Aliasing, how does it work?
Intuition from the Fourier Transform

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September 9, 2010

Motivation

- Someone gives you a filter.
- What does it do?
- Output depends on convolution; difficult to visualize
- We need an alternative method for analysis
- Today will be about 1-dimensional continuous signals
- But it can be easily extended to 2-dimensional discrete signals (Images!)
- These slides will make you accept some assumptions (too difficult to prove in one lecture)
- Ultimately, I will explain why aliasing occurs from a freq. perspective

Fourier Transform

- **Assumption 1** The Fourier transform exists and is defined as: $\mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$ where $\int_{-\infty}^{\infty} |x(t)|dt < \infty$
- **Assumption 2** The Fourier transform inverse is: $\mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$
- I will denote *Fourier transform pairs* as $x(t) \iff X(\omega)$
- We may not formally prove these, but we can use them to provide intuition.

Convolution and Multiplication

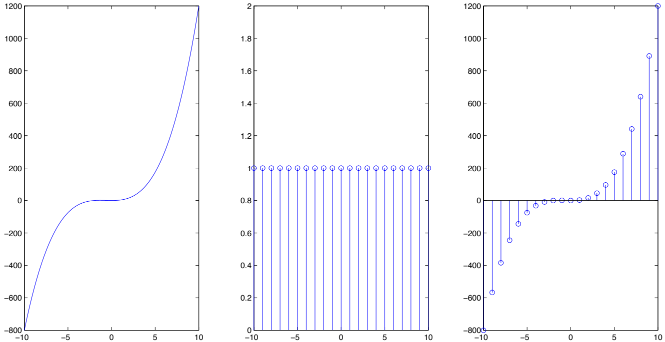
- $f(t) * g(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau$
- We flip g(t) and slide it along f(t)
- Let's apply the Fourier tranform
- $\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau e^{-j\omega t}dt$
- Some math later...
- $\mathcal{F}\{f(t) * g(t)\} = F(\omega)G(\omega)$
- Similarly, $\mathcal{F}\{f(t)g(t)\} = F(\omega) * G(\omega)$

Dirac delta function

- Also known as the unit impulse
- $\delta(t) = \begin{cases} \infty & : t = 0 \\ 0 & : \textit{otherwise} \end{cases}$
- $\int_{-\infty}^{\infty} \delta(t) dt = 1$
- $\int_a^b \delta(t - t_0) x(t) dt = \begin{cases} x(t_0) & : a < t_0 < b \\ 0 & : \textit{otherwise} \end{cases}$
- Note: This isn't really a function! Assume we can treat it like one. (Yay, engineering.)

Sampling

- Build a *Dirac comb* out of impulses spaced by the *sampling period* T_s
  $(f_s = \frac{1}{T_s}, \omega_s = 2\pi f_s)$
- $s(t) = \Sigma_{n=-\infty}^{\infty} \delta(t - nT_s)$
- $y(t) = s(t)x(t)$



Sampling (cont.)

- We need a friendlier form to work with
- And some more math gets us there...
- $s(t) = \Sigma_{n=-\infty}^{\infty} \delta(t - nT_s) = \Sigma_{n=-\infty}^{\infty} \frac{1}{T_s} e^{j2\pi nt/T_s}$
- $\mathcal{F}\{s(t)\} = S(\omega) = \int_{-\infty}^{\infty} \Sigma_{n=-\infty}^{\infty} \frac{1}{T_s} e^{j2\pi nt/T_s} e^{-j\omega t} dt$
- $S(\omega) = \frac{2\pi}{T_s} \Sigma_{n=-\infty}^{\infty} \delta(\omega - 2\pi nt/T_s)$

Sampling (cont.)

- $y(t) = s(t)x(t)$
- $Y(\omega) = S(\omega) * X(\omega)$
- $S(\omega) = \frac{2\pi}{T_s}\Sigma_{n=-\infty}^{\infty}\delta(\omega - 2\pi nt/T_s)$
- $Y(\omega) = \frac{2\pi}{T_s}\Sigma_{n=-\infty}^{\infty}\delta(\omega - 2\pi nt/T_s) * X(\omega)$
- Try graphically convolving offset impulse with a function; what happens? (It shifts!)

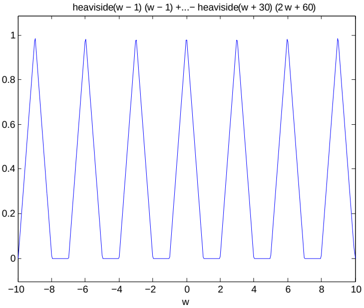
The Intuition

Sampling a signal with sampling angular frequency ω_s in the time/spacial domain leads to infinite replication and shifting by ω_s in the frequency domain (times a constant)

Nyquist Rate

Case 1: $\omega_s > 2\omega_{bw}$

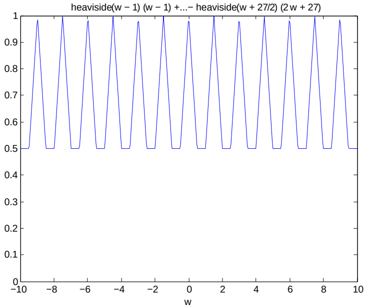
Original signal in frequency domain is triangle function for these next examples.



Nyquist Rate (cont.)

Case 2: $\omega_s < 2\omega_{bw}$

Note what a postfilter would leave behind; interaction between replicas.



Prefilter

Same signal as the last slide; prefilter leads to less distortion