1. (25 points)

Assume that $A \in \mathbb{R}^{n\times n}$ is nonsingular, $C \in \mathbb{R}^{n\times n}$, $c \in \mathbb{R}^n$, and $d \in \mathbb{R}^n$. How would you use the LU factorization with pivoting to solve for the vectors $y \in \mathbb{R}^n$ and $z \in \mathbb{R}^n$ in

$$\begin{bmatrix} C & A \\ A^T & 0 \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} = \begin{bmatrix} c \\ d \end{bmatrix}$$

It is fine to express your answer in MATLAB.

Solution

Block back-substitution idea:

(i) Solve $A^T y = d$ for y.

(ii) Solve $Az = c - Cy$ for z

Carry this out with a single LU factorization

Compute $PA = LU$.
Since $A^T P^T = U^T L^T$, $d = A^T y = (A^T P^T)(Py) = U^T(L^T(Py)$. So

```
[L,U,P] = lu(A);
y = P'*(L'\(U'\d)
z = U\(L\(P*(c - C*y)))
```

2. (20 points)

Suppose $A \in \mathbb{R}^{m\times n}$ and that rank$(A) = m < n$. Assume that we have applied QR with column pivoting to this matrix obtaining

$$A\Pi \;=\; QR$$

where $\Pi \in \mathbb{R}^{n\times n}$ is a permutation, $Q \in \mathbb{R}^{m\times m}$ is orthogonal, and $R \in \mathbb{R}^{m\times n}$ is upper triangular. How would you compute a solution to the underdetermined linear system $Ax = b$? No \ allowed in your answer except for square, triangular systems.

Solution

$b = Ax = (A\Pi)(\Pi^T x)$ so we need $Q^T b \;=\; R\Pi^T x$. Partitioning

$$R = [R_1 R_2] \qquad R_1 \in \mathbb{R}^{m\times m},\; R_{12} \in \mathbb{R}^{m\times n-m}$$

and

$$\Pi^T x \;=\; \begin{bmatrix} y \\ z \end{bmatrix} \qquad y \in \mathbb{R}^m, z \in \mathbb{R}^{n-m}$$

we require $Q^T b \;=\; R_1 y + R_{12} z$. Once z is picked, we solve a triangular system for y. If we set $z = 0$ then

```
y = R(1:m,1:m)\(Q'*b);

z = zeros(n-m,1);

x = P*[y;z]
```

3. (20 points)

Suppose we have the factorization

$$A = LDL^T$$

where $L \in \mathbb{R}^{n\times n}$ is lower triangular and $D = \text{diag}(d_1, \ldots, d_n)$. Assume that $d_k < 0$ for some index k and that $k << n$. How would you compute a unit 2-norm vector $x \in \mathbb{R}^n$ so that $x^T Ax < 0$.? Express your answer in MATLAB assuming that `L`, `D`, and `k` are initialized. You may use the \ operator.

Solution

We want $0 > x^T LDL^T x = (L^T x)^T D(L^T x)$. Note that $e_k^T De_k = d_k < 0$. Thus, if $L^T z = e_k$ then $z^T Az < 0$.

```
z = L(1:k,1:k)'\[zeros(k-1,1);1];

x = z/norm(z);

x = [x;zeros(n-k,1)];
```

4. (20 points)

(a) Suppose the floating point addition of 8000 and 2^{-13} is 8000 on a computer that does base-2 floating point arithmetic. Give a reasonable upper bound on the number of bits that are used to represent the mantissa. Explain

Solution

$8000 = 5^3 2^6$. 125 requires about 7 bits to represent. $125 + 2^{-13}$ requires about 20 bits to represent. Since there isn't enough mantissa hardware to do this we may conclude that there are at most 19 or 20 bits used for the mantissa.

(b) Suppose $A \in \mathbb{R}^{n\times n}$ is nonsingular with SVD $A = U\Sigma V^T$. What is the 2-norm condition of

$$A_\epsilon = A - \epsilon u_n v_n^T$$

where $u_n = U(:, n)$, $v_n = V(:, n)$, and $0 < \epsilon < \sigma_n$.

Solution

$$U^T A_\epsilon V = \Sigma - \epsilon(U^T u_n)(V^T v_n) = \Sigma - \epsilon e_n e_n^T.$$

Thus, the largest singular value of A_ϵ is σ_1 and the smallest singular value is $\sigma_n - \epsilon$. The condition is $\sigma_1/(\sigma_n - \epsilon)$.

5. (15 points)

Complete the following function so that it performs as specified. You may use the \ operator.

```
 function x = LS3(A1,b1,A2,b2,A3,b3)
% A1, A2, and A3 are each m-by-n with rank n
% b1, b2, and b3 are each m-by-1.
% x minimizes norm(A1*x-b1,2)^2 + norm(A2*x-b2,2)^2 + norm(A3*x-b3,2)^2
```

Solution

Since

$$\| A_1x - b_1 \|_2^2 + \| A_2x - b_2 \|_2^2 + \| A_3x - b_3 \|_2^2 \;=\; \left\| \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix} x - \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \right\|_2^2$$

```
x = [A1;A2;A3]\[b1;b2;b3]
```

Or the small example $(m = 3, n = 2)$ route:

$$\begin{aligned}
\| A1 * x - b1 \|_2^2 \;&=\; |A1(1,:) * x - b1(1)|^2 + |A1(2,:) * x - b1(2)|^2 + |A1(3,:) * x - b1(3)|^2 \\
\| A2 * x - b2 \|_2^2 \;&=\; |A2(1,:) * x - b2(1)|^2 + |A2(2,:) * x - b2(2)|^2 + |A2(3,:) * x - b2(3)|^2 \\
\| A3 * x - b3 \|_2^2 \;&=\; |A3(1,:) * x - b3(1)|^2 + |A3(2,:) * x - b3(2)|^2 + |A3(3,:) * x - b3(3)|^2
\end{aligned}$$

so the sum of these 9 terms is the 2-norm squared of the vector

$$\begin{bmatrix} A1(1,:)x - b1(1) \\ A1(2,:)x - b1(2) \\ A1(3,:)x - b1(3) \\ A2(1,:)x - b2(1) \\ A2(2,:)x - b2(2) \\ A2(3,:)x - b2(3) \\ A3(1,:)x - b3(1) \\ A3(2,:)x - b3(2) \\ A3(3,:)x - b3(3) \end{bmatrix} = \begin{bmatrix} A1 * x - b1 \\ A2 * x - b2 \\ A3 * x - b3 \end{bmatrix} = \begin{bmatrix} A1 \\ A2 \\ A3 \end{bmatrix} x - \begin{bmatrix} b1 \\ b2 \\ b3 \end{bmatrix}$$