CS 4110

Programming Languages & Logics

Lecture 17
Fixed-Point Combinators

Review: Church Booleans

We can encode TRUE, FALSE, and IF, as:

$$\text{TRUE} \triangleq \lambda x.\, \lambda y.\, x$$
$$\text{FALSE} \triangleq \lambda x.\, \lambda y.\, y$$
$$\text{IF} \triangleq \lambda b.\, \lambda t.\, \lambda f.\, b\ t\ f$$

This way, IF behaves how it ought to:

$$\text{IF TRUE}\ v_t\ v_f \rightarrow^* v_t$$
$$\text{IF FALSE}\ v_t\ v_f \rightarrow^* v_f$$

Review: Church Numerals

Church numerals encode a number n as a function that takes f and x, and applies f to x n times.

$$\begin{aligned} \overline{0} &\triangleq \lambda f.\, \lambda x.\, x \\ \overline{1} &\triangleq \lambda f.\, \lambda x.\, f\, x \\ \overline{2} &\triangleq \lambda f.\, \lambda x.\, f\,(f\, x) \end{aligned}$$

We can define other functions on integers:

$$\text{SUCC} \triangleq \lambda n.\, \lambda f.\, \lambda x.\, f\,(n\, f\, x)$$

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\end{aligned}
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\text{SUCC} &\triangleq \lambda n.\, \lambda f.\, \lambda x.\, f\,(n\,f\,x) \\
\text{PLUS} &\triangleq \lambda n_1.\, \lambda n_2.\, n_1\ \text{SUCC}\ n_2 \\
\text{TIMES} &\triangleq \lambda n_1.\, \lambda n_2.\, n_1\,(\text{PLUS}\ \mathrm{n}_2)\ \overline{0}
\end{aligned}
$$

Review: Church Numerals

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Recursive Functions

How would we write recursive functions like factorial?

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We'd like to write it like this...

$$\text{FACT} \triangleq \lambda n.\, \text{IF}\ (\text{ISZERO}\ n)\ 1\ (\text{TIMES}\ n\ (\text{FACT}\ (\text{PRED}\ n)))$$

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In slightly more readable notation this is...

$$\text{FACT} \triangleq \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times \text{FACT}\ (n - 1)$$

...but this is an equation, not a definition!

Recursion removal trick

We can perform a “trick” to define a function FACT that satisfies the recursive equation on the previous slide.

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Define a new function FACT′ that takes a function *f* as an argument. Then, for "recursive" calls, it uses *f f*:

$$\text{FACT}' \triangleq \lambda f.\, \lambda n.\, \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f\, f)\,(n - 1))$$

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Then define FACT as FACT′ applied to itself:

$$\text{FACT} \triangleq \text{FACT}'\ \text{FACT}'$$

Example

Let's try evaluating FACT on 3...

FACT 3

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$$\begin{aligned}\text{FACT } 3 &= (\text{FACT}'\ \text{FACT}')\ 3 \\ &= ((\lambda f.\ \lambda n.\ \textbf{if } n = 0 \textbf{ then } 1 \textbf{ else } n \times ((f\,f)\ (n-1)))\ \text{FACT}')\ 3\end{aligned}$$

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&\to \textbf{if}\ 3 = 0\ \textbf{then}\ 1\ \textbf{else}\ 3 \times ((\mathsf{FACT}'\ \mathsf{FACT}')\ (3-1))
\end{aligned}
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&= 3 \times (\text{FACT}\, (3-1))
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&\to^{*} 6
\end{aligned}
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Fixed point combinators

Our "trick" requires following human-readable instructions. Write a different function f' that takes itself as an argument and uses self-application for recursive calls, and then define f as $f' \; f'$.

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Consider factorial again. It is a fixed point of the following:

$$G \triangleq \lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\,(n-1))$$

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Recall that if g if a fixed point of G, then $G\,g = g$. To see that any fixed point g is a real factorial function, try evaluating it:

$$g\,5 = (G\,g)\,5$$

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$$\begin{aligned} g\,5 &= (G\,g)\,5 \\ &\to^* 5 \times (g\,4) \\ &= 5 \times ((G\,g)\,4) \end{aligned}$$

Fixed point combinators

How can we generate the fixed point of G?

In denotational semantics, finding fixed points took a lot of math. In the λ-calculus, we just need a suitable combinator...

Y Combinator

The (infamous) Y combinator is defined as

$$\mathsf{Y} \triangleq \lambda f.\, (\lambda x.\, f\,(x\,x))\,(\lambda x.\, f\,(x\,x))$$

We say that Y is a *fixed point combinator* because Y f is a fixed point of f (for any lambda term f).

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What happens when we evaluate Y G under CBV?

Z Combinator

To avoid this issue, we'll use a slight variant of the Y combinator, called Z, which is easier to use under CBV.

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Example

Let's see Z in action, on our function G.

FACT

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\end{aligned}
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= & (\lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\, (n - 1))) \\
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=_\beta & \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\lambda y.\, (\mathrm{Z}\, G)\, y)\, (n-1)
\end{array}
$$

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Let's see Z in action, on our function G.

$$
\begin{array}{ll}
 & \mathrm{FACT} \\
= & \mathrm{Z}\,G \\
= & (\lambda f.\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,f\,(\lambda y.\,x\,x\,y)))\,G \\
\to & (\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y)) \\
\to & G\,(\lambda y.\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,(\lambda x.\,G\,(\lambda y.\,x\,x\,y))\,y) \\
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= & \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\text{FACT}\, (n-1))
\end{array}
$$

Other fixed point combinators

There are many (indeed infinitely many) fixed-point combinators. Here's a cute one:

$$\mathsf{Y}_k \triangleq (\mathsf{L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L\,L})$$

where

$$\begin{aligned}\mathsf{L} \triangleq\ & \lambda abcdefghijklmnopqstuvwxyzr. \\ & (r\,(t\,h\,i\,s\,i\,s\,a\,f\,i\,x\,e\,d\,p\,o\,i\,n\,t\,c\,o\,m\,b\,i\,n\,a\,t\,o\,r))\end{aligned}$$

Turing's Fixed Point Combinator

To gain some more intuition for fixed point combinators, let's derive a combinator Θ originally discovered by Turing.

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We know that $\Theta\, f$ is a fixed point of f, so we have

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We can write the following recursive equation:

$$\Theta = \lambda f.\, f\,(\Theta\, f)$$

Now use the recursion removal trick:

$$\begin{aligned} \Theta' &\triangleq \lambda t.\, \lambda f.\, f\,(t\, t\, f) \\ \Theta &\triangleq \Theta'\, \Theta' \end{aligned}$$

θ Example

$\text{FACT} = \Theta\ G$

θ Example

$$\begin{aligned} \text{FACT} &= \Theta\, G \\ &= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\,G \end{aligned}$$

θ Example

$$
\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\, G \\
&\rightarrow (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\, f))\, G
\end{aligned}
$$

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&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\,G \\
&\to (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,f))\,G \\
&\to G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,G)
\end{aligned}$$

θ Example

$$
\begin{aligned}
\mathsf{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\, G \\
&\to (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\, f))\, G \\
&\to G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\, G) \\
&= G\,(\Theta\, G)
\end{aligned}
$$

θ Example

$$
\begin{aligned}
\text{FACT} &= \Theta\, G \\
&= ((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f)))\,G \\
&\rightarrow (\lambda f.\, f\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,f))\,G \\
&\rightarrow G\,((\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,(\lambda t.\, \lambda f.\, f\,(t\,t\,f))\,G) \\
&= G\,(\Theta\, G) \\
&= (\lambda f.\, \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (f\,(n-1)))\,(\Theta\, G) \\
&\rightarrow \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times ((\Theta\, G)\,(n-1)) \\
&= \lambda n.\, \textbf{if}\ n = 0\ \textbf{then}\ 1\ \textbf{else}\ n \times (\text{FACT}\,(n-1))
\end{aligned}
$$