CS 4110

Programming Languages & Logics

Lecture 17
Definitional Translation & Continuations

Definitional Translation

We know how to encode Booleans, conditionals, natural numbers, and recursion in λ-calculus.

Can we define a *real* programming language by translating everything in it into the λ-calculus?

Definitional Translation

We know how to encode Booleans, conditionals, natural numbers, and recursion in λ-calculus.

Can we define a *real* programming language by translating everything in it into the λ-calculus?

In definitional translation, we define a denotational semantics where the target is a simpler programming language instead of mathematical objects.

Multi-Argument λ-calculus

Let's define a version of the λ-calculus that allows functions to take multiple arguments.

$$e ::= x \mid \lambda x_1, \ldots, x_n.\, e \mid e_0\; e_1\; \ldots\; e_n$$

Multi-Argument λ-calculus

We can define a CBV operational semantics:

$$E ::= [\cdot] \mid v_0 \ \dots \ v_{i-1} \, E \, e_{i+1} \ \dots \ e_n$$

$$\frac{e \to e'}{E[e] \to E[e']}$$

CONTEXT

$$\frac{}{(\lambda x_1, \dots, x_n.\, e_0)\, v_1 \ \dots \ v_n \to e_0\{v_1/x_1\}\{v_2/x_2\} \dots \{v_n/x_n\}} \ \beta$$

The evaluation contexts ensure that we evaluate multi-argument applications $e_0 \, e_1 \ \dots \ e_n$ from left to right.

Definitional Translation

The multi-argument λ-calculus isn't any more expressive that the pure λ-calculus.

Definitional Translation

The multi-argument λ-calculus isn't any more expressive that the pure λ-calculus.

We can define a translation $\mathcal{T}[\![\cdot]\!]$ that takes an expression in the multi-argument λ-calculus and returns an equivalent expression in the pure λ-calculus.

Definitional Translation

The multi-argument λ-calculus isn't any more expressive that the pure λ-calculus.

We can define a translation $\mathcal{T}[\![\cdot]\!]$ that takes an expression in the multi-argument λ-calculus and returns an equivalent expression in the pure λ-calculus.

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &= x \\
\mathcal{T}[\![\lambda x_1, \ldots, x_n.\, e]\!] &= \lambda x_1. \;\ldots\; \lambda x_n.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_0\; e_1\; e_2\; \ldots\; e_n]\!] &= (\ldots((\mathcal{T}[\![e_0]\!]\; \mathcal{T}[\![e_1]\!])\; \mathcal{T}[\![e_2]\!]) \ldots\; \mathcal{T}[\![e_n]\!])
\end{aligned}
$$

This translation *curries* the multi-argument λ-calculus.

Products (Pairs) and Let

Syntax

$$
\begin{aligned}
e ::= {} & x \\
& \mid \lambda x.\, e \\
& \mid e_1\ e_2 \\
& \mid (e_1, e_2) \\
& \mid \#1\ e \\
& \mid \#2\ e \\
& \mid \mathsf{let}\ x = e_1\ \mathsf{in}\ e_2
\end{aligned}
$$

$$
\begin{aligned}
v ::= {} & \lambda x.\, e \\
& \mid (v_1, v_2)
\end{aligned}
$$

Products (Pairs) and Let

Evaluation Contexts

$$
\begin{aligned}
E ::= {} & [\cdot] \\
& \mid E\ e \\
& \mid v\ E \\
& \mid (E, e) \\
& \mid (v, E) \\
& \mid \#1\ E \\
& \mid \#2\ E \\
& \mid \mathsf{let}\ x = E\ \mathsf{in}\ e_2
\end{aligned}
$$

Products (Pairs) and Let

Semantics

$$\frac{e \to e'}{E[e] \to E[e']}$$

$$\frac{}{(\lambda x.\, e)\; v \to e\{v/x\}}\;\beta$$

$$\frac{}{\#1\;(v_1, v_2) \to v_1} \qquad \frac{}{\#2\;(v_1, v_2) \to v_2}$$

$$\frac{}{\mathsf{let}\; x = v\; \mathsf{in}\; e \to e\{v/x\}}$$

Products (Pairs) and Let

Translation

$$
\begin{aligned}
\mathcal{T}[\![x]\!] &= x \\
\mathcal{T}[\![\lambda x.\, e]\!] &= \lambda x.\, \mathcal{T}[\![e]\!] \\
\mathcal{T}[\![e_1\ e_2]\!] &= \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!] \\
\mathcal{T}[\![(e_1, e_2)]\!] &= (\lambda x.\, \lambda y.\, \lambda f.\, f\, x\, y)\ \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!] \\
\mathcal{T}[\![\#1\ e]\!] &= \mathcal{T}[\![e]\!]\ (\lambda x.\, \lambda y.\, x) \\
\mathcal{T}[\![\#2\ e]\!] &= \mathcal{T}[\![e]\!]\ (\lambda x.\, \lambda y.\, y) \\
\mathcal{T}[\![\mathsf{let}\ x = e_1\ \mathsf{in}\ e_2]\!] &= (\lambda x.\, \mathcal{T}[\![e_2]\!])\ \mathcal{T}[\![e_1]\!]
\end{aligned}
$$

Laziness

Consider the call-by-name λ-calculus...

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & e_1\ e_2 \\
\mid {} & \lambda x.\ e
\end{aligned}
$$

$$v ::= \lambda x.\ e$$

Semantics

$$\frac{e_1 \to e_1'}{e_1\ e_2 \to e_1'\ e_2} \qquad \frac{}{(\lambda x.\ e_1)\ e_2 \to e_1\{e_2/x\}}\ \beta$$

Laziness

Translation

$$\mathcal{T}[\![x]\!] = x\,(\lambda y.\, y)$$
$$\mathcal{T}[\![\lambda x.\, e]\!] = \lambda x.\, \mathcal{T}[\![e]\!]$$
$$\mathcal{T}[\![e_1\; e_2]\!] = \mathcal{T}[\![e_1]\!]\,(\lambda z.\, \mathcal{T}[\![e_2]\!]) \qquad z \text{ is not a free variable of } e_2$$

$$\mathcal{T}[\![(\lambda x.\, x)\;((\lambda y.\, y)\;(\lambda z.\, z))]\!]$$
$$= \mathcal{T}[\![\lambda x.\, x]\!]\;(\lambda a.\, \mathcal{T}[\![(\lambda y.\, y)\;(\lambda z.\, z)]\!])$$
$$= (\lambda x.\, x\; ID)\;(\lambda a.\, (\lambda y.\, y\; ID)\; \lambda b.\, [\![\lambda z.\, z]\!])$$
$$= \quad " \quad (\lambda a.\; (\lambda y.\, y\; ID)(\lambda b.\, \lambda z.\, z\; ID))$$

References

Syntax

$$
\begin{aligned}
e ::= & \; x \\
& \mid \lambda x.\, e \\
& \mid e_0\; e_1
\end{aligned}
$$

$$
v ::= \lambda x.\, e
$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e
\end{aligned}
$$

$$v ::= \lambda x.\, e$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathrm{ref}\ e \\
\mid {} & !e
\end{aligned}
$$

malloc

*e = ...

*e

$$v ::= \lambda x.\, e$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathrm{ref}\ e \\
\mid {} & !e \\
\mid {} & e_1 := e_2
\end{aligned}
$$

$*e_1 = e_2$

$$v ::= \lambda x.\, e$$

References

Syntax

$\ell \in Loc$

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e \\
\mid {} & !e \\
\mid {} & e_1 := e_2 \\
\mid {} & \ell
\end{aligned}
$$

SURFACE

$$v ::= \lambda x.\, e$$

References

Syntax

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & \lambda x.\, e \\
\mid {} & e_0\ e_1 \\
\mid {} & \mathsf{ref}\ e \\
\mid {} & \ !e \\
\mid {} & e_1 := e_2 \\
\mid {} & \ell
\end{aligned}
$$

$$
\begin{aligned}
v ::= {} & \lambda x.\, e \\
\mid {} & \ell
\end{aligned}
$$

References

Evaluation Contexts

$$\begin{aligned} E ::= & \ [\cdot] \\ & \mid E\ e \\ & \mid v\ E \end{aligned}$$

References

Evaluation Contexts

$$\begin{aligned} E ::= {} & [\cdot] \\ \mid {} & E\ e \\ \mid {} & v\ E \\ \mid {} & \text{ref}\ E \end{aligned}$$

References

Evaluation Contexts

$$
\begin{aligned}
E ::= &\ [\cdot] \\
 \mid &\ E\ e \\
 \mid &\ v\ E \\
 \mid &\ \text{ref}\ E \\
 \mid &\ !E
\end{aligned}
$$

References

Evaluation Contexts

$$
\begin{aligned}
E ::= {} & [\cdot] \\
\mid {} & E\ e \\
\mid {} & v\ E \\
\mid {} & \text{ref}\ E \\
\mid {} & \ !E \\
\mid {} & E := e
\end{aligned}
$$

References

Evaluation Contexts

$$
\begin{aligned}
E ::= \ & [\cdot] \\
| \ & E \ e \\
| \ & v \ E \\
| \ & \text{ref } E \\
| \ & !E \\
| \ & E := e \\
| \ & v := E
\end{aligned}
$$

References

Semantics

$\langle \sigma, e \rangle \to \langle \sigma', e' \rangle \quad \sigma : Loc \to Val$

OLD

$$\frac{\langle \sigma, e \rangle \to \langle \sigma', e' \rangle}{\langle \sigma, E[e] \rangle \to \langle \sigma', E[e'] \rangle} \qquad \frac{}{\langle \sigma, (\lambda x.\, e)\; v \rangle \to \langle \sigma, e\{v/x\} \rangle}\ \beta$$

$$\frac{\ell \notin dom(\sigma)}{\langle \sigma, \mathsf{ref}\; v \rangle \to \langle \sigma[\ell \mapsto v], \ell \rangle} \qquad \frac{\sigma(\ell) = v}{\langle \sigma,\ !\ell \rangle \to \langle \sigma, v \rangle}$$

$$\frac{}{\langle \sigma, \ell := v \rangle \to \langle \sigma[\ell \mapsto v], v \rangle}$$

References

Translation

...left as an exercise to the reader. ;-)

Adequacy

How do we know if a translation is correct?

Adequacy

How do we know if a translation is correct?

Every target evaluation should represent a source evaluation...

Definition (Soundness)

$\forall e \in \mathbf{Exp}_{\mathrm{src}}.$ if $\mathcal{T}[\![e]\!] \rightarrow^{*}_{\mathrm{trg}} v'$ then $\exists v.\ e \rightarrow^{*}_{\mathrm{src}} v$
and v' equivalent to v

Adequacy

How do we know if a translation is correct?

Every target evaluation should represent a source evaluation...

Definition (Soundness)

$\forall e \in \mathbf{Exp}_{\mathrm{src}}.$ if $\mathcal{T}[\![e]\!] \rightarrow^*_{\mathrm{trg}} v'$ then $\exists v.\ e \rightarrow^*_{\mathrm{src}} v$ and v' equivalent to v

...and every source evaluation should have a target evaluation:

Definition (Completeness)

$\forall e \in \mathbf{Exp}_{\mathrm{src}}.$ if $e \rightarrow^*_{\mathrm{src}} v$ then $\exists v'.\ \mathcal{T}[\![e]\!] \rightarrow^*_{\mathrm{trg}} v'$ and v' equivalent to v

Continuations

In the preceding translations, the control structure of the source language was translated directly into the corresponding control structure in the target language.

For example:

$$\mathcal{T}[\![\lambda x.\, e]\!] = \lambda x.\, \mathcal{T}[\![e]\!]$$
$$\mathcal{T}[\![e_1\ e_2]\!] = \mathcal{T}[\![e_1]\!]\ \mathcal{T}[\![e_2]\!]$$

What can go wrong with this approach?

Continuations

- A snippet of code that represents “the rest of the program”
- Can be used directly by programmers…
- …or in program transformations by a compiler
- Make the control flow of the program explicit
- Also useful for defining the meaning of features like exceptions

Example

Consider the following expression:

$$(\lambda x.\, x)\, ((1 + 2) + 3) + 4$$

Example

Consider the following expression:

$$(\lambda x.\, x)\; ((1 + 2) + 3) + 4$$

If we make all of the continuations explicit, we obtain:

$$k_0 = \lambda v.\, (\lambda x.\, x)\; v$$

Example

Consider the following expression:

$$(\lambda x.\, x)\ ((1 + 2) + 3) + 4$$

If we make all of the continuations explicit, we obtain:

$$k_0 = \lambda v.\, (\lambda x.\, x)\ v$$
$$k_1 = \lambda a.\, k_0\ (a + 4)$$

Example

Consider the following expression:

$$(\lambda x.\, x)\; ((1 + 2) + 3) + 4$$

If we make all of the continuations explicit, we obtain:

$$\begin{aligned} k_0 &= \lambda v.\, (\lambda x.\, x)\; v \\ k_1 &= \lambda a.\, k_0\; (a + 4) \\ k_2 &= \lambda b.\, k_1\; (b + 3) \end{aligned}$$

Example

Consider the following expression:

$$(\lambda x.\, x)\, ((1 + 2) + 3) + 4$$

If we make all of the continuations explicit, we obtain:

$$\begin{aligned} k_0 &= \lambda v.\, (\lambda x.\, x)\, v \\ k_1 &= \lambda a.\, k_0\, (a + 4) \\ k_2 &= \lambda b.\, k_1\, (b + 3) \\ k_3 &= \lambda c.\, k_2\, (c + 2) \end{aligned}$$

Example

Consider the following expression:

$$(\lambda x.\, x)\; ((1 + 2) + 3) + 4$$

If we make all of the continuations explicit, we obtain:

$$\begin{aligned}
k_0 &= \lambda v.\, (\lambda x.\, x)\; v \\
k_1 &= \lambda a.\, k_0\; (a + 4) \\
k_2 &= \lambda b.\, k_1\; (b + 3) \\
k_3 &= \lambda c.\, k_2\; (c + 2)
\end{aligned}$$

$$(\lambda x.\, e_2)\; e_1$$

The original expression is equivalent to $k_3\; 1$, or:

$$(\lambda c.\, (\lambda b.\, (\lambda a.\, (\lambda v.\, (\lambda x.\, x)\; v)\; (a + 4))\; (b + 3))\; (c + 2))\; 1$$

Example (Continued)

Recall that let $x = e$ in e' is syntactic sugar for $(\lambda x.\, e')\ e$.

Hence, we can rewrite the expression with continuations more succinctly as

$$
\begin{array}{l}
\text{let } c = 1 \text{ in} \\
\text{let } b = c + 2 \text{ in} \\
\text{let } a = b + 3 \text{ in} \\
\text{let } v = a + 4 \text{ in} \\
(\lambda x.\, x)\ v
\end{array}
$$

CPS Transformation

$\mathcal{CPS}[\![\lambda x.\, e]\!]\, k \triangleq$ $\quad x \quad e \quad k$

We write $\mathcal{CPS}[\![e]\!]\; k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh."

CPS Transformation

$$\mathcal{CPS}[\![n]\!]\, k = k\, n$$

$\mathcal{CPS}[\![5]\!] = \lambda k.\, k\, 5$

We write $\mathcal{CPS}[\![e]\!]\, k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh."

CPS Transformation

$$\mathcal{CPS}[\![n]\!]\,k = k\,n$$

$$\mathcal{CPS}[\![e_1 + e_2]\!]\,k = \mathcal{CPS}[\![e_1]\!]\,(\lambda n.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda m.\,k\,(n+m)))$$

We write $\mathcal{CPS}[\![e]\!]\,k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh."

CPS Transformation

$$\begin{aligned}
\mathcal{CPS}[\![n]\!]\, k &= k\, n \\
\mathcal{CPS}[\![e_1 + e_2]\!]\, k &= \mathcal{CPS}[\![e_1]\!]\, (\lambda n.\, \mathcal{CPS}[\![e_2]\!]\, (\lambda m.\, k\, (n + m))) \\
\mathcal{CPS}[\![(e_1, e_2)]\!]\, k &= \mathcal{CPS}[\![e_1]\!]\, (\lambda v.\, \mathcal{CPS}[\![e_2]\!]\, (\lambda w.\, k\, (v, w)))
\end{aligned}$$

We write $\mathcal{CPS}[\![e]\!]\, k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh."

CPS Transformation

$$\mathcal{CPS}[\![n]\!]\,k = k\,n$$
$$\mathcal{CPS}[\![e_1 + e_2]\!]\,k = \mathcal{CPS}[\![e_1]\!]\,(\lambda n.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda m.\,k\,(n+m)))$$
$$\mathcal{CPS}[\![(e_1, e_2)]\!]\,k = \mathcal{CPS}[\![e_1]\!]\,(\lambda v.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda w.\,k\,(v, w)))$$
$$\mathcal{CPS}[\![\#1\;e]\!]\,k = \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#1\;v))$$

We write $\mathcal{CPS}[\![e]\!]\,k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are “fresh.”

CPS Transformation

$$\mathcal{CPS}[\![n]\!]\,k = k\,n$$
$$\mathcal{CPS}[\![e_1 + e_2]\!]\,k = \mathcal{CPS}[\![e_1]\!]\,(\lambda n.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda m.\,k\,(n+m)))$$
$$\mathcal{CPS}[\![(e_1, e_2)]\!]\,k = \mathcal{CPS}[\![e_1]\!]\,(\lambda v.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda w.\,k\,(v, w)))$$
$$\mathcal{CPS}[\![\#1\;e]\!]\,k = \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#1\;v))$$
$$\mathcal{CPS}[\![\#2\;e]\!]\,k = \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#2\;v))$$

We write $\mathcal{CPS}[\![e]\!]\,k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh."

CPS Transformation

$$\mathcal{CPS}[\![n]\!]\,k = k\,n$$
$$\mathcal{CPS}[\![e_1 + e_2]\!]\,k = \mathcal{CPS}[\![e_1]\!]\,(\lambda n.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda m.\,k\,(n+m)))$$
$$\mathcal{CPS}[\![(e_1, e_2)]\!]\,k = \mathcal{CPS}[\![e_1]\!]\,(\lambda v.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda w.\,k\,(v, w)))$$
$$\mathcal{CPS}[\![\#1\;e]\!]\,k = \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#1\;v))$$
$$\mathcal{CPS}[\![\#2\;e]\!]\,k = \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#2\;v))$$
$$\mathcal{CPS}[\![x]\!]\,k = k\,x$$

We write $\mathcal{CPS}[\![e]\!]\,k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are “fresh.”

CPS Transformation

$$
\begin{aligned}
\mathcal{CPS}[\![n]\!]\,k &= k\,n \\
\mathcal{CPS}[\![e_1 + e_2]\!]\,k &= \mathcal{CPS}[\![e_1]\!]\,(\lambda n.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda m.\,k\,(n+m))) \\
\mathcal{CPS}[\![(e_1, e_2)]\!]\,k &= \mathcal{CPS}[\![e_1]\!]\,(\lambda v.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda w.\,k\,(v, w))) \\
\mathcal{CPS}[\![\#1\;e]\!]\,k &= \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#1\;v)) \\
\mathcal{CPS}[\![\#2\;e]\!]\,k &= \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#2\;v)) \\
\mathcal{CPS}[\![x]\!]\,k &= k\,x \\
\mathcal{CPS}[\![\lambda x.\,e]\!]\,k &= k\,(\lambda x.\,\lambda k'.\,\mathcal{CPS}[\![e]\!]\,k')
\end{aligned}
$$

We write $\mathcal{CPS}[\![e]\!]\,k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are “fresh.”

CPS Transformation

$$
\begin{aligned}
\mathcal{CPS}[\![n]\!]\, k &= k\, n \\
\mathcal{CPS}[\![e_1 + e_2]\!]\, k &= \mathcal{CPS}[\![e_1]\!]\, (\lambda n.\, \mathcal{CPS}[\![e_2]\!]\, (\lambda m.\, k\, (n + m))) \\
\mathcal{CPS}[\![(e_1, e_2)]\!]\, k &= \mathcal{CPS}[\![e_1]\!]\, (\lambda v.\, \mathcal{CPS}[\![e_2]\!]\, (\lambda w.\, k\, (v, w))) \\
\mathcal{CPS}[\![\#1\, e]\!]\, k &= \mathcal{CPS}[\![e]\!]\, (\lambda v.\, k\, (\#1\, v)) \\
\mathcal{CPS}[\![\#2\, e]\!]\, k &= \mathcal{CPS}[\![e]\!]\, (\lambda v.\, k\, (\#2\, v)) \\
\mathcal{CPS}[\![x]\!]\, k &= k\, x \\
\mathcal{CPS}[\![\lambda x.\, e]\!]\, k &= k\, (\lambda x.\, \lambda k'.\, \mathcal{CPS}[\![e]\!]\, k') \\
\mathcal{CPS}[\![e_1\, e_2]\!]\, k &= \mathcal{CPS}[\![e_1]\!]\, (\lambda f.\, \mathcal{CPS}[\![e_2]\!]\, (\lambda v.\, f\, v\, k))
\end{aligned}
$$

We write $\mathcal{CPS}[\![e]\!]\, k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are “fresh.”

CPS Transformation

$$
\begin{aligned}
\mathcal{CPS}[\![n]\!]\,k &= k\,n \\
\mathcal{CPS}[\![e_1 + e_2]\!]\,k &= \mathcal{CPS}[\![e_1]\!]\,(\lambda n.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda m.\,k\,(n+m))) \\
\mathcal{CPS}[\![(e_1, e_2)]\!]\,k &= \mathcal{CPS}[\![e_1]\!]\,(\lambda v.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda w.\,k\,(v, w))) \\
\mathcal{CPS}[\![\#1\;e]\!]\,k &= \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#1\;v)) \\
\mathcal{CPS}[\![\#2\;e]\!]\,k &= \mathcal{CPS}[\![e]\!]\,(\lambda v.\,k\,(\#2\;v)) \\
\mathcal{CPS}[\![x]\!]\,k &= k\,x \\
\mathcal{CPS}[\![\lambda x.\,e]\!]\,k &= k\,(\lambda x.\,\lambda k'.\,\mathcal{CPS}[\![e]\!]\,k') \\
\mathcal{CPS}[\![e_1\;e_2]\!]\,k &= \mathcal{CPS}[\![e_1]\!]\,(\lambda f.\,\mathcal{CPS}[\![e_2]\!]\,(\lambda v.\,f\,v\,k))
\end{aligned}
$$

We write $\mathcal{CPS}[\![e]\!]\,k = \ldots$ instead of $\mathcal{CPS}[\![e]\!] = \lambda k.\ \ldots$

We assume that the new variables introduced are "fresh."

$$\text{CPS}[\![(\lambda x.\, x)(((1+2)+3)+4)]\!]\, k =$$

$$\text{CPS}[\![\lambda x.\, x]\!]\, (\lambda f.\; \text{CPS}[\![\ldots]\!]\, (\lambda v.\; f\, v\, k))$$

$$\text{CPS}[\![\lambda x.\, x]\!] \to \lambda k.\, k\, (\lambda x.\, \lambda k'.\; \text{CPS}[\![x]\!]\, k')$$

$$\text{CPS}[\![x]\!]\, k' \to k'\, x$$

$$\lambda k.\, k\, (\lambda x.\, \lambda k'.\, k'\, x)$$

$$\text{CPS}[\![\ldots]\!] \to (\lambda k.\; \text{CPS}[\![\ldots]\!]\, (\lambda n.\; \text{CPS}[\![4]\!]\, (\lambda m.\, k\, (n+m))))$$

$$(\lambda k.\; \text{CPS}[\![\ldots]\!]\, (\lambda n.\, (\lambda m.\, k(n+m))\, 4))$$

$$(\lambda k.\, k\, (\lambda x.\, \lambda k'.\, k'\, x))\, (\lambda f.\, (\lambda k.\; \text{CPS}[\![\ldots]\!])\, (\lambda n.\, (\lambda m.\, k\, (n+m)))\, (4))\; \lambda v.\, f\, v\, k)$$