CS 4220: Prelim 1 Solution Guide

```
95-100  x x x x x x x x x x x

90-94   x x x x x x

85-89   x x x x x

80-84   x x x x x x x x x

75-79   x x

70-74   x x x x

65-69   x x x x x x x

60-64   x x x x

55-59   x x x x

50-54   x x x x

<50     x x x
```

Rough grade guidelines

A = [85,100]

B = [65,75]

C = [50,60]

1. (25 points) The product of two upper triangular matrices is upper triangular:

$$\begin{bmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \end{bmatrix} \begin{bmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \end{bmatrix} = \begin{bmatrix} x & x & x & x & x \\ 0 & x & x & x & x \\ 0 & 0 & x & x & x \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & 0 & x \end{bmatrix}.$$

The following function correctly implements this operation:

```
  function C = TriProd(A,B)
% A and B and nxn upper triangular matrices and C = AB
[n,n] = size(A);
C = zeros(n,n);
for j=1:n
   for k = 1:n
      C(:,j) = C(:,j) + B(k,j)*A(:,k);
   end
end
```

Modify the this implementation so that it is flop efficient. As with the given implementation, the body of the inner loop in your solution should feature an operation of the form

$$\text{vector} \leftarrow \text{vector} + \text{scalar} \times \text{vector}$$

Solution:

```
function C = TriProd(A,B)
% A and B and nxn upper triangular matrices and C = AB
[n,n] = size(A);
C = zeros(n,n);
for j=1:n
   for k = 1:j
      C(1:k,j) = C(1:k,j) + B(k,j)*A(1:k,k);
   end
end
```

To see what is going on, look at 3rd column in the example:

```
  x                  x               x               x
  x                  0               x               x
  x      =   B(1,3)* 0   + B(2,3) *  0   +  B(3,3) * x
  0                  0               0               0
  0                  0               0               0
```

12 points for abbreviated loop range
13 points for shortened vector operation

2. (15 points) Assume that $A \in \mathbb{R}^{n\times n}$ is symmetric positive definite and that its Cholesky factorization $A = GG^T$ is available where G is lower triangular. Give an efficient algorithm for computing $\alpha = v^T A^{-1} v$ where v is the last column of the n-by-n identity matrix. You may use \ if necessary.

20 points:

$$\alpha = v^T w$$

where $Aw = v$. So

```
w = G'\(G\v);
alfa = w(n)
```

23 points:

$$\alpha = v^T (GG^T)^{-1} v = (G^{-1}v)^T (G^{-1}v) = y^T y$$

where $Gy = v$. So

```
y = G\v;
alfa = y'*y
```

25 points: If $Gy = v$ then because G is lower triangular, then $y = v/g_{nn}$. Since $v^T v = 1$,

```
alfa = 1/G(n,n)^2
```

Solution that explicitly compute A^{-1} of G^{-1} receive about 7 points. NEVER COMPUTE THE EXPLICIT INVERSE

3. (25 points) Consider the following linear system

$$\begin{bmatrix} A & I & 0 \\ 0 & A & I \\ 0 & 0 & A \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d \\ e \\ f \end{bmatrix}$$

where we assume that $A \in \mathbb{R}^{n \times n}$ is nonsingular and $d, e, f \in \mathbb{R}^n$ are given. Write a MATLAB fragment that assigns to —tt x, `y`, and `z` the solution vectors $x, y, z \in \mathbb{R}^n$. Make effective use of `[L,U,P] = lu(A)`.

```
Solution:
```

```
[L,U,P] = lu(A);

z = U\(L\(P*f));

y = U\L\(P*(e - z)))

x = U\L\(P*(d - y)))
```

```
5 points  Az = f
5 points  Ay = e-z
5 points  Ax = d-y
10 points for using [L,U,P] = lu(A) correctly

-3 for ignporing P or using P' instead of P
```

4. (25 points) Answer each of the following questtions with at most one or two sentences.

(a) What can you say about the SVD of A if A is close to singular with respect to the unit roundoff?

9 points for things like "the smallest singular value is about the unit roundoff" or "cond(A) =)(1/eps)".

(b) What can you say about the SVD of $A \in \mathbb{R}^{n \times n}$ if $\text{rank}(A) = r < n$?

8 points fo "All but the first r singular values are zero" or "A has a zero singular value"

(c) What can you say about the SVD of A if $\| Ax \|_2$ is constant for all unit 2-norm vectors x?

8 points for "All the singular values of A are the same" or even "Sigma = I"