CS 4110

Programming Languages & Logics

Lecture 28
Propositions as Types

Propositions as Types

Logics = Type Systems

Inference Rules for Logic

We have used inference rules to build up inductively defined sets of PL concepts: operational steps, valid Hoare triples, associations between terms and types, etc.

Logicians use the same kind of notation to build up the set of true logical formulas.

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Logicians use the same kind of notation to build up the set of true logical formulas.

Here's a rule from natural deduction, a *constructive* logic invented by logician Gerhard Gentzen in 1935:

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\ \wedge\text{-INTRO}$$

Given a proof of ϕ and a proof of ψ, the rule lets you *construct* a proof of $\phi \wedge \psi$.

Natural Deduction

Let's use our usual 4110 tools to define the set of true formulas ("theorems").

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We'll start with a grammar for formulas:

$$
\begin{array}{rcl}
\phi & ::= & \top \\
& \mid & \bot \\
& \mid & X \\
& \mid & \phi \wedge \psi \\
& \mid & \phi \vee \psi \\
& \mid & \phi \rightarrow \psi \\
& \mid & \neg \phi \\
& \mid & \forall X.\ \phi
\end{array}
$$

where X ranges over Boolean variables
and $\neg \phi$ is an abbreviation for $\phi \rightarrow \bot$.

Natural Deduction

Let's define a judgment that that a formula is true given a set of assumptions Γ:

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Examples:

- $\vdash A \land B \to A$

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Examples:

- $\vdash A \wedge B \to A$
- $\vdash \neg(A \wedge B) \to \neg A \vee \neg B$

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Examples:

- $\vdash A \wedge B \rightarrow A$
- $\vdash \neg(A \wedge B) \rightarrow \neg A \vee \neg B$
- $A, B, C \vdash B$ $B \wedge C$ $A \vee B$

Natural Deduction

Let’s write the rules for our judgment:

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \land \psi}\ \land\text{-INTRO}$$

$$\frac{\phi \land \psi}{\phi}$$

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$$\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}\ \wedge\text{-ELIM1} \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}\ \wedge\text{-ELIM2}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}$$

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$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\ \to\text{-INTRO}$$

...and so on.

Natural Deduction

$$\frac{}{\Gamma, \phi \vdash \phi}\ \textsc{Axiom}$$

$$\frac{\Gamma, \phi \vdash \psi}{\Gamma \vdash \phi \to \psi}\ \to\text{-INTRO} \qquad \frac{\Gamma \vdash \phi \to \psi \qquad \Gamma \vdash \phi}{\Gamma \vdash \psi}\ \to\text{-ELIM}$$

$$\frac{\Gamma \vdash \phi \qquad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}\ \wedge\text{-INTRO} \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}\ \wedge\text{-ELIM1} \qquad \frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}\ \wedge\text{-ELIM2}$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi}\ \vee\text{-INTRO1} \qquad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}\ \vee\text{-INTRO2}$$

$$\frac{\Gamma \vdash \phi \vee \psi \qquad \Gamma \vdash \phi \to \chi \qquad \Gamma \vdash \psi \to \chi}{\Gamma \vdash \chi}\ \vee\text{-ELIM}$$

$$\frac{\Gamma, P \vdash \phi}{\Gamma \vdash \forall P.\, \phi}\ \forall\text{-INTRO} \qquad \frac{\Gamma \vdash \forall P.\, \phi}{\Gamma \vdash \phi\{\psi/P\}}\ \forall\text{-ELIM}$$

Natural Deduction

Let's try a proof! We can write a proof that $A \land B \to B \land A$ is a theorem.

$$\dfrac{\dfrac{A \land B \vdash A \land B}{A \land B \vdash B}\ \land\text{-ELIM-R} \qquad \dfrac{A \land B \vdash A \land B}{A \land B \vdash A}\ \land\text{-ELIM-L}}{\dfrac{A \land B \vdash B \land A}{\vdash A \land B \to B \land A}}\ \land\text{-INTRO}$$

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$$\dfrac{\dfrac{\dfrac{\dfrac{}{A \land B \vdash A \land B}\ \textsf{AXIOM}}{A \land B \vdash B}\ \land\textsf{-ELIM2} \qquad \dfrac{\dfrac{}{A \land B \vdash A \land B}\ \textsf{AXIOM}}{A \land B \vdash A}\ \land\textsf{-ELIM1}}{A \land B \vdash B \land A}\ \land\textsf{-INTRO}}{\vdash A \land B \to B \land A}\ \to\textsf{-INTRO}$$

Natural Deduction

Let's try a proof! We can write a proof that $A \wedge B \rightarrow B \wedge A$ is a theorem.

$$\dfrac{\dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\ \textsf{AXIOM}}{A \wedge B \vdash B}\ \wedge\textsf{-ELIM2} \qquad \dfrac{\dfrac{}{A \wedge B \vdash A \wedge B}\ \textsf{AXIOM}}{A \wedge B \vdash A}\ \wedge\textsf{-ELIM1}}{\dfrac{A \wedge B \vdash B \wedge A}{\vdash A \wedge B \rightarrow B \wedge A}\ \rightarrow\textsf{-INTRO}}\ \wedge\textsf{-INTRO}$$

Does this look familiar?

Natural Deduction

Let's try a proof! We can write a proof that $A \wedge B \rightarrow B \wedge A$ is a theorem.

$$
\dfrac{
  \dfrac{
    \dfrac{}{A \wedge B \vdash A \wedge B}\;\textsf{AXIOM}
  }{A \wedge B \vdash B}\;\wedge\textsf{-ELIM2}
  \qquad
  \dfrac{
    \dfrac{}{A \wedge B \vdash A \wedge B}\;\textsf{AXIOM}
  }{A \wedge B \vdash A}\;\wedge\textsf{-ELIM1}
}{
  \dfrac{A \wedge B \vdash B \wedge A}{\vdash A \wedge B \rightarrow B \wedge A}\;\rightarrow\textsf{-INTRO}
}\;\wedge\textsf{-INTRO}
$$

Does this look familiar?

$$
\dfrac{
  \dfrac{
    \dfrac{}{x : A \times B \vdash x : A \times B}\;\textsf{T-VAR}
  }{x : A \times B \vdash \#2\ x : B}\;\textsf{T-\#1}
  \qquad
  \dfrac{
    \dfrac{}{x : A \times B \vdash x : A \times B}\;\textsf{T-VAR}
  }{x : A \times B \vdash \#1\ x : A}\;\textsf{T-\#2}
}{
  \dfrac{x : A \times B \vdash (\#2\ x, \#1\ x) : B \times A}{\vdash \lambda x.(\#2\ x, \#1\ x) : A \times B \rightarrow B \times A}\;\textsf{T-ABS}
}\;\textsf{T-PAIR}
$$

Propositions as Types

Every natural deduction proof tree has a corresponding type tree in System F with product and sum types! And vice-versa!

Type Systems		Formal Logic	
τ	Type	ϕ	Formula
τ	is inhabited	ϕ	is a theorem
e	Well-typed expression	π	Proof

A program with a given type acts as a *witness* that the type's corresponding formula is true.

Propositions as Types

Every type rule in System F with product and sum types corresponds 1-1 with a proof rule in natural deduction:

Type Systems		**Formal Logic**	
$\rightarrow$	Function	$\rightarrow$	Implication
$\times$	Product	$\wedge$	Conjunction
$+$	Sum	$\vee$	Disjunction
$\forall$	Universal	$\forall$	Quantifier

You can even add existential types to correspond to existential quantification. It still works!

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Is this a coincidence? Natural deduction was invented by a German logician in 1935. Types for the λ-calculus were invented by Church at Princeton in 1940.

Propositions as Types Through the Ages

Natural Deduction Gentzen (1935)	⇔	**Typed λ-Calculus** Church (1940)
Type Schemes Hindley (1969)	⇔	**ML's Type System** Milner (1975)
System F Girard (1972)	⇔	**Polymorphic λ-Calculus** Reynolds (1974)
Modal Logic Lewis (1910)	⇔	**Monads** Kleisli (1965), Moggi (1987)
Classical–Intuitionistic Embedding Gödel (1933)	⇔	**Continuation Passing Style** Reynolds (1972)

Term Assignment

This all means that we have a new way of proving theorems: writing programs!

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To prove a formula ϕ:

1. Convert the ϕ into its corresponding type τ.
2. Find some program *v* that has the type τ.
3. Realize that the existence of *v* implies a type tree for $\vdash v : \tau$, which implies a proof tree for $\vdash \phi$.

$\forall X. X \simeq \bot$

Negation and Continuations

Let's explore one extension. We'd like to use this rule from classical logic:

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \neg\neg\phi} \leftarrow (\phi \to \bot) \to \bot$$

but there's no obvious correspondence in System F.

Negation and Continuations

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$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \neg\neg\phi}$$

CPS⟦n⟧ k ≜ k n

CPS⟦n⟧ ≜ λk. k n (k : int → ⊥)

but there's no obvious correspondence in System F.

Recall that $\neg\phi$ is shorthand for $\phi \to \bot$. So $\neg\neg\phi$ corresponds to the System F function type $(\tau \to \bot) \to \bot$.

So what we need is a way to take any program of any type τ and turn it into a program of type $(\tau \to \bot) \to \bot$.

Negation and Continuations

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int → void
(int → int) → void

but there's no obvious correspondence in System F.

Recall that $\neg\phi$ is shorthand for $\phi \to \bot$. So $\neg\neg\phi$ corresponds to the System F function type $(\tau \to \bot) \to \bot$.

So what we need is a way to take any program of any type τ and turn it into a program of type $(\tau \to \bot) \to \bot$.

Shockingly, that's exactly what the CPS transform does! A τ becomes a function that takes a continuation $\tau \to \bot$ and invokes it, producing $\bot$.