



1 Lambda calculus evaluation

There are many different evaluation strategies for the λ -calculus. The most permissive is *full β reduction*, which allows any *redex*—i.e., any expression of the form $(\lambda x. e_1) e_2$ —to step to $e_1\{e_2/x\}$ at any time. It is defined formally by the following small-step operational semantics rules:

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \frac{e_2 \rightarrow e'_2}{e_1 e_2 \rightarrow e_1 e'_2} \quad \frac{e_1 \rightarrow e'_1}{\lambda x. e_1 \rightarrow \lambda x. e'_1} \quad \beta \frac{}{(\lambda x. e_1) e_2 \rightarrow e_1\{e_2/x\}}$$

The *call by value* (CBV) strategy enforces a more restrictive strategy: it only allows an application to reduce after its argument has been reduced to a value (i.e., a λ -abstraction) and does not allow evaluation under a λ . It is described by the following small-step operational semantics rules (here we show a left-to-right version of CBV):

$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \frac{e_2 \rightarrow e'_2}{v_1 e_2 \rightarrow v_1 e'_2} \quad \beta \frac{}{(\lambda x. e_1) v_2 \rightarrow e_1\{v_2/x\}}$$

Finally, the *call by name* (CBN) strategy allows an application to reduce even when its argument is not a value but does not allow evaluation under a λ . It is described by the following small-step operational semantics rules:

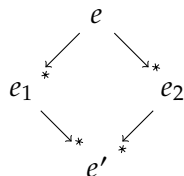
$$\frac{e_1 \rightarrow e'_1}{e_1 e_2 \rightarrow e'_1 e_2} \quad \beta \frac{}{(\lambda x. e_1) e_2 \rightarrow e_1\{e_2/x\}}$$

2 Confluence

It is not hard to see that the full β reduction strategy is non-deterministic. This raises an interesting question: does the choices made during the evaluation of an expression affect the final result? The answer turns out to be no: full β reduction is *confluent* in the following sense:

Theorem (Confluence). *If $e \rightarrow^* e_1$ and $e \rightarrow^* e_2$ then there exists e' such that $e_1 \rightarrow^* e'$ and $e_2 \rightarrow^* e'$.*

Confluence can be depicted graphically as follows:



Confluence is often also called the Church–Rosser property.

3 Substitution

Each of the evaluation relations for λ -calculus has a β defined in terms of a substitution operation on expressions. Because the expressions involved in the substitution may share some variable names (and because we are working up to α -equivalence) the definition of this operation is slightly subtle and defining it precisely turns out to be trickier than might first appear.

As a first attempt, consider an obvious (but incorrect) definition of the substitution operator. Here we are substituting e for x in some other expression:

$$\begin{aligned} y\{e/x\} &= \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases} \\ (e_1 e_2)\{e/x\} &= (e_1\{e/x\}) (e_2\{e/x\}) \\ (\lambda y.e_1)\{e/x\} &= \lambda y.e_1\{e/x\} \quad \text{where } y \neq x \end{aligned}$$

The intuitive idea is that the last rule relies on α -equivalence to “rewrite” abstractions that use x so they do not conflict. Unfortunately, this definition produces the wrong results when we substitute an expression with free variables under a λ . For example,

$$(\lambda y.x)\{y/x\} = (\lambda y.y)$$

To fix this problem, we need to revise our definition so that when we substitute under a λ we do not accidentally bind variables in the expression we are substituting. The following definition correctly implements *capture-avoiding substitution*:

$$\begin{aligned} y\{e/x\} &= \begin{cases} e & \text{if } y = x \\ y & \text{otherwise} \end{cases} \\ (e_1 e_2)\{e/x\} &= (e_1\{e/x\}) (e_2\{e/x\}) \\ (\lambda y.e_1)\{e/x\} &= \lambda y.(e_1\{e/x\}) \quad \text{where } y \neq x \text{ and } y \notin fv(e) \end{aligned}$$

Note that in the case for λ -abstractions, we require that the bound variable y be different from the variable x we are substituting for and that y not appear in the free variables of e , the expression we are substituting. Because we work up to α -equivalence, we can always pick y to satisfy these side conditions. For example, to calculate $(\lambda z.x z)\{(w y z)/x\}$ we first rewrite $\lambda z.x z$ to $\lambda u.x u$ and then apply the substitution, obtaining $\lambda u.(w y z) u$ as the result.