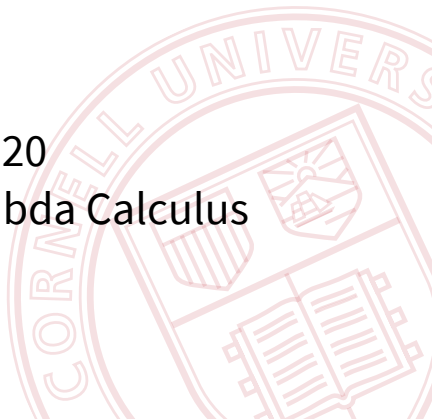


CS 4110

Programming Languages & Logics

Lecture 20

Simply-Typed Lambda Calculus



Simply-Typed Lambda Calculus

Syntax

expressions	$e ::= x \mid \lambda x:\tau. e \mid e_1 e_2 \mid n \mid e_1 + e_2 \mid ()$
values	$v ::= \lambda x:\tau. e \mid n \mid ()$
types	$\tau ::= \mathbf{int} \mid \mathbf{unit} \mid \tau_1 \rightarrow \tau_2$

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Dynamic Semantics

$$E ::= [\cdot] \mid E e \mid v E \mid E + e \mid v + E$$

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x:\tau. e) v \rightarrow e\{v/x\}}$$

$$\frac{n = n_1 + n_2}{n_1 + n_2 \rightarrow n}$$

Simply-Typed Lambda Calculus

Static Semantics

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$$\frac{}{\Gamma \vdash n : \mathbf{int}} \text{T-INT}$$

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$$\frac{}{\Gamma \vdash n : \mathbf{int}} \text{T-INT}$$

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Static Semantics

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$$\frac{}{\Gamma \vdash () : \mathbf{unit}} \text{T-UNIT}$$

$$\frac{\Gamma \vdash e_1 : \mathbf{int} \quad \Gamma \vdash e_2 : \mathbf{int}}{\Gamma \vdash e_1 + e_2 : \mathbf{int}} \text{T-ADD}$$

Simply-Typed Lambda Calculus

Static Semantics

$$\frac{}{\Gamma \vdash n : \mathbf{int}} \text{T-INT}$$

$$\frac{}{\Gamma \vdash () : \mathbf{unit}} \text{T-UNIT}$$

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$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{T-VAR}$$

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$$\frac{\Gamma \vdash e_1 : \mathbf{int} \quad \Gamma \vdash e_2 : \mathbf{int}}{\Gamma \vdash e_1 + e_2 : \mathbf{int}} \text{T-ADD}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{T-VAR}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau. e : \tau \rightarrow \tau'} \text{T-ABS}$$

Simply-Typed Lambda Calculus

Static Semantics

$$\frac{}{\Gamma \vdash n : \mathbf{int}} \text{T-INT}$$

$$\frac{}{\Gamma \vdash () : \mathbf{unit}} \text{T-UNIT}$$

$$\frac{\Gamma \vdash e_1 : \mathbf{int} \quad \Gamma \vdash e_2 : \mathbf{int}}{\Gamma \vdash e_1 + e_2 : \mathbf{int}} \text{T-ADD}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{T-VAR}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau. e : \tau \rightarrow \tau'} \text{T-ABS}$$

$$\frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 e_2 : \tau'} \text{T-APP}$$

Properties

Theorem (Type soundness)

If $\vdash e : \tau$ and $e \rightarrow^ e'$ and $e' \not\rightarrow$ then e' is a value and $\vdash e' : \tau$.*

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Lemma (Preservation)

If $\vdash e : \tau$ and $e \rightarrow e'$ then $\vdash e' : \tau$.

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Theorem (Type soundness)

If $\vdash e : \tau$ and $e \rightarrow^ e'$ and $e' \not\rightarrow$ then e' is a value and $\vdash e' : \tau$.*

Lemma (Preservation)

If $\vdash e : \tau$ and $e \rightarrow e'$ then $\vdash e' : \tau$.

Lemma (Progress)

If $\vdash e : \tau$ then either e is a value or there exists an e' such that $e \rightarrow e'$.

Extra Lemmas for Preservation

Lemma (Substitution)

If $x:\tau' \vdash e:\tau$ and $\vdash v:\tau'$ then $\vdash e\{v/x\}:\tau$.

Lemma (Context)

If $\vdash E[e]:\tau$ and $\vdash e:\tau'$ and $\vdash e':\tau'$ then $\vdash E[e']:\tau$.

Extra Lemma for Progress

Lemma (Canonical Forms)

If $\vdash v : \tau$, then

- 1. If τ is **int**, then v is a constant, i.e., some c .*
- 2. If τ is $\tau_1 \rightarrow \tau_2$, then v is an abstraction, i.e., $\lambda x : \tau_1. e$ for some x and e .*