CS 4110

Programming Languages & Logics

Lecture 14
More λ-calculus

Review: λ-calculus

Syntax

$$
\begin{array}{rcl}
e & ::= & x \mid e_1\, e_2 \mid \lambda x.\, e \\
v & ::= & \lambda x.\, e
\end{array}
$$

Semantics (call by value)

$$
\frac{e_1 \rightarrow e_1'}{e_1\, e_2 \rightarrow e_1'\, e_2} \qquad \frac{e \rightarrow e'}{v\, e \rightarrow v\, e'}
$$

$$
\frac{}{(\lambda x.\, e)\, v \rightarrow e\{v/x\}}\ \beta
$$

Example: Twice

Consider the function defined by $\mathit{double}\; x = x + x$.

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We can abstract this pattern using a generic function:

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Now the functions above can be written as

$$\begin{aligned} \mathit{quadruple} &= \mathit{twice}\ \mathit{double} \\ \mathit{hexadecatuple} &= \mathit{twice}\ \mathit{quadruple} \\ \mathit{256uple} &= \mathit{twice}\ \mathit{hexadecatuple} \\ & \ (\text{or } (\mathit{twice}\,(\lambda x.\ \mathit{twice}\, x))\ \mathit{double}) \end{aligned}$$

Evaluation

The essence of λ-calculus evaluation is the β-reduction rule, which says how to apply a function to an argument.

$$\frac{}{(\lambda x.\, e)\; v \rightarrow e\{v/x\}}\ \beta\text{-REDUCTION}$$

But there are many different evaluation strategies, each corresponding to particular ways of using β-reduction:

- Call-by-value
- Call-by-name
- “Full” β-reduction
- ...

Call by value

$$\frac{e_1 \rightarrow e_1'}{e_1\ e_2 \rightarrow e_1'\ e_2} \qquad \frac{e_2 \rightarrow e_2'}{v_1\ e_2 \rightarrow v_1\ e_2'}$$

$$\frac{}{(\lambda x.\, e_1)\ v_2 \rightarrow e_1\{v_2/x\}}\ \beta$$

Key characteristics:

- Arguments evaluated fully before they are supplied to functions
- Evaluation goes from left to right (in this presentation)
- We don't evaluate "under a λ"

Call by name

$$\frac{e_1 \rightarrow e_1'}{e_1\, e_2 \rightarrow e_1'\, e_2}$$

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Key characteristics:

- Arguments supplied immediately to functions
- Evaluation still goes from left to right (in this presentation)
- We still don't evaluate "under a λ"

Full β reduction

$$\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad \frac{e_2 \to e_2'}{e_1\ e_2 \to e_1\ e_2'}$$

$$\frac{e \to e'}{\lambda x.\, e \to \lambda x.\, e'}$$

$$\frac{}{(\lambda x.\, e_1)\ e_2 \to e_1\{e_2/x\}}\ \beta$$

Key characteristics:

- Use the β rule anywhere...
- ...including “under a λ”...
- ...nondeterministically.

Confluence

Full β reduction has this property:

$$\begin{tikzcd}
 & e \arrow[dl] \arrow[dr] & \\
e_1 \arrow[dr] & & e_2 \arrow[dl] \\
 & e' &
\end{tikzcd}$$

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Theorem (Confluence)

If $e \rightarrow^* e_1$ and $e \rightarrow^* e_2$ then $e_1 \rightarrow^* e'$ and $e_2 \rightarrow^* e'$ for some e'.

Substitution

The main workhorse in the β rule is substitution, which replaces free occurrences of a variable x with a term e.

However, defining substitution $e_1\{e_2/x\}$ correctly is tricky...

"Substitution"

As a first attempt, consider:

$$y\{e/x\} \quad = \quad \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases}$$

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What's wrong with this definition?

It substitutes bound variables too!

$$(\lambda y.y)\{3/y\}$$

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We *assume away* abstractions over x. (Thanks, α-equivalence!)

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Real Substitution

The correct definition is *capture-avoiding substitution:*

$$
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(e_1\, e_2)\{e/x\} & = & (e_1\{e/x\})\,(e_2\{e/x\}) & \\
(\lambda y.e_1)\{e/x\} & = & \lambda y.(e_1\{e/x\}) & \text{where } y \neq x \text{ and } y \notin \mathit{fv}(e)
\end{array}
$$

where $\mathit{fv}(e)$ is the *free variables* of a term e.