# CS 4110

# Programming Languages & Logics

## Lecture 25
## Type Inference

# Review: Polymorphic $\lambda$-Calculus

## Syntax

$$\begin{aligned} e &::= n \mid x \mid \lambda x\!:\!\tau.\, e \mid e_1\, e_2 \mid \Lambda X.\, e \mid e\, [\tau] \\ v &::= n \mid \lambda x\!:\!\tau.\, e \mid \Lambda X.\, e \end{aligned}$$

## Dynamic Semantics

$$E ::= [\cdot] \mid E\, e \mid v\, E \mid E\, [\tau]$$

$$\frac{e \to e'}{E[e] \to E[e']} \qquad \frac{}{(\lambda x\!:\!\tau.\, e)\, v \to e\{v/x\}} \qquad \frac{}{(\Lambda X.\, e)\, [\tau] \to e\{\tau/X\}}$$

# Review: Polymorphic $\lambda$-Calculus

$$\frac{}{\Delta, \Gamma \vdash n : \mathbf{int}} \qquad \frac{\Gamma(x) = \tau}{\Delta, \Gamma \vdash x : \tau}$$

$$\frac{\Delta, \Gamma, x : \tau \vdash e : \tau' \quad \Delta \vdash \tau \text{ ok}}{\Delta, \Gamma \vdash \lambda x : \tau.\, e : \tau \to \tau'} \qquad \frac{\Delta, \Gamma \vdash e_1 : \tau \to \tau' \quad \Delta, \Gamma \vdash e_2 : \tau}{\Delta, \Gamma \vdash e_1\, e_2 : \tau'}$$

$$\frac{\Delta \cup \{X\}, \Gamma \vdash e : \tau}{\Delta, \Gamma \vdash \Lambda X.\, e : \forall X.\, \tau} \qquad \frac{\Delta, \Gamma \vdash e : \forall X.\, \tau' \quad \Delta \vdash \tau \text{ ok}}{\Delta, \Gamma \vdash e\, [\tau] : \tau'\{\tau / X\}}$$

# Review: Polymorphic $\lambda$-Calculus

Polymorphism let us write a doubling function that works for *any* type of function:

$$\text{double} \triangleq \Lambda X.\, \lambda f{:}X \to X.\, \lambda x{:}X.\, f\,(f\,x).$$

The type of this expression is:

$$\forall X.\, (X \to X) \to X \to X$$

You can use the polymorphic function by providing a type:

$$\text{double}\,[\mathbf{int}]\,(\lambda n{:}\mathbf{int}.\, n + 1)\; 7$$

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In languages like OCaml, programmers don't have to annotate their programs with $\forall X.\ \tau$ or $e\ [\tau]$.

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For example, we can write:

```
let double f x = f (f x)
```

and OCaml will figure out that the type is:

```
('a → 'a) → 'a → 'a
```

which is equivalent to the same System F type:

$$\forall A.\ (A \to A) \to A \to A$$

# Type Inference

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We can also write

```
double (fun x → x+1) 7
```

and OCaml will infer that the polymorphic function `double` is instantiated at the type `int`.

# ML Polymorphism

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- Not prenex: $(\forall \alpha.\, \alpha \to \alpha) \to \textbf{int}$

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## Examples

- Prenex: $\forall \alpha.\, \alpha \to \alpha$
- Not prenex: $(\forall \alpha.\, \alpha \to \alpha) \to \textbf{int}$

These restrictions have the following practical ramifications:

- Can't instantiate type variables with polymorphic types
- Can't put a polymorphic type on the left of an arrow

# Example

These restrictions mean that certain terms that are typeable in System F are not typeable in ML!

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```
         OCaml version 4.01.0

# fun x -> x x;;
Error: This expression has type 'a -> 'b
    but an expression was expected of type 'a
    The type variable 'a occurs inside 'a -> 'b
```

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Type inference may be undecidable for the polymorphic $\lambda$-calculus and OCaml, but it is possible for the simply-tpyed $\lambda$-calculus!

Type inference for the STLC means guessing a $\tau$ in every abstraction in an *untyped* version:

$$\lambda x.\, e$$

to produce a *typed* program:

$$\lambda x : \tau.\, e$$

that we can use in the typing rule for functions.

# Example

Here's an untyped program:

$\lambda a.\, \lambda b.\, \lambda c.\, \text{if } a\ (b + 1) \text{ then } b \text{ else } c$

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- the argument type of $a$ must be the same as $b + 1$
- the result type of $a$ must be **bool**
- the type of $c$ must be the same as $b$

Putting all these pieces together:

$\lambda a : \mathbf{int} \to \mathbf{bool}.\ \lambda b : \mathbf{int}.\ \lambda c : \mathbf{int}.\ \text{if } a\ (b + 1) \text{ then } b \text{ else } c$

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Given a typing context $\Gamma$ and an expression $e$, it generates a set of *constraints*—equations between types.

If these constraints are solvable, then $e$ can be well-typed in $\Gamma$.

A solution to a set of constraints is a *type substitution* $\sigma$ that, for each equation, makes both sides syntactically equal.

# STLC for Type Inference

Let's define the type inference judgment for this STLC language:

$$e ::= x \mid \lambda x\!:\!\tau.\, e \mid e_1\; e_2 \mid n \mid e_1 + e_2$$
$$\tau ::= \textbf{int} \mid X \mid \tau_1 \to \tau_2$$

You can use a type variable $X$ wherever you want to have a type inferred.

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$$\frac{\Gamma \vdash e_1 : \tau_1 \mid C_1 \quad \Gamma \vdash e_2 : \tau_2 \mid C_2}{\Gamma \vdash e_1 + e_2 : \textbf{int} \mid C_1 \cup C_2 \cup \{\tau_1 = \textbf{int}, \tau_2 = \textbf{int}\}}\ \textsc{CT-Add}$$

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$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \mid C}{\Gamma \vdash \lambda x : \tau_1 .\, e : \tau_1 \to \tau_2 \mid C}\ \textsc{CT-Abs}$$

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$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \tau_1 \mid C_1 \quad \Gamma \vdash e_2 : \tau_2 \mid C_2 \\ X \text{ fresh} \quad C' = C_1 \cup C_2 \cup \{\tau_1 = \tau_2 \to X\}\end{array}}{\Gamma \vdash e_1\, e_2 : X \mid C'}\ \textsc{CT-App}$$

# Solving Constraints

A *type substitution* is a finite map from type variables to types.

Example: The substitution

$$[X \mapsto \textbf{int}, Y \mapsto \textbf{int} \to \textbf{int}]$$

maps type variable $X$ to **int** and $Y$ to $\textbf{int} \to \textbf{int}$.

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We don't need to worry about avoiding variable capture: all type variables are "free."

Given two substitutions $\sigma_1$ and $\sigma_2$, we write $\sigma_1 \circ \sigma_2$ for their composition: $(\sigma_1 \circ \sigma_2)(\tau) = \sigma_1(\sigma_2(\tau))$.

# Unification

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We say that a substitution $\sigma$ *unifies* constraint $\tau = \tau'$ if $\sigma(\tau) = \sigma(\tau')$.

We say that substitution $\sigma$ *satisfies* (or *unifies*) set of constraints *C* if $\sigma$ unifies every constraint in *C*.

# Unification

If:

- $\Gamma \vdash e : \tau \mid C$, and
- $\sigma$ satisfies $C$,

then $e$ has type $\tau'$ under $\Gamma$,
where $\sigma(\tau) = \tau'$.

If there are no substitutions that satisfy $C$, then $e$ is not typeable.

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So let's find a substitution $\sigma$ that unifies a set of constraints $C$!

# Unification Algorithm

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  $unify(C'\{\tau'/X\}) \circ [X \mapsto \tau']$
else if $\tau' = X$ and $X$ not a free variable of $\tau$ then
  $unify(C'\{\tau/X\}) \circ [X \mapsto \tau]$
else if $\tau = \tau_o \rightarrow \tau_1$ and $\tau' = \tau'_o \rightarrow \tau'_1$ then
  $unify(C' \cup \{\tau_0 = \tau'_0, \tau_1 = \tau'_1\})$

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else if $\tau = \tau_o \to \tau_1$ and $\tau' = \tau'_o \to \tau'_1$ then
    *unify*$(C' \cup \{\tau_0 = \tau'_0, \tau_1 = \tau'_1\})$
else
    *fail*

# Unification Properties

The unification algorithm always terminates.

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The solution, if it exists, is the most general solution: if $\sigma = \mathit{unify}(C)$ and $\sigma'$ is a solution to $C$, then there is some $\sigma''$ such that $\sigma' = (\sigma'' \circ \sigma)$.