CS 4110

Programming Languages & Logics

Lecture 15
De Bruijn, Combinators, Encodings

Review: λ-calculus

Syntax

$$
\begin{array}{rcl}
e & ::= & x \mid e_1\, e_2 \mid \lambda x.\, e \\
v & ::= & \lambda x.\, e
\end{array}
$$

Semantics

$$
\frac{e_1 \rightarrow e_1'}{e_1\, e_2 \rightarrow e_1'\, e_2} \qquad \frac{e \rightarrow e'}{v\, e \rightarrow v\, e'}
$$

$$
\frac{}{(\lambda x.\, e)\, v \rightarrow e\{v/x\}}\ \beta
$$

Rewind: Currying

This is just a function that returns a function:

$$\text{ADD} \triangleq \lambda x.\, \lambda y.\, x + y$$

$$\text{ADD } 38 \rightarrow \lambda y.\, 38 + y$$

$$\text{ADD } 38\ 4 = (\text{ADD } 38)\ 4 \rightarrow 42$$

Informally, you can think of it as a *curried* function that takes two arguments, one after the other.

But that's just a way to get intuition. The λ-calculus only has one-argument functions.

de Bruijn Notation

Another way to avoid the tricky issues with substitution is to use a *nameless* representation of terms.

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Abstractions have lost their variables!

Variables are replaced with numerical indices!

Examples

Here are some terms written in standard and de Bruijn notation:

Standard	**de Bruijn**
$\lambda x.\ x$	$\lambda.\ 0$
$\lambda z.\ z$	

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$\lambda x.\ \lambda y.\ x$	

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$\lambda x.\ x$	$\lambda.\ 0$
$\lambda z.\ z$	$\lambda.\ 0$
$\lambda x.\ \lambda y.\ x$	$\lambda.\ \lambda.\ 1$
$\lambda x.\ \lambda y.\ \lambda s.\ \lambda z.\ x\ s\ (y\ s\ z)$	

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$(\lambda x.\ x\ x)\ (\lambda x.\ x\ x)$	$(\lambda.\ 0\ 0)\ (\lambda.\ 0\ 0)$
$(\lambda x.\ \lambda x.\ x)\ (\lambda y.\ y)$	$(\lambda.\ \lambda.\ 0)\ (\lambda.\ 0)$

Free variables

To represent a λ-expression that contains free variables in de Bruijn notation, we need a way to map the free variables to integers.

We will work with respect to a map Γ from variables to integers called a *context*.

Examples:

Suppose that Γ maps x to 0 and y to 1.

- Representation of $x\ y$ is 0 1
- Representation of $\lambda z.\ x\ y\ z$ $\lambda.$ 1 2 0

Shifting

To define substitution, we will need an operation that shifts by i the variables above a cutoff c:

$$
\begin{aligned}
\uparrow^i_c (n) &= \begin{cases} n & \text{if } n < c \\ n + i & \text{otherwise} \end{cases} \\
\uparrow^i_c (\lambda.e) &= \lambda.(\uparrow^i_{c+1} e) \\
\uparrow^i_c (e_1\, e_2) &= (\uparrow^i_c e_1)\,(\uparrow^i_c e_2)
\end{aligned}
$$

The cutoff c keeps track of the variables that were bound in the original expression and so should not be shifted.

The cutoff is 0 initially.

Substitution

Now we can define substitution:

$$
\begin{aligned}
n\{e/m\} &= \begin{cases} e & \text{if } n = m \\ n & \text{otherwise} \end{cases} \\
(\lambda.e_1)\{e/m\} &= \lambda.e_1\{(\uparrow^1_0 e)/m+1\} \\
(e_1\, e_2)\{e/m\} &= (e_1\{e/m\})\,(e_2\{e/m\})
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$$

The β rule for terms in de Bruijn notation is just:

$$
\frac{}{(\lambda.e_1)\, e_2 \;\rightarrow\; \uparrow^{-1}_0 (e_1\{\uparrow^1_0 e_2/0\})}\;\beta
$$

Example

Consider the term $(\lambda u.\lambda v.u\,x)\,y$ with respect to a context where $\Gamma(x) = 0$ and $\Gamma(y) = 1$.

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$$\begin{array}{ll} & (\lambda.\lambda.1\;2)\;1 \\ \rightarrow & \uparrow_0^{-1} ((\lambda.1\;2)\{(\uparrow_0^1 1)/0\}) \end{array}$$

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$$\begin{array}{rl} & (\lambda.\lambda.1\,2)\,1 \\ \to & \uparrow_0^{-1} ((\lambda.1\,2)\{(\uparrow_0^1 1)/0\}) \\ = & \uparrow_0^{-1} ((\lambda.1\,2)\{2/0\}) \end{array}$$

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= & \uparrow_0^{-1} \lambda.3\,2
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= & \uparrow_0^{-1}\,\lambda.3\,2 \\
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\end{array}
$$

which, in standard notation (with respect to Γ), is the same as $\lambda v.y\,x$.

Combinators

Another way to avoid the issues having to do with free and bound variable names in the λ-calculus is to work with closed expressions or *combinators*.

With just three combinators, we can encode the entire λ-calculus.

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With just three combinators, we can encode the entire λ-calculus.

$$\mathsf{K} = \lambda x.\lambda y.\, x$$
$$\mathsf{S} = \lambda x.\lambda y.\lambda z.\, x\, z\, (y\, z)$$
$$\mathsf{I} = \lambda x.\, x$$

Combinators

We can even define independent evaluation rules that don't depend on the λ-calculus at all.

Behold the "SKI-calculus":

$$\mathsf{K}\ e_1\ e_2 \to e_1$$
$$\mathsf{S}\ e_1\ e_2\ e_3 \to e_1\ e_3\ (e_2\ e_3)$$
$$\mathsf{I}\ e \to e$$

You would never want to program in this language—it doesn't even have variables!—but it's exactly as powerful as the λ-calculus.

Bracket Abstraction

The function $[x]$ that takes a combinator term M and builds another term that behaves like $\lambda x.M$:

$$
\begin{array}{rcll}
[x]\, x & = & \mathsf{I} & \\
[x]\, N & = & \mathsf{K}\, N & \text{where } x \notin fv(N) \\
[x]\, N_1\, N_2 & = & \mathsf{S}\, ([x]\, N_1)\, ([x]\, N_2) &
\end{array}
$$

The idea is that $([x]\, M)\, N \to M\{N/x\}$ for every term N.

Bracket Abstraction

We then define a function $(e)*$ that maps a λ-calculus expression to a combinator term:

$$
\begin{aligned}
(x)* &= x \\
(e_1\ e_2)* &= (e_1)*\ (e_2)* \\
(\lambda x.e)* &= [x]\ (e)*
\end{aligned}
$$

Example

As an example, the expression $\lambda x. \lambda y.\, x$ is translated as follows:

$$\begin{array}{rl} & (\lambda x. \lambda y.\, x)* \\ = & [x]\,(\lambda y.\, x)* \\ = & [x]\,([y]\,x) \\ = & [x]\,(\mathsf{K}\,x) \\ = & (\mathsf{S}\,([x]\,\mathsf{K})\,([x]\,x)) \\ = & \mathsf{S}\,(\mathsf{K}\,\mathsf{K})\,\mathsf{I} \end{array}$$

No variables in the final combinator term!

Example

We can check that this behaves the same as our original λ-expression by seeing how it evaluates when applied to arbitrary expressions e_1 and e_2.

$$\begin{array}{ll} & (\lambda x. \lambda y.\, x)\; e_1\; e_2 \\ \rightarrow & (\lambda y.\, e_1)\; e_2 \\ \rightarrow & e_1 \end{array}$$

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\begin{array}{ll}
 & (\lambda x.\lambda y.\, x)\; e_1\; e_2 \\
\to & (\lambda y.\, e_1)\; e_2 \\
\to & e_1
\end{array}
$$

and

$$
\begin{array}{ll}
 & (\mathsf{S}\;(\mathsf{K}\;\mathsf{K})\;\mathsf{I})\; e_1\; e_2 \\
\to & (\mathsf{K}\;\mathsf{K}\; e_1)\;(\mathsf{I}\; e_1)\; e_2 \\
\to & \mathsf{K}\; e_1\; e_2 \\
\to & e_1
\end{array}
$$

SKI Without I

Looking back at our definitions...

$$\begin{aligned} &\mathsf{K}\ e_1\ e_2 \rightarrow e_1 \\ &\mathsf{S}\ e_1\ e_2\ e_3 \rightarrow e_1\ e_3\ (e_2\ e_3) \\ &\mathsf{I}\ e \rightarrow e \end{aligned}$$

...*I* isn't strictly necessary. It behaves the same as S K K.

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Our example becomes:

$$\mathsf{S}\ (\mathsf{K}\ \mathsf{K})\ (\mathsf{S}\ \mathsf{K}\ \mathsf{K})$$

Encodings

The pure λ-calculus contains only functions as values. It is not exactly easy to write large or interesting programs in the pure λ-calculus. We can however encode objects, such as booleans, and integers.

Booleans

We need to define functions TRUE, FALSE, AND, NOT, IF, and other operators that behave as follows:

$$\begin{aligned} \text{AND TRUE FALSE} &= \text{FALSE} \\ \text{NOT FALSE} &= \text{TRUE} \\ \text{IF TRUE } e_1\ e_2 &= e_1 \\ \text{IF FALSE } e_1\ e_2 &= e_2 \end{aligned}$$

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$$\begin{aligned} \text{TRUE} &\triangleq \lambda x.\, \lambda y.\, x \\ \text{FALSE} &\triangleq \lambda x.\, \lambda y.\, y \end{aligned}$$

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We want the function IF to behave like

$$\lambda b.\ \lambda t.\ \lambda f.\ \text{if } b \text{ is our term TRUE then } t\text{, otherwise } f$$

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We can also write the standard Boolean operators.

$$\text{NOT} \triangleq$$

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We can also write the standard Boolean operators.

$$\text{NOT} \triangleq \lambda b.\ b\ \text{FALSE TRUE}$$

$$\text{AND} \triangleq \lambda b_1.\ \lambda b_2.\ b_1\ b_2\ \text{FALSE}$$

$$\text{OR} \triangleq \lambda b_1.\ \lambda b_2.\ b_1\ \text{TRUE}\ b_2$$

Church Numerals

Let's encode the natural numbers!

We'll write $\overline{n}$ for the encoding of the number n. The central function we'll need is a *successor* operation:

$$\text{SUCC}\ \overline{n} = \overline{n+1}$$

Church Numerals

Church numerals encode a number *n* as a function that takes *f* and *x*, and applies *f* to *x* *n* times.

$$
\begin{aligned}
\overline{0} &\triangleq \lambda f.\, \lambda x.\, x \\
\overline{1} &\triangleq \lambda f.\, \lambda x.\, f\, x \\
\overline{2} &\triangleq \lambda f.\, \lambda x.\, f\, (f\, x)
\end{aligned}
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We can write a successor function that "inserts" another application of *f*:

$$\text{SUCC} \triangleq \lambda n.\, \lambda f.\, \lambda x.\, f\,(n\, f\, x)$$

Addition

Given the definition of SUCC, we can define addition. Intuitively, the natural number $n_1 + n_2$ is the result of applying the successor function n_1 times to n_2.

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$$\text{PLUS} \triangleq \lambda n_1.\, \lambda n_2.\, n_1 \;\text{SUCC}\; n_2$$