### CS 4110

# **Programming Languages & Logics**

# Lecture 29 Propositions as Types

### Propositions as Types

Logics = Type Systems

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Logicians use the same kind of notation to build up the set of true logical formulas.

Here's a rule from natural deduction, a *constructive* logic invented by logician Gerhard Gentzen in 1935:

$$rac{\phi \quad \psi}{\phi \wedge \psi} \wedge \text{-intro}$$

Given a proof of  $\phi$  and a proof of  $\psi$ , the rule lets you construct a proof of  $\phi \wedge \psi.$ 

Let's use our usual 4110 tools to define the set of true formulas ("theorems").

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We'll start with a grammar for formulas:

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$$\begin{array}{cccc} ::= & \top \\ & \mid & \bot \\ & \mid & X \\ & \mid & \phi \land \psi \\ & \mid & \phi \lor \psi \\ & \mid & \phi \to \psi \\ & \mid & \neg \phi \\ & \mid & \forall X. \phi \end{array}$$

where X ranges over Boolean variables and  $\neg \phi$  is an abbreviation for  $\phi \rightarrow \bot$ .

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Examples:

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- $\vdash \neg (A \land B) \rightarrow \neg A \lor \neg B$
- $A, B, C \vdash B$

Let's write the rules for our judgment:

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...and so on.

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$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi} \xrightarrow{\Gamma \vdash \phi} \varphi \rightarrow \text{-INTRO} \qquad \frac{\Gamma \vdash \phi \rightarrow \psi}{\Gamma \vdash \psi} \xrightarrow{\Gamma \vdash \phi} \rightarrow \text{-ELIM}$$

$$\frac{\vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi} \land \text{-INTRO} \qquad \frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \phi} \land \text{-ELIM1} \qquad \frac{\Gamma \vdash \phi \land \psi}{\Gamma \vdash \psi} \land \text{-ELIM2}$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \lor \psi} \lor \text{-INTRO1} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \lor \psi} \lor \text{-INTRO2}$$

$$\frac{\Gamma \vdash \phi \lor \psi}{\Gamma \vdash \chi} \lor \text{-INTRO1} \qquad \frac{\Gamma \vdash \psi \rightarrow \chi}{\Gamma \vdash \chi} \lor \text{-ELIM}$$

$$\frac{\Gamma, P \vdash \phi}{\Gamma \vdash \forall P, \phi} \lor \text{-INTRO} \qquad \frac{\Gamma \vdash \forall P, \phi}{\Gamma \vdash \phi \{\psi/P\}} \lor \text{-ELIM}$$

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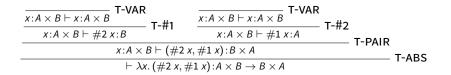


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#### Does this look familiar?



Every natural deduction proof tree has a corresponding type tree in System F with product and sum types! And vice-versa!

	Type Systems	Formal Logic
$\tau$	Туре	$\phi$ Formula
$\tau$	is inhabited	$\phi$ is a theorem
е	Well-typed expression	$\pi$ Proof

A program with a given type acts as a *witness* that the type's corresponding formula is true.

Every type rule in System F with product and sum types corresponds 1-1 with a proof rule in natural deduction:

Type Systems		Formal Logic	
$\rightarrow$	Function	$\rightarrow$	Implication
×	Product	$\wedge$	Conjunction
+	Sum	$\vee$	Disjunction
$\forall$	Universal	$\forall$	Quantifier

You can even add existential types to correspond to existential quantification. It still works!

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Is this a coincidence? Natural deduction was invented by a German logician in 1935. Types for the  $\lambda$ -calculus were invented by Church at Princeton in 1940.

# Propositions as Types Through the Ages

# Natural Deduction

Gentzen (1935)

**Type Schemes** Hindley (1969)

**System F** Girard (1972)

**Modal Logic** Lewis (1910)

**Classical-Intuitionistic Embedding** Gödel (1933)

- $\Leftrightarrow \quad \textbf{Typed } \lambda \textbf{-Calculus} \\ \text{Church (1940)} \\$
- ↔ ML's Type System Milner (1975)
- $\Leftrightarrow \quad \begin{array}{l} \textbf{Polymorphic } \lambda \textbf{-Calculus} \\ \text{Reynolds (1974)} \end{array}$

↔ Monads Kleisli (1965), Moggi (1987)

⇔ Continuation Passing Style Reynolds (1972)

### Term Assignment

This all means that we have a new way of proving theorems: writing programs!

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To prove a formula  $\phi$ :

- 1. Convert the  $\phi$  into its corresponding type  $\tau$ .
- 2. Find some program v that has the type  $\tau$ .
- 3. Realize that the existence of v implies a type tree for  $\vdash v:\tau$ , which implies a proof tree for  $\vdash \phi$ .

#### Negation and Continuations

Let's explore one extension. We'd like to use this rule from classical logic:

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Recall that  $\neg \phi$  is shorthand for  $\phi \rightarrow \bot$ . So  $\neg \neg \phi$  corresponds to the System F function type  $(\tau \rightarrow \bot) \rightarrow \bot$ .

So what we need is a way to take any program of any type  $\tau$  and turn it into a program of type  $(\tau \rightarrow \bot) \rightarrow \bot$ .

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So what we need is a way to take any program of any type  $\tau$  and turn it into a program of type  $(\tau \rightarrow \bot) \rightarrow \bot$ .

Shockingly, that's exactly what the CPS transform does! A  $\tau$  becomes a function that takes a continuation  $\tau \to \bot$  and invokes it, producing  $\bot$ .