\mathbf{p}}) = 0$$

$$\tilde{\mathbf{q}}^T\mathbf{R}\,[\mathbf{t}]_\times\,\tilde{\mathbf{p}} = 0$$

Fundamental matrix

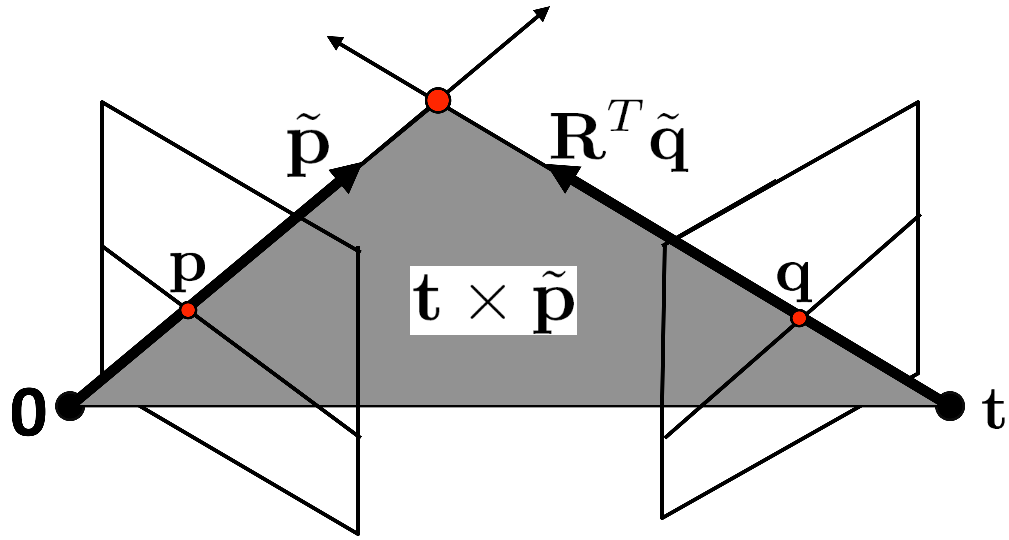


$$\tilde{\mathbf{q}}^T \underbrace{\mathbf{R}\,[\mathbf{t}]_\times}_{\mathbf{E}} \tilde{\mathbf{p}} = 0$$

$\mathbf{E}$ ← the Essential matrix

$$\tilde{\mathbf{q}}^T \mathbf{E} \tilde{\mathbf{p}} = 0$$

Fundamental matrix

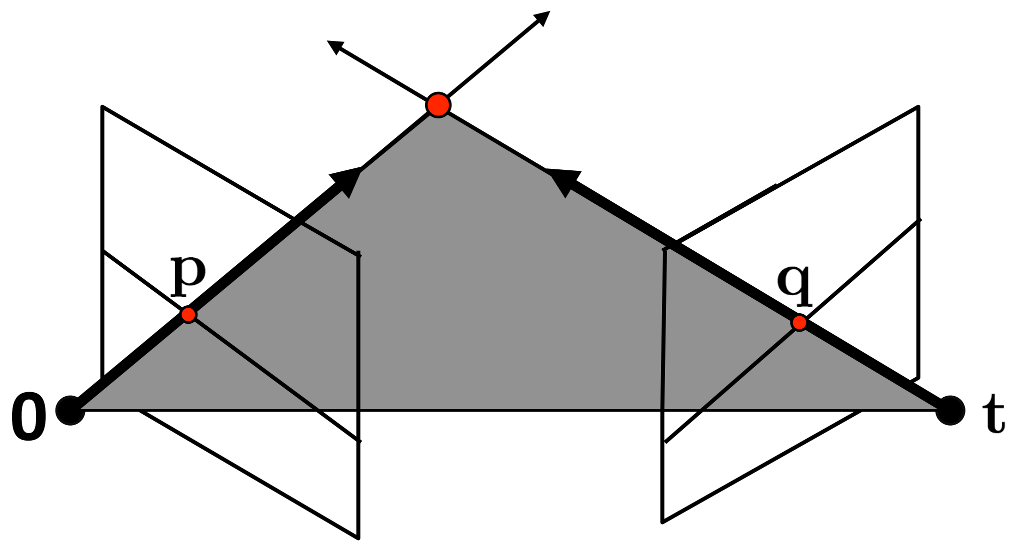


$$\tilde{\mathbf{q}}^T \mathbf{R} \left[\mathbf{t}\right]_\times \tilde{\mathbf{p}} = 0$$

$$\mathbf{q}^T \underbrace{\mathbf{K}_2^{-T} \mathbf{R} \left[\mathbf{t}\right]_\times \mathbf{K}_1^{-1}}_{\mathbf{F}} \mathbf{p} = 0$$

$\mathbf{F}$ ← the Fundamental matrix

Fundamental matrix



$\mathbf{K}_1$: intrinsics of camera 1

$\mathbf{K}_2$: intrinsics of camera 2

$\mathbf{R}$: rotation of image 2 w.r.t. camera 1

$$\mathbf{q}^T \underbrace{\mathbf{K}_2^{-T} \mathbf{R} \left[\mathbf{t}\right]_\times \mathbf{K}_1^{-1}}_{\mathbf{F}} \mathbf{p} = 0$$

$\mathbf{F}$ ← the Fundamental matrix

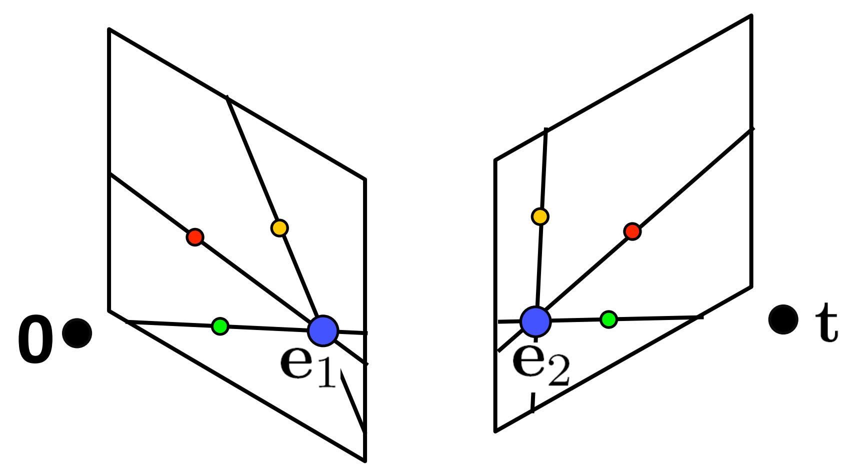
Fundamental matrix result

$$\mathbf{q}^T \mathbf{F} \mathbf{p} = 0$$

(Longuet-Higgins, 1981)

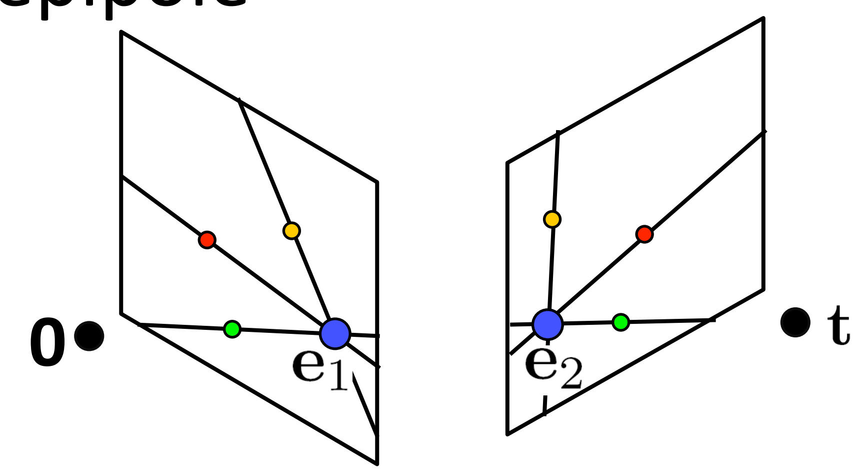
Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$



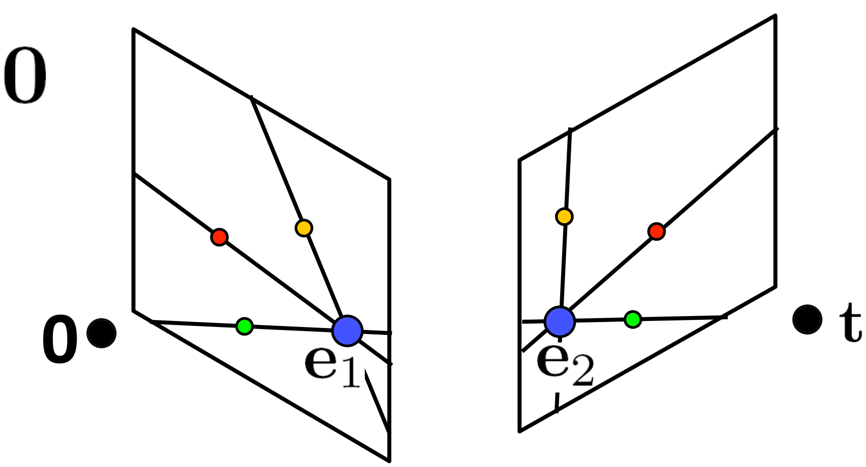
Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$
- $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{F}^T\mathbf{e}_2 = \mathbf{0}$
- All epipolar lines contain epipole

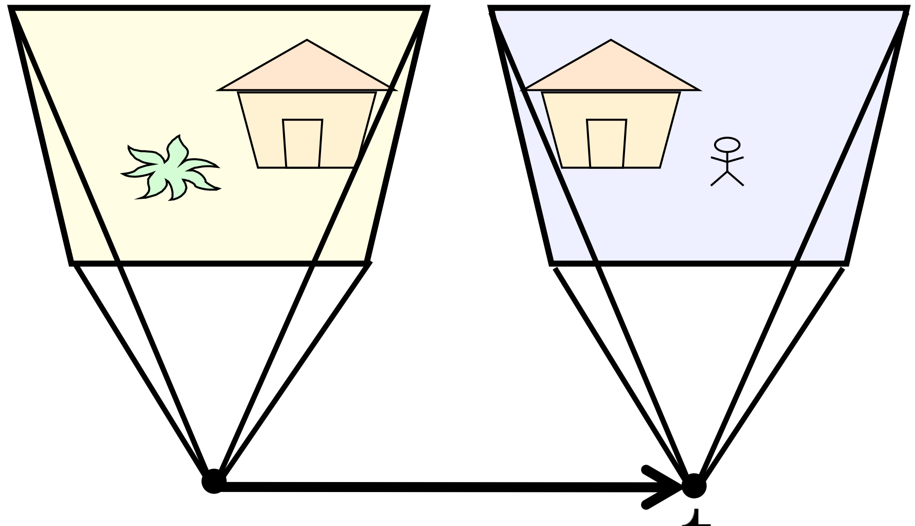


Properties of the Fundamental Matrix

- $\mathbf{F}\mathbf{p}$ is the epipolar line associated with $\mathbf{p}$
- $\mathbf{F}^T\mathbf{q}$ is the epipolar line associated with $\mathbf{q}$
- $\mathbf{F}\mathbf{e}_1 = \mathbf{0}$ and $\mathbf{F}^T\mathbf{e}_2 = \mathbf{0}$
- $\mathbf{F}$ is rank 2



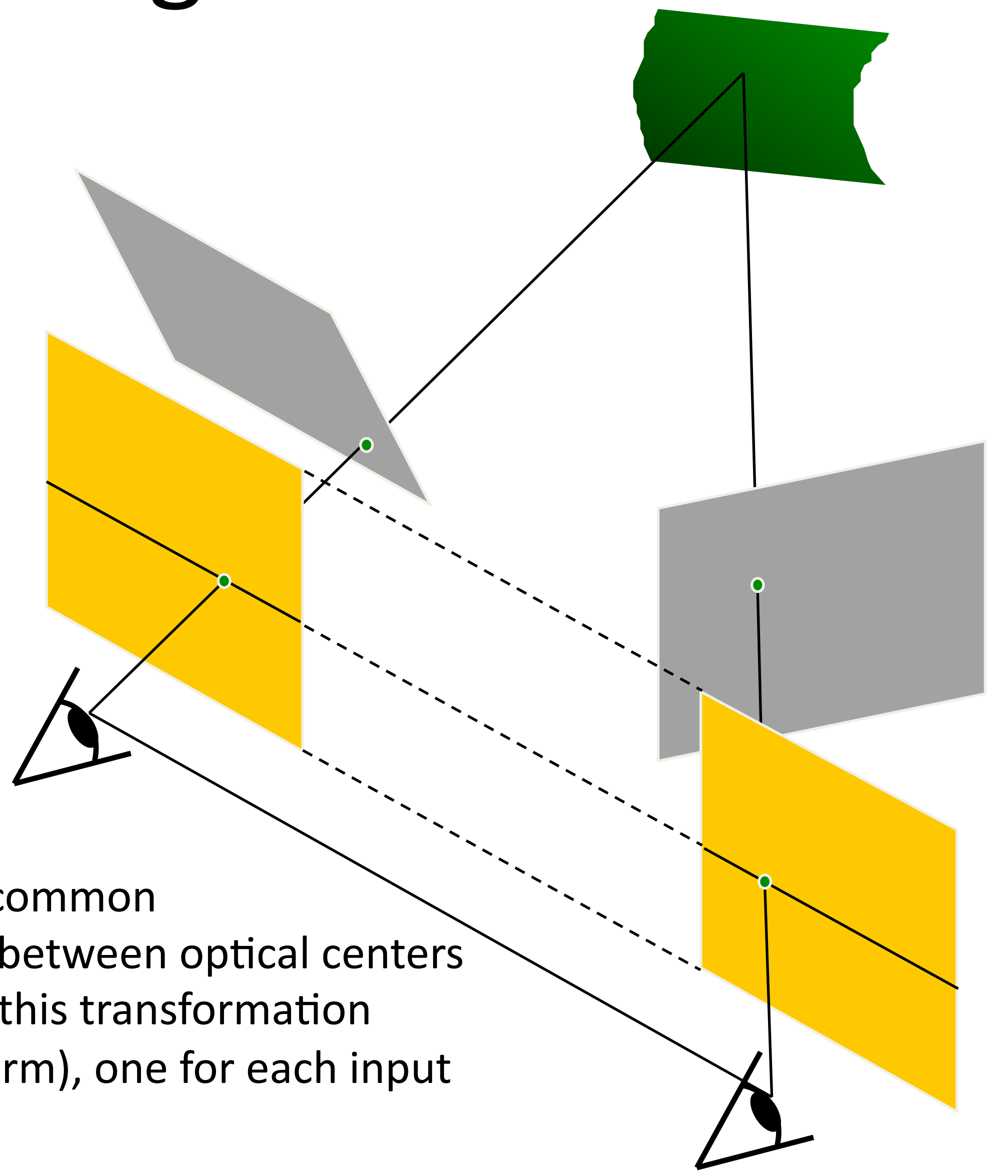
Rectified case



$$\mathbf{R} = \mathbf{I}_{3\times 3}$$
$$\mathbf{t} = [\ 1 \quad 0 \quad 0\ ]^T$$

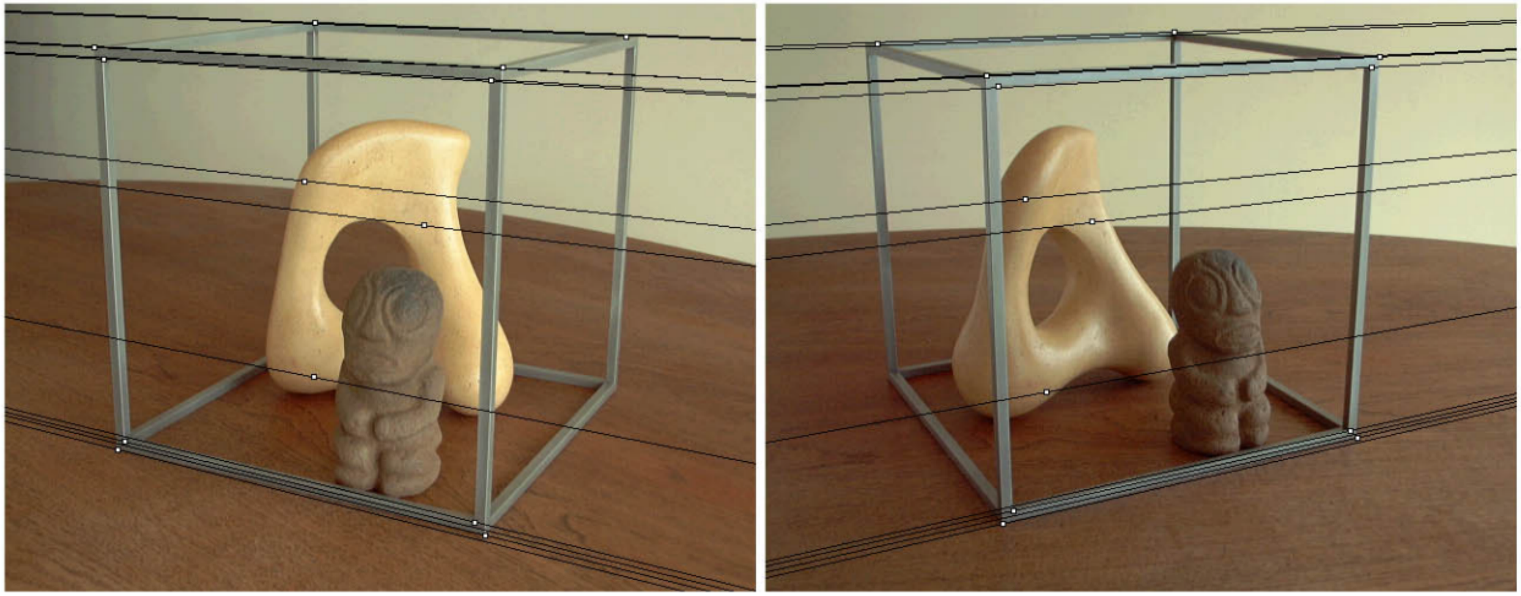
$$\mathbf{E} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

Stereo image rectification

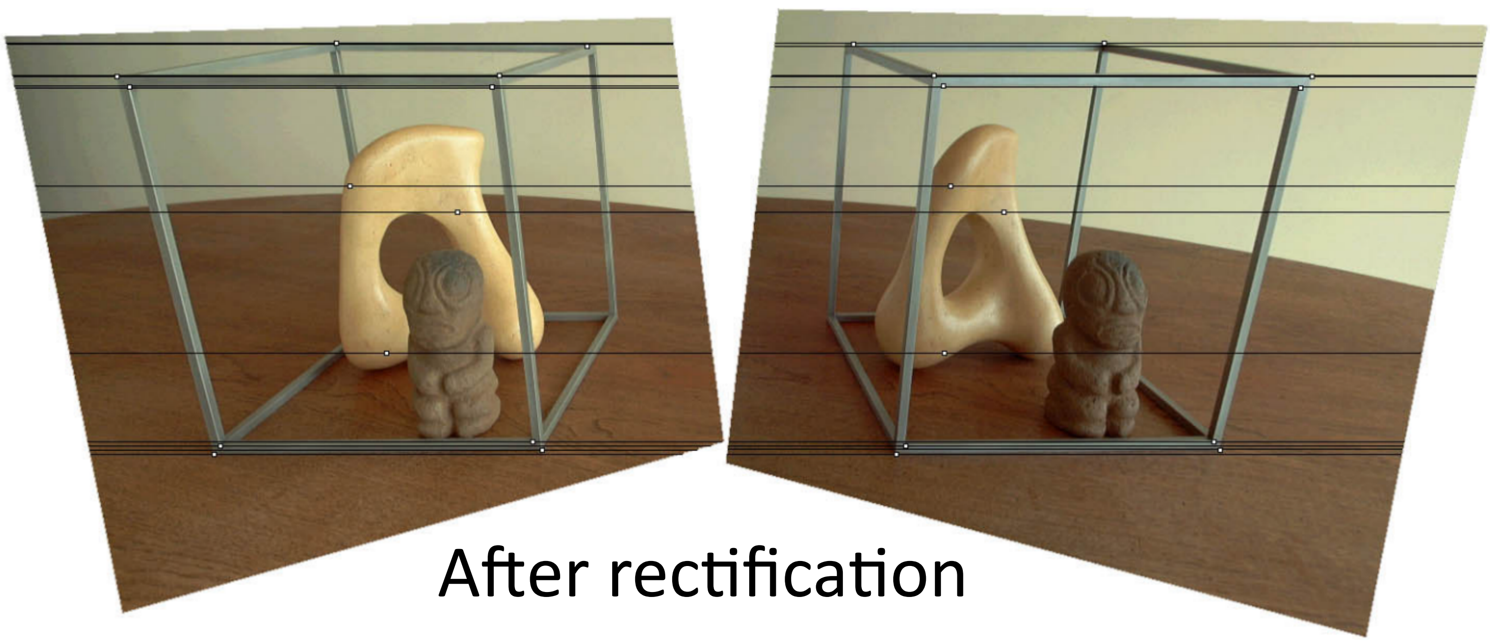


- reproject image planes onto a common
- plane parallel to the line between optical centers
- pixel motion is horizontal after this transformation
- two homographies (3x3 transform), one for each input image reprojection

➢ C. Loop and Z. Zhang. [Computing Rectifying Homographies for Stereo Vision](). IEEE Conf. Computer Vision and Pattern Recognition, 1999.



Original stereo pair



After rectification

Questions?

Alternative Formulation

Homogeneous notation for lines

Recall that a point (x, y) in 2D is represented by the homogeneous 3-vector $\mathbf{x} = (x_1, x_2, x_3)^\top$, where $x = x_1/x_3, y = x_2/x_3$

A line in 2D is represented by the homogeneous 3-vector

$$\mathbf{l} = \begin{pmatrix} l_1 \\ l_2 \\ l_3 \end{pmatrix}$$

which is the line $l_1 x + l_2 y + l_3 = 0$.

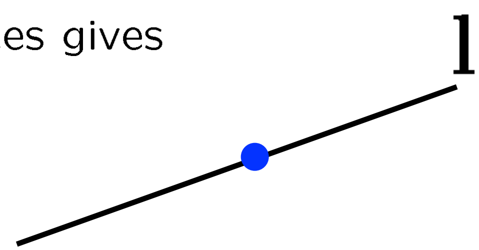
Example represent the line $y = 1$ as a homogeneous vector.

Write the line as $-y + 1 = 0$ then $l_1 = 0, l_2 = -1, l_3 = 1$, and $\mathbf{l} = (0, -1, 1)^\top$.

Note that $\mu(l_1 x + l_2 y + l_3) = 0$ represents the same line (only the ratio of the homogeneous line coordinates is significant).

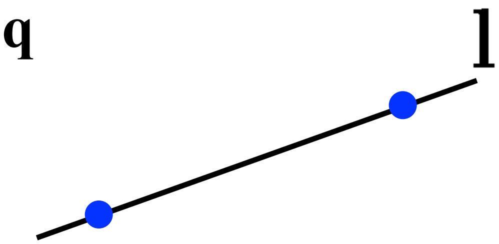
Writing both the point and line in homogeneous coordinates gives

$$l_1 x_1 + l_2 x_2 + l_3 x_3 = 0$$



- point on line $\mathbf{l}.\mathbf{x} = 0$ or $\mathbf{l}^\top \mathbf{x} = 0$ or $\mathbf{x}^\top \mathbf{l} = 0$

- The line $\mathbf{l}$ through the two points $\mathbf{p}$ and $\mathbf{q}$ is $\mathbf{l} = \mathbf{p} \times \mathbf{q}$



Proof

$$\mathbf{l}.\mathbf{p} = (\mathbf{p} \times \mathbf{q}).\mathbf{p} = 0 \qquad \mathbf{l}.\mathbf{q} = (\mathbf{p} \times \mathbf{q}).\mathbf{q} = 0$$

- The intersection of two lines $\mathbf{l}$ and $\mathbf{m}$ is the point $\mathbf{x} = \mathbf{l} \times \mathbf{m}$

Algebraic representation of epipolar geometry

We know that the epipolar geometry defines a mapping

$$\mathbf{x} \quad \rightarrow \quad \mathbf{l}'$$

point in first image — epipolar line in second image

- the map ony depends on the cameras $\mathtt{P}, \mathtt{P}'$ (not on structure)

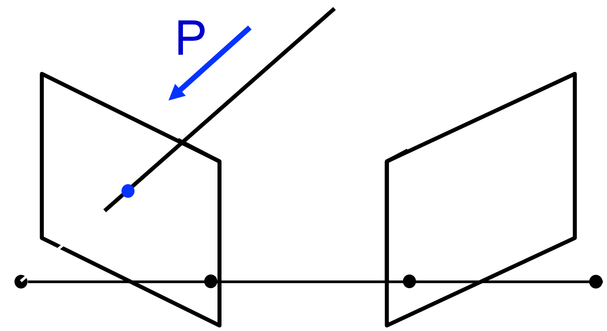
- it will be shown that the map is linear and can be written as $\mathbf{l}' = \mathtt{F}\mathbf{x}$, where $\mathtt{F}$ is a 3×3 matrix called the fundamental matrix

$$\mathbf{l}' = F\mathbf{x}$$

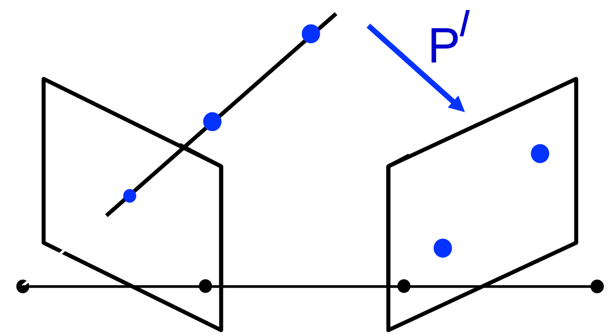
Derivation of the algebraic expression

<u>Outline</u>

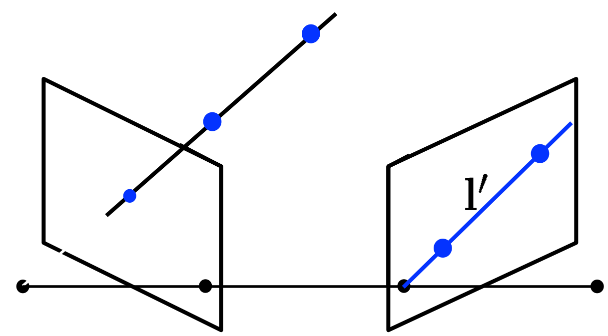
Step 1: for a point x in the first image back project a ray with camera P



Step 2: choose two points on the ray and project into the second image with camera P'



Step 3: compute the line through the two image points using the relation $\mathbf{l}' = \mathbf{p} \times \mathbf{q}$

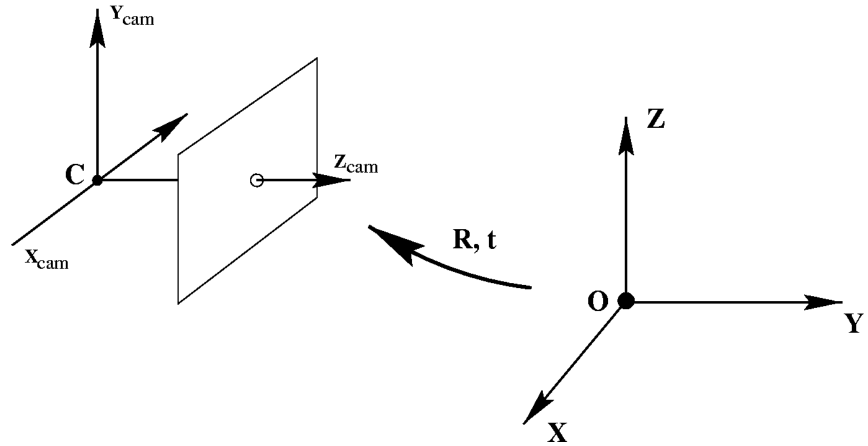


- choose camera matrices

$$\mathtt{P} = \mathtt{K}[\mathtt{R} \mid \mathtt{t}]$$

internal calibration (K) — rotation (R) — translation (t)

from world to camera coordinate frame

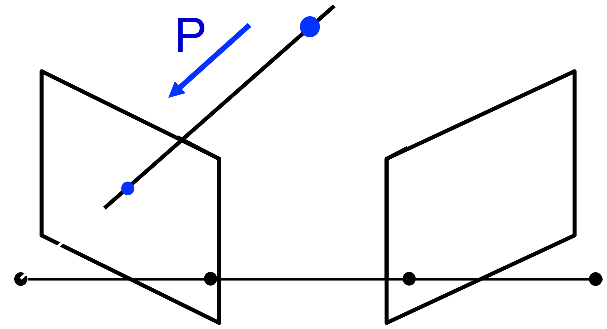


- first camera $\mathtt{P} = \mathtt{K}[\mathtt{I} \mid \mathbf{0}]$

world coordinate frame aligned with first camera

- second camera $\mathtt{P}' = \mathtt{K}'[\mathtt{R} \mid \mathtt{t}]$

Step 1: for a point x in the first image
back project a ray with camera $\mathtt{P} = \mathtt{K}[\,\mathtt{I} \mid \mathbf{0}\,]$



A point $\mathbf{x}$ back projects to a ray

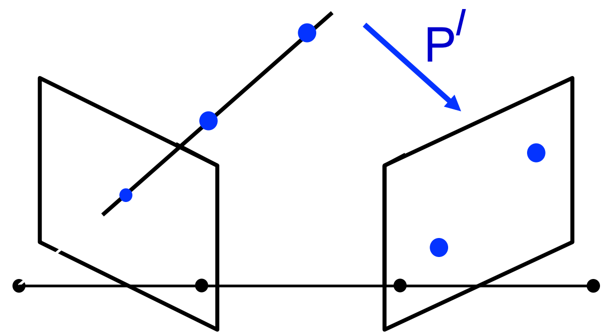
$$\begin{pmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{pmatrix} = \mathrm{Z}\mathtt{K}^{-1} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \mathrm{Z}\mathtt{K}^{-1}\mathbf{x}$$

where Z is the point's depth, since

$$\mathbf{X}(\mathrm{Z}) = \begin{pmatrix} \mathrm{Z}\mathtt{K}^{-1}\mathbf{x} \\ 1 \end{pmatrix}$$

satisfies

$$\mathtt{P}\mathbf{X}(\mathrm{Z}) = \mathtt{K}[\mathtt{I} \mid \mathbf{0}]\mathbf{X}(\mathrm{Z}) = \mathbf{x}$$



Step 2: choose two points on the ray and project into the second image with camera P'

Consider two points on the ray $\mathtt{X}(z) = \begin{pmatrix} z\mathtt{K}^{-1}\mathbf{x} \\ 1 \end{pmatrix}$

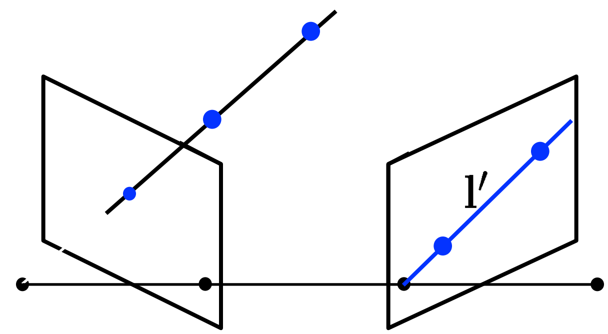
- $\mathbf{Z}$ = 0 is the camera centre $\begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix}$

- $\mathbf{Z} = \infty$ is the point at infinity $\begin{pmatrix} \mathtt{K}^{-1}\mathbf{x} \\ 0 \end{pmatrix}$

Project these two points into the second view

$$\mathtt{P}' \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}] \begin{pmatrix} \mathbf{0} \\ 1 \end{pmatrix} = \mathtt{K}'\mathbf{t} \qquad \mathtt{P}' \begin{pmatrix} \mathtt{K}^{-1}\mathbf{x} \\ 0 \end{pmatrix} = \mathtt{K}'[\mathtt{R} \mid \mathbf{t}] \begin{pmatrix} \mathtt{K}^{-1}\mathbf{x} \\ 0 \end{pmatrix} = \mathtt{K}'\mathtt{R}\mathtt{K}^{-1}\mathbf{x}$$

Step 3: compute the line through the two image points using the relation $\mathbf{l}' = \mathbf{p} \times \mathbf{q}$



Compute the line through the points $\mathbf{l}' = (\mathtt{K}'\mathbf{t}) \times (\mathtt{K}'\mathtt{R}\mathtt{K}^{-1}\mathbf{x})$

Using the identity $(\mathtt{M}\mathbf{a}) \times (\mathtt{M}\mathbf{b}) = \mathtt{M}^{-\top}(\mathbf{a} \times \mathbf{b})$ where $\mathtt{M}^{-\top} = (\mathtt{M}^{-1})^{\top} = (\mathtt{M}^{\top})^{-1}$

$$\mathbf{l}' = \mathtt{K}'^{-\top}\left(\mathbf{t} \times (\mathtt{R}\mathtt{K}^{-1}\mathbf{x})\right) = \underbrace{\mathtt{K}'^{-\top}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1}}_{F}\mathbf{x}$$

F is the fundamental matrix

$$\mathbf{l}' = \mathbf{F}\mathbf{x} \qquad \mathbf{F} = \mathtt{K}'^{-\top}[\mathbf{t}]_{\times}\mathtt{R}\mathtt{K}^{-1}$$

Points $\mathbf{x}$ and $\mathbf{x}'$ correspond ($\mathbf{x} \leftrightarrow \mathbf{x}'$) then $\mathbf{x}'^{\top}\mathbf{l}' = 0$

$$\mathbf{x}'^{\top}\mathbf{F}\mathbf{x} = 0$$