# CS 4110

# Programming Languages & Logics

## Lecture 8
## Denotational Semantics Proofs

# Kleene Fixed-Point Theorem

## Definition (Scott Continuity)

A function $F$ is *Scott-continuous* if for every chain $X_1 \subseteq X_2 \subseteq \dots$ we have $F(\bigcup_i X_i) = \bigcup_i F(X_i)$.

# Kleene Fixed-Point Theorem

## Definition (Scott Continuity)

A function $F$ is *Scott-continuous* if for every chain $X_1 \subseteq X_2 \subseteq \dots$ we have $F(\bigcup_i X_i) = \bigcup_i F(X_i)$.

## Theorem (Kleene Fixed Point)

*Let $F$ be a Scott-continuous function. The least fixed point of $F$ is $\bigcup_i F^i(\emptyset)$.*

# Denotational Semantics for IMP Commands

$$\mathcal{C}[\![\textbf{skip}]\!] = \{(\sigma, \sigma)\}$$

$$\mathcal{C}[\![x := a]\!] = \{(\sigma, \sigma[x \mapsto n]) \mid (\sigma, n) \in \mathcal{A}[\![a]\!]\}$$

$$\mathcal{C}[\![c_1; c_2]\!] = \{(\sigma, \sigma') \mid \exists \sigma''.\ ((\sigma, \sigma'') \in \mathcal{C}[\![c_1]\!] \wedge (\sigma'', \sigma') \in \mathcal{C}[\![c_2]\!])\}$$

$$\mathcal{C}[\![\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2]\!] = \{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in \mathcal{C}[\![c_1]\!]\} \ \cup \{(\sigma, \sigma') \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!] \wedge (\sigma, \sigma') \in \mathcal{C}[\![c_2]\!]\}$$

$$\mathcal{C}[\![\textbf{while } b \textbf{ do } c]\!] = \mathit{fix}(f)$$

where $F(f) = \{(\sigma, \sigma) \mid (\sigma, \textbf{false}) \in \mathcal{B}[\![b]\!]\} \ \cup \{(\sigma, \sigma') \mid (\sigma, \textbf{true}) \in \mathcal{B}[\![b]\!] \wedge \exists \sigma''.\ ((\sigma, \sigma'') \in \mathcal{C}[\![c]\!] \wedge (\sigma'', \sigma') \in f\}$

# Exercises

**skip**; $c$ and $c$; **skip** are equivalent.

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$\mathcal{C}[\![\textbf{while false do } c]\!]$ is equivalent to **skip**.

$\mathcal{C}[\![\textbf{while true do skip}]\!] = ?$