CS 4110

Programming Languages & Logics

Lecture 21
Advanced Types

Review

We've developed a type system for the λ-calculus and mathematical tools for proving its type soundness.

We also know how to extend the λ-calculus with new language features.

Today, we'll extend our *type system* with features commonly found in real-world languages: products, sums, references, and exceptions.

Products (Pairs)

Syntax

$$e ::= \cdots \mid (e_1, e_2) \mid \#1\, e \mid \#2\, e$$
$$v ::= \cdots \mid (v_1, v_2)$$

Products (Pairs)

Syntax

$\tau ::= \text{int} \mid \text{unit} \mid \tau_1 \to \tau_2$

$$e ::= \cdots \mid (e_1, e_2) \mid \#1\, e \mid \#2\, e$$
$$v ::= \cdots \mid (v_1, v_2)$$

Semantics

$$E ::= \cdots \mid (E, e) \mid (v, E) \mid \#1\, E \mid \#2\, E$$

$$\frac{}{\#1\,(v_1, v_2) \to v_1} \qquad \frac{}{\#2\,(v_1, v_2) \to v_2}$$

Product Types

$\tau ::= \ldots \mid \tau_1 \times \tau_2$

int × int

(int → int × int)

× int

Product Types

$$\tau_1 \times \tau_2$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#1\, e : \tau_1}$$

Product Types

type t =
  Foo of int * int
| Bar of int

$$\tau_1 \times \tau_2$$

$$\frac{\Gamma \vdash e_1 : \tau_1 \quad \Gamma \vdash e_2 : \tau_2}{\Gamma \vdash (e_1, e_2) : \tau_1 \times \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#1\, e : \tau_1}$$

$$\frac{\Gamma \vdash e : \tau_1 \times \tau_2}{\Gamma \vdash \#2\, e : \tau_2}$$

Sums (Tagged Unions)

Syntax

$$e ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ e \mid \mathsf{inr}_{\tau_1+\tau_2}\ e \mid (\mathsf{case}\ e_1\ \mathsf{of}\ e_2 \mid e_3)$$
$$v ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ v \mid \mathsf{inr}_{\tau_1+\tau_2}\ v$$

$$\text{int} + (\text{unit} + (\text{int} \to \text{int}))$$
$$\stackrel{?}{=} (\text{int} + \text{unit}) + (\text{int} \to \text{int})$$

$$\text{int} + \text{bool} \stackrel{?}{=} \text{bool} + \text{int}$$

Sums (Tagged Unions)

Syntax

$$e ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ e \mid \mathsf{inr}_{\tau_1+\tau_2}\ e \mid (\mathsf{case}\ e_1\ \mathsf{of}\ e_2 \mid e_3)$$
$$v ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ v \mid \mathsf{inr}_{\tau_1+\tau_2}\ v$$

Semantics

$$E ::= \cdots \mid \mathsf{inl}_{\tau_1+\tau_2}\ E \mid \mathsf{inr}_{\tau_1+\tau_2}\ E \mid (\mathsf{case}\ E\ \mathsf{of}\ e_2 \mid e_3)$$

$$\frac{}{\mathsf{case}\ \mathsf{inl}_{\tau_1+\tau_2}\ v\ \mathsf{of}\ e_2 \mid e_3 \rightarrow e_2\ v}$$

$$\frac{}{\mathsf{case}\ \mathsf{inr}_{\tau_1+\tau_2}\ v\ \mathsf{of}\ e_2 \mid e_3 \rightarrow e_3\ v}$$

Sum Types

$((),())$ ← unit × unit : 1

unit + unit : 2

$$\tau ::= \cdots \mid \tau_1 + \tau_2$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathrm{inl}_{\tau_1 + \tau_2}\, e : \tau_1 + \tau_2}$$

$\mathrm{inl}_{unit+unit}\ ()$

$\mathrm{inr}\ \ $ " $\ ()$

Sum Types

$$\tau ::= \cdots \mid \tau_1 + \tau_2$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathsf{inl}_{\tau_1+\tau_2}\, e : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathsf{inr}_{\tau_1+\tau_2}\, e : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e_1 : \tau_1 + \tau_2 \qquad \Gamma \vdash e_2 : \tau_1 \to \tau \qquad \Gamma \vdash e_3 : \tau_2 \to \tau}{\Gamma \vdash \mathsf{case}\ e_1\ \mathsf{of}\ e_2 \mid e_3 : \tau}$$

Sum Types

$$\tau ::= \cdots \mid \tau_1 + \tau_2$$

$$\frac{\Gamma \vdash e : \tau_1}{\Gamma \vdash \mathsf{inl}_{\tau_1 + \tau_2}\, e : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_2}{\Gamma \vdash \mathsf{inr}_{\tau_1 + \tau_2}\, e : \tau_1 + \tau_2}$$

$$\frac{\Gamma \vdash e : \tau_1 + \tau_2 \quad \Gamma \vdash e_1 : \tau_1 \to \tau \quad \Gamma \vdash e_2 : \tau_2 \to \tau}{\Gamma \vdash \mathsf{case}\ e\ \mathsf{of}\ e_1 \mid e_2 : \tau}$$

Example

$\text{let } f = \lambda a : \mathbf{int} + (\mathbf{int} \to \mathbf{int}).\ \text{case } a \text{ of } (\lambda \overset{:\,\mathrm{int}}{y}.\ y + 1) \mid (\lambda \overset{:\,\mathrm{int} \to \mathrm{int}}{g}.\ g\ 35) \text{ in}$

$\text{let } h = \lambda x : \mathbf{int}.\ x + 7 \text{ in}$

$f\,(\text{inr}_{\mathbf{int}+(\mathbf{int}\to\mathbf{int})}\ h)$

References

Syntax

$\ell \in Loc$

$$
\begin{aligned}
e &::= \cdots \mid \mathsf{ref}\, e \mid\ !e \mid e_1 := e_2 \mid \ell \\
v &::= \cdots \mid \ell
\end{aligned}
$$

References

Syntax

$$
\begin{aligned}
e &::= \cdots \mid \mathsf{ref}\; e \mid\; !e \mid e_1 := e_2 \mid \ell \\
v &::= \cdots \mid \ell
\end{aligned}
$$

Semantics

$$E ::= \cdots \mid \mathsf{ref}\; E \mid\; !E \mid E := e \mid v := E$$

$$\frac{\ell \notin dom(\sigma)}{\langle \sigma, \mathsf{ref}\; v\rangle \to \langle \sigma[\ell \mapsto v], \ell\rangle} \qquad \frac{\sigma(\ell) = v}{\langle \sigma,\ !\ell\rangle \to \langle \sigma, v\rangle}$$

$$\frac{}{\langle \sigma, \ell := v\rangle \to \langle \sigma[\ell \mapsto v], v\rangle}$$

Reference Types

$$\tau ::= \cdots \mid \tau\ \textbf{ref}$$

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$$\frac{\Gamma \vdash e : \tau}{\Gamma \vdash \mathsf{ref}\ e : \tau\ \textbf{ref}}$$

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$$\frac{\Gamma \vdash e : \tau\ \textbf{ref}}{\Gamma \vdash\ !e : \tau}$$

Reference Types

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$$\frac{\Gamma \vdash e : \tau\ \textbf{ref}}{\Gamma \vdash\ !e : \tau}$$

$$\frac{\Gamma \vdash e_1 : \tau\ \textbf{ref} \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1 := e_2 : \tau}$$

Question

Is this type system sound?

Question

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}$$

Is this type system sound?

Well... what is the type of a location ℓ?

$$\langle \sigma, \text{ref } 5 \rangle \to \langle \sigma', \ell_{20} \rangle$$

$$\frac{}{\Gamma \vdash \ell :}$$

Question

Is this type system sound?

Well... what is the type of a location ℓ? (Oops!)

Store Typings

Let Σ range over partial functions from locations to types.

$\Sigma(\ell) = \mathrm{int}$

$\Sigma, \ell \mapsto \mathrm{int}$

$\Sigma[\ell \mapsto \mathrm{int}]$

$\Sigma, \Gamma \vdash e : \tau$

Store Typings

Let Σ range over partial functions from locations to types.

$$\frac{\Gamma, \Sigma \vdash e : \tau}{\Gamma, \Sigma \vdash \mathsf{ref}\, e : \tau\ \mathbf{ref}}$$

Store Typings

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Store Typings

Let Σ range over partial functions from locations to types.

$$\frac{\Gamma, \Sigma \vdash e : \tau}{\Gamma, \Sigma \vdash \mathrm{ref}\, e : \tau\ \mathbf{ref}}$$

$$\frac{\Gamma, \Sigma \vdash e : \tau\ \mathbf{ref}}{\Gamma, \Sigma \vdash\ !e : \tau}$$

$$\frac{\Gamma, \Sigma \vdash e_1 : \tau\ \mathbf{ref} \quad \Gamma, \Sigma \vdash e_2 : \tau}{\Gamma, \Sigma \vdash e_1 := e_2 : \tau}$$

Store Typings

Let Σ range over partial functions from locations to types.

[ ]

ref 5

$$\frac{\Gamma, \Sigma \vdash e : \tau}{\Gamma, \Sigma \vdash \mathrm{ref}\, e : \tau\ \mathbf{ref}}$$

$[\ell_A \mapsto \tau]$

$\rightarrow \ell_B$

$$\frac{\Gamma, \Sigma \vdash e : \tau\ \mathbf{ref}}{\Gamma, \Sigma \vdash !e : \tau}$$

$$\frac{\Gamma, \Sigma \vdash e_1 : \tau\ \mathbf{ref} \quad \Gamma, \Sigma \vdash e_2 : \tau}{\Gamma, \Sigma \vdash e_1 := e_2 : \tau}$$

$$\frac{\Sigma(\ell) = \tau}{\Gamma, \Sigma \vdash \ell : \tau\ \mathbf{ref}}$$

Reference Types Metatheory

Definition

Store σ is *well-typed* with respect to typing context Γ and store typing Σ, written $\Gamma, \Sigma \vdash \sigma$, if $dom(\sigma) = dom(\Sigma)$ and for all $\ell \in dom(\sigma)$ we have $\Gamma, \Sigma \vdash \sigma(\ell) : \Sigma(\ell)$.

$\exists \tau . \Sigma, \Gamma \vdash e : \tau$

Reference Types Metatheory

Definition

Store σ is *well-typed* with respect to typing context Γ and store typing Σ, written $\Gamma, \Sigma \vdash \sigma$, if $\mathit{dom}(\sigma) = \mathit{dom}(\Sigma)$ and for all $\ell \in \mathit{dom}(\sigma)$ we have $\Gamma, \Sigma \vdash \sigma(\ell) : \Sigma(\ell)$.

Theorem (Type soundness)

If $\cdot, \Sigma \vdash e : \tau$ and $\cdot, \Sigma \vdash \sigma$ and $\langle e, \sigma \rangle \to^* \langle e', \sigma' \rangle$ then either e' is a value, or there exists e'' and σ'' such that $\langle e', \sigma' \rangle \to \langle e'', \sigma'' \rangle$.

Reference Types Metatheory

Definition

Store σ is *well-typed* with respect to typing context Γ and store typing Σ, written $\Gamma, \Sigma \vdash \sigma$, if $dom(\sigma) = dom(\Sigma)$ and for all $\ell \in dom(\sigma)$ we have $\Gamma, \Sigma \vdash \sigma(\ell) : \Sigma(\ell)$.

Theorem (Type soundness)

If $\emptyset, \Sigma \vdash e : \tau$ *and* $\emptyset, \Sigma \vdash \sigma$ *and* $\langle e, \sigma \rangle \to^* \langle e', \sigma' \rangle$ *then either* e' *is a value, or there exists* e'' *and* σ'' *such that* $\langle e', \sigma' \rangle \to \langle e'', \sigma'' \rangle$.

Lemma (Preservation)

If $\Gamma, \Sigma \vdash e : \tau$ *and* $\Gamma, \Sigma \vdash \sigma$ *and* $\langle e, \sigma \rangle \to \langle e', \sigma' \rangle$ *then there exists some* $\Sigma' \supseteq \Sigma$ *such that* $\Gamma, \Sigma' \vdash e' : \tau$ *and* $\Gamma, \Sigma' \vdash \sigma'$.

Landin's Knot

Using references, we (re)gain the ability define recursive functions!

$$\textbf{let } r = \text{ref } \lambda x\,{:}\,\textbf{int}.\ 0\ \textbf{in}$$

Landin's Knot

Using references, we (re)gain the ability define recursive functions!

$$\begin{array}{l}\textbf{let}\ r = \mathrm{ref}\ \lambda x{:}\,\textbf{int}.\ 0\ \textbf{in} \\ \textbf{let}\ f = (\lambda x{:}\,\textbf{int}.\ \textbf{if}\ x = 0\ \textbf{then}\ 1\ \textbf{else}\ x \times (!r)\,(x - 1))\ \ \textbf{in}\end{array}$$

Landin's Knot

Using references, we (re)gain the ability define recursive functions!

$$\begin{aligned}
&\textbf{let } r = \text{ref } \lambda x\!:\!\textbf{int}.\ 0 \textbf{ in}\\
&\textbf{let } f = (\lambda x\!:\!\textbf{int}.\ \textbf{if } x = 0 \textbf{ then } 1 \textbf{ else } x \times (!r)\,(x-1))\ \ \textbf{in}\\
&\textbf{let } a = (r := f) \textbf{ in}
\end{aligned}$$

Landin's Knot

Using references, we (re)gain the ability define recursive functions!

$$
\begin{array}{l}
\textbf{let}\ r = \mathrm{ref}\ \lambda x{:}\,\textbf{int}.\ 0\ \textbf{in} \\
\textbf{let}\ f = (\lambda x{:}\,\textbf{int}.\ \textbf{if}\ x = 0\ \textbf{then}\ 1\ \textbf{else}\ x \times (!r)\,(x - 1))\ \ \textbf{in} \\
\textbf{let}\ a = (r := f)\ \textbf{in} \\
f\ 5
\end{array}
$$

Fixed Points

Syntax

$$e ::= \cdots \mid \mathsf{fix}\ e$$

Fixed Points

Syntax

$$e ::= \cdots \mid \text{fix}\ e$$

Semantics

$$E ::= \cdots \mid \text{fix}\ E$$

$$\frac{}{\text{fix}\ \lambda x\!:\!\tau.\, e \to e\{(\text{fix}\ \lambda x\!:\!\tau.\, e)/x\}}$$

Fixed Points

Syntax

$$e ::= \cdots \mid \mathsf{fix}\; e$$

Semantics

$$E ::= \cdots \mid \mathsf{fix}\; E$$

$$\frac{}{\mathsf{fix}\; \lambda x{:}\tau.\, e \rightarrow e\{(\mathsf{fix}\; \lambda x{:}\tau.\, e)/x\}}$$

The typing rule for fix is on the homework...