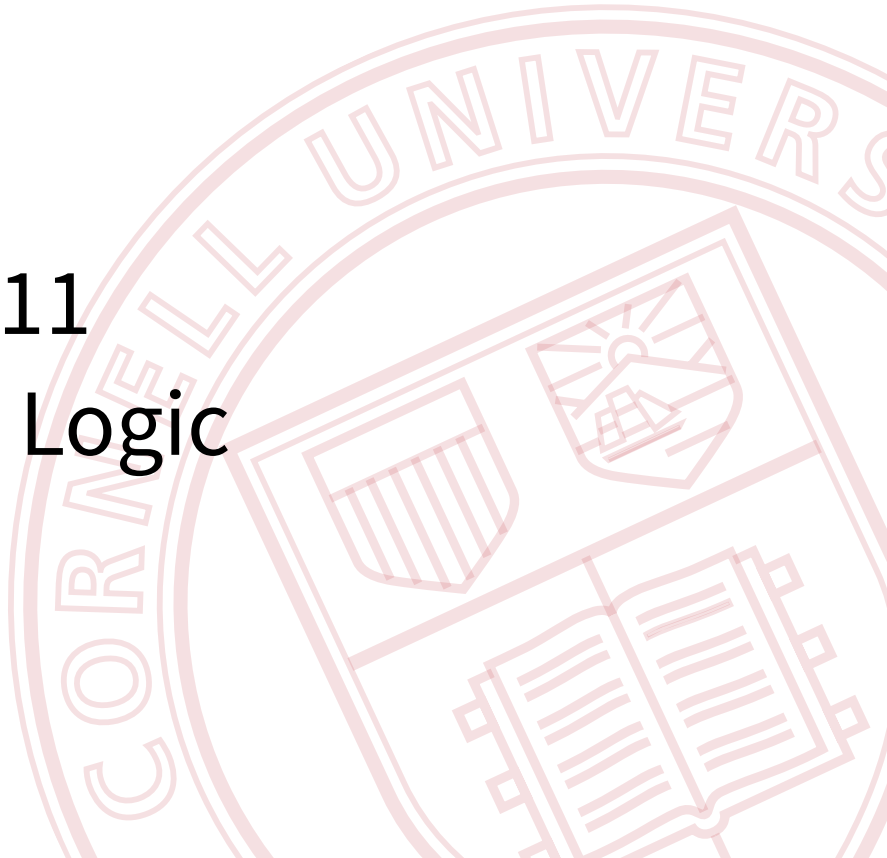
CS 4110

Programming Languages & Logics



Lecture 11

More Hoare Logic

Overview

Last time

- Hoare Logic

Today

- “Decorated” programs
- Weakest Preconditions

Review: Hoare Logic

$$\frac{}{\vdash \{P\}\ \textbf{skip}\ \{P\}}\ \textsc{Skip} \qquad \frac{}{\vdash \{P[a/x]\}\ x := a\ \{P\}}\ \textsc{Assign}$$

$$\frac{\vdash \{P\}\ c_1\ \{R\} \qquad \vdash \{R\}\ c_2\ \{Q\}}{\vdash \{P\}\ c_1; c_2\ \{Q\}}\ \textsc{Seq}$$

$$\frac{\vdash \{P \wedge b\}\ c_1\ \{Q\} \qquad \vdash \{P \wedge \neg b\}\ c_2\ \{Q\}}{\vdash \{P\}\ \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2\ \{Q\}}\ \textsc{If}$$

$$\frac{\vdash \{P \wedge b\}\ c\ \{P\}}{\vdash \{P\}\ \textbf{while}\ b\ \textbf{do}\ c\ \{P \wedge \neg b\}}\ \textsc{While}$$

$$\frac{\models P \Rightarrow P' \qquad \vdash \{P'\}\ c\ \{Q'\} \qquad \models Q' \Rightarrow Q}{\vdash \{P\}\ c\ \{Q\}}\ \textsc{Consequence}$$

Decorated Programs

Observation: Once we've identified loop invariants and uses of consequence, the structure of a Hoare logic is determined!

Notation: Can write proofs by "decorating" programs with:

- A precondition ($\{P\}$)
- A postcondition ($\{Q\}$)
- Invariants ($\{I\}$**while** b **do** c)
- Uses of consequence $\{R\} \Rightarrow \{S\}$
- Assertions between sequences $c_1; \{T\} c_2$

A decorated program describes a valid Hoare logic proof if the rest of the proof tree's structure is implied. (Caveats: Invariants are constrained, etc.)

Example: Decorated Factorial

$$\{\mathsf{x} = n \wedge n > 0\}$$

$$\begin{array}{l} \mathsf{y} := 1; \\ \textbf{while}\ \mathsf{x} > 0\ \textbf{do}\ \{ \\ \qquad \mathsf{y} := \mathsf{y} * \mathsf{x}; \\ \qquad \mathsf{x} := \mathsf{x} - 1 \\ \} \end{array}$$

$$\{\mathsf{y} = n!\}$$

Example: Decorated Factorial

$\{\mathsf{x} = n \wedge n > 0\} \Rightarrow$
$\{1 = 1 \wedge \mathsf{x} = n \wedge n > 0\}$
$\mathsf{y} := 1;$
$\{\mathsf{y} = 1 \wedge \mathsf{x} = n \wedge n > 0\} \Rightarrow$
$\{\mathsf{y} * \mathsf{x}! = n! \wedge x \geq 0\}$
while $\mathsf{x} > 0$ **do** $\{$
$\quad \{\mathsf{y} * \mathsf{x}! = n! \wedge x > 0 \wedge x \geq 0\} \Rightarrow$
$\quad \{\mathsf{y} * \mathsf{x} * (\mathsf{x} - 1)! = n! \wedge (x - 1) \geq 0\}$
$\quad \mathsf{y} := \mathsf{y} * \mathsf{x};$
$\quad \{\mathsf{y} * (\mathsf{x} - 1)! = n! \wedge (x - 1) \geq 0\}$
$\quad \mathsf{x} := \mathsf{x} - 1$
$\quad \{\mathsf{y} * \mathsf{x}! = n! \wedge x \geq 0\}$
$\}$
$\{\mathsf{y} * \mathsf{x}! = n! \wedge (x \geq 0) \wedge \neg(x > 0)\} \Rightarrow$
$\{\mathsf{y} = n!\}$

Informal Rules for Decoration

Check whether a decorated program represents a valid proof using local consistency checks.

Informal Rules for Decoration

Check whether a decorated program represents a valid proof using local consistency checks.

For **skip**, the precondition and postcondition should be the same:

$$\{P\}$$
skip
$$\{P\}$$

Informal Rules for Decoration

For sequences, $\{P\}\ c_1\ \{R\}$ and $\{R\}\ c_2\ \{Q\}$ must be (recursively) locally consistent:

$$
\begin{array}{l}
\{P\} \\
c_1; \\
\{R\} \\
c_2 \\
\{Q\}
\end{array}
$$

Informal Rules for Decoration

Assignment should use the substitution from the rule:

$$\{P[a/x]\}$$
$$x := a$$
$$\{P\}$$

Informal Rules for Decoration

An **if** is locally consistent when both branches are locally consistent after adding the branch condition to each:

$\{P\}$
if b **then**
  $\{P \wedge b\}$
  c_1
  $\{Q\}$
else
  $\{P \wedge \neg b\}$
  c_2
  $\{Q\}$
$\{Q\}$

Informal Rules for Decoration

Decorate a **while** with the loop invariant:

$$
\begin{array}{l}
\{P\} \\
\textbf{while } b \textbf{ do} \\
\quad \{P \wedge b\} \\
\quad c \\
\quad \{P\} \\
\{P \wedge \neg b\}
\end{array}
$$

Informal Rules for Decoration

To capture the CONSEQUENCE rule, you can always write a (valid) implication:

$$\{P\} \Rightarrow \{Q\}$$

Example

$\{ \qquad\qquad\qquad\qquad \}$

while $(0 < y)$ **do** (

$\quad x := x + 1;$

$\quad y := y - 1$

)

$\{ \qquad\qquad\qquad\qquad \}$

Example

$\{\mathsf{x} = m \ \wedge \ \mathsf{y} = n \wedge 0 \leq n\}$

while $(0 < \mathsf{y})$ **do** (

$\quad \mathsf{x} := \mathsf{x} + 1;$

$\quad \mathsf{y} := \mathsf{y} - 1$

)

$\{\mathsf{x} = m + n\}$

Example

$\{\mathsf{x} = m \ \wedge \ \mathsf{y} = n \ \wedge \ 0 \leq n\} \Rightarrow$
$\{I\}$
while $(0 < \mathsf{y})$ **do** (
  $\{I \wedge 0 < y\} \Rightarrow$
  $\{I[\mathsf{y} - 1/\mathsf{y}][\mathsf{x} + 1/\mathsf{x}]\}$
  $\mathsf{x} := \mathsf{x} + 1;$
  $\{I[\mathsf{y} - 1/\mathsf{y}]\}$
  $\mathsf{y} := \mathsf{y} - 1$
  $\{I\}$
)
$\{I \wedge 0 \not< \mathsf{y}\} \Rightarrow$
$\{\mathsf{x} = m + n\}$

Where I is $(\mathsf{x} = m + n - \mathsf{y}) \wedge 0 \leq \mathsf{y}$.

Example

$\{\quad\}$

while $(x \neq 0)$ **do** (

$\quad x := x - 1$

)

$\{\quad\}$

Example

$\{\textbf{true}\}$

while $(x \neq 0)$ **do** (

$\quad x := x - 1$

)

$\{x = 0\}$

Example

$\{ \qquad\qquad \}$

$y := 1$
while $(0 < x)$ **do** (
$\quad x := x - 1;$
$\quad y := y * 2$
)

$\{ \qquad\quad \}$

Example

$\{\mathsf{x} = n \wedge 0 \leq n\}$

$\mathsf{y} := 1$
while $(0 < \mathsf{x})$ **do** (
  $\mathsf{x} := \mathsf{x} - 1;$
  $\mathsf{y} := \mathsf{y} * 2$
)

$\{\mathsf{y} = 2^n\}$