CS 4110

Programming Languages & Logics

Lecture 15
Encodings

Review: λ-calculus

Syntax

$$
\begin{array}{lcl}
e & ::= & x \mid e_1\, e_2 \mid \lambda x.\, e \\
v & ::= & \lambda x.\, e
\end{array}
$$

Semantics

$$
\frac{e_1 \to e_1'}{e_1\, e_2 \to e_1'\, e_2} \qquad \frac{e \to e'}{v\, e \to v\, e'}
$$

$$
\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta
$$

Rewind: Currying

This is just a function that returns a function:

$$\text{ADD} \triangleq \lambda x.\, \lambda y.\, x + y$$

$$\text{ADD } 38 \to \lambda y.\, 38 + y$$

$$\text{ADD } 38\ 4 = (\text{ADD } 38)\ 4 \to 42$$

Informally, you can think of it as a *curried* function that takes two arguments, one after the other.

But that's just a way to get intuition. The λ-calculus only has one-argument functions.

Review: Call-by-Value

Here are the syntax and CBV semantics of λ-calculus:

$$
\begin{aligned}
e &::= x \mid \lambda x.\, e \mid e_1\, e_2 \\
v &::= \lambda x.\, e
\end{aligned}
$$

$$
\frac{e_1 \to e_1'}{e_1\, e_2 \to e_1'\, e_2} \qquad \frac{e \to e'}{v\, e \to v\, e'}
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\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta
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There are two kinds of rules: *congruence rules* that specify evaluation order and *computation rules* that specify the "interesting" reductions.

Evaluation Contexts

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An evaluation context E is an expression with a "hole" in it: a single occurrence of the special symbol $[\cdot]$ in place of a subexpression.

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$$E ::= [\cdot] \mid E\, e \mid v\, E$$

We write $E[e]$ to mean the evaluation context E where the hole has been replaced with the expression e.

Examples

$$E_1 = [\cdot]\,(\lambda x.\, x)$$
$$E_1[\lambda y.\, y\, y] = (\lambda y.\, y\, y)\ \lambda x.\, x$$

Examples

$$
\begin{aligned}
E_1 &= [\cdot]\,(\lambda x.\, x) \\
E_1[\lambda y.\, y\, y] &= (\lambda y.\, y\, y)\; \lambda x.\, x
\end{aligned}
$$

$$
\begin{aligned}
E_2 &= (\lambda z.\, z\, z)\,[\cdot] \\
E_2[\lambda x.\, \lambda y.\, x] &= (\lambda z.\, z\, z)\,(\lambda x.\, \lambda y.\, x)
\end{aligned}
$$

Examples

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\end{aligned}
$$

$$
\begin{aligned}
E_3 &= ([\cdot]\; \lambda x.\, x\, x)\; ((\lambda y.\, y)\; (\lambda y.\, y)) \\
E_3[\lambda f.\, \lambda g.\, f\, g] &= ((\lambda f.\, \lambda g.\, f\, g)\; \lambda x.\, x\, x)\; ((\lambda y.\, y)\; (\lambda y.\, y))
\end{aligned}
$$

CBV With Evaluation Contexts

With evaluation contexts, we can define the evaluation semantics for the CBV λ-calculus with just two rules: one for evaluation contexts, and one for β-reduction.

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With this syntax:

$$E ::= [\cdot] \mid E\, e \mid v\, E$$

The small-step rules are:

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\, e)\, v \to e\{v/x\}}\ \beta$$

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For call-by-name, the syntax for evaluation contexts is different:

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But the small-step rules are the same:

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']}$$

$$\frac{}{(\lambda x.\, e)\ e' \rightarrow e\{e'/x\}}\ \beta$$

Encodings

The pure λ-calculus contains only functions as values. It is not exactly easy to write large or interesting programs in the pure λ-calculus. We can however encode objects, such as booleans, and integers.

Booleans

We need to define functions TRUE, FALSE, AND, NOT, IF, and other operators that behave as follows:

$$\begin{aligned} \text{AND TRUE FALSE} &= \text{FALSE} \\ \text{NOT FALSE} &= \text{TRUE} \\ \text{IF TRUE } e_1\, e_2 &= e_1 \\ \text{IF FALSE } e_1\, e_2 &= e_2 \end{aligned}$$

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Let's start by defining TRUE and FALSE:

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$$\begin{aligned} \text{TRUE} &\triangleq \lambda x.\, \lambda y.\, x \\ \text{FALSE} &\triangleq \lambda x.\, \lambda y.\, y \end{aligned}$$

Booleans

We want the function IF to behave like

$$\lambda b.\ \lambda t.\ \lambda f.\ \text{if } b \text{ is our term TRUE then } t\text{, otherwise } f$$

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We can also write the standard Boolean operators.

$$\begin{aligned} \text{NOT} &\triangleq \\ \text{AND} &\triangleq \\ \text{OR} &\triangleq \end{aligned}$$

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We want the function IF to behave like

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$$\text{IF} \triangleq \lambda b.\, \lambda t.\, \lambda f.\, b\ t\ f$$

We can also write the standard Boolean operators.

$$\text{NOT} \triangleq \lambda b.\, b\ \text{FALSE TRUE}$$

$$\text{AND} \triangleq \lambda b_1.\, \lambda b_2.\, b_1\ b_2\ \text{FALSE}$$

$$\text{OR} \triangleq \lambda b_1.\, \lambda b_2.\, b_1\ \text{TRUE}\ b_2$$

Church Numerals

Let's encode the natural numbers!

We'll write $\overline{n}$ for the encoding of the number n. The central function we'll need is a *successor* operation:

$$\text{SUCC}\ \overline{n} = \overline{n+1}$$

Church Numerals

Church numerals encode a number n as a function that takes f and x, and applies f to x n times.

$$
\begin{aligned}
\overline{0} &\triangleq \lambda f.\, \lambda x.\, x \\
\overline{1} &\triangleq \lambda f.\, \lambda x.\, f\, x \\
\overline{2} &\triangleq \lambda f.\, \lambda x.\, f\,(f\, x)
\end{aligned}
$$

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Church numerals encode a number *n* as a function that takes *f* and *x*, and applies *f* to *x* *n* times.

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\end{aligned}$$

We can write a successor function that "inserts" another application of *f*:

$$\text{SUCC} \triangleq \lambda n.\, \lambda f.\, \lambda x.\, f\,(n\, f\, x)$$

Addition

Given the definition of SUCC, we can define addition. Intuitively, the natural number $n_1 + n_2$ is the result of applying the successor function n_1 times to n_2.

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$$\mathsf{PLUS} \triangleq \lambda n_1.\, \lambda n_2.\, n_1\ \mathsf{SUCC}\ n_2$$