CS 4110

Programming Languages & Logics

Lecture 11
Weakest Preconditions

Review: Decorating Programs

$\{\textbf{true}\}$

$$
\begin{array}{l}
x := m; \\
y := 0; \\
\textbf{while } (n < x) \textbf{ do } (\\
\quad x := x - n; \\
\quad y := y + 1 \\
)
\end{array}
$$

$\{ \qquad\qquad\qquad\qquad \}$

Review: Decorating Programs

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$\textbf{while}\ (n < x)\ \textbf{do}\ ($
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$\quad y := y + 1$
$)$

$\{n \times y + x = m\}$

In other words, the program divides m by n, so y is the quotient and x is the remainder.

Generating Preconditions

To fill in a precondition:

$$\{\quad\}\ c\ \{Q\}$$

there are many possible preconditions—and some are more useful than others.

Weakest Preconditions

Intuition: The weakest liberal precondition for c and Q is the weakest assertion P such that $\{P\}\ c\ \{Q\}$ is valid.

Weakest Preconditions

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More formally...

Definition (Weakest Liberal Precondition)

P is a weakest liberal precondition of c and Q written $wlp(c, Q)$ if:

$$\forall \sigma, I.\ \sigma \vDash_I P \iff (\mathcal{C}[\![c]\!]\ \sigma) \text{ undefined } \vee (\mathcal{C}[\![c]\!]\sigma) \vDash_I Q$$

Weakest Preconditions

POST

PRE

$$wlp(\textbf{skip}, P) \triangleq P$$

Weakest Preconditions

$$
\begin{aligned}
wlp(\textbf{skip}, P) &= P \\
wlp(\mathsf{x} := a, P) &= P[a/\mathsf{x}]
\end{aligned}
$$

$$(x < y+1 \land 2 < x+2)\boxed{[x+2/x]}$$

$$= x+2 < y+1 \land 2 < (x+2)+2$$

Weakest Preconditions

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\begin{array}{rcl}
wlp(\textbf{skip}, P) & = & P \\
wlp(\mathsf{x} := a, P) & = & P[a/\mathsf{x}] \\
wlp((c_1; c_2), P) & = & wlp(c_1, wlp(c_2, P))
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wlp(\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2, P) & = & (b \implies wlp(c_1, P)) \wedge \\
 & & (\neg b \implies wlp(c_2, P))
\end{array}
$$

$$
P := \mathrm{true} \mid \mathrm{false} \mid P_1 \wedge P_2 \mid P_1 \Rightarrow P_2
$$

Weakest Preconditions

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$$

where

$$
\begin{array}{rcl}
F_0(P) & = & \mathbf{true} \\
F_{i+1}(P) & = & (\neg b \implies P) \wedge (b \implies wlp(c, F_i(P)))
\end{array}
$$

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

p := getPacket();
processPacket(p);
assert P_{safe}

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

$\mathrm{p} := \mathrm{getPacket}();$
$\mathrm{processPacket}(\mathrm{p});$
$\{P_{\text{safe}}\}$

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

p := getPacket();
$\{P_{\text{filter}}(p)\}$;
processPacket(p);
$\{P_{\text{safe}}\}$

Applications of Weakest Preconditions

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p := getPacket();
assert $P_{\text{filter}}(p)$;
processPacket(p);

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

p := getPacket();
assert $P_{\text{filter}}(p)$;
processPacket(p);

P_{filter} should be the *weakest* precondition to avoid ruling out legitimate inputs.

David Brumley, Hao Wang, Somesh Jha, and Dawn Song. “Creating Vulnerability Signatures Using Weakest Preconditions.” In *Computer Security Foundations* (CSF), 2007.

Properties of Weakest Preconditions

Lemma (Correctness of Weakest Preconditions)

$$
\begin{aligned}
&\forall c \in \textbf{Com}, Q \in \textbf{Assn}. \\
&\qquad \models \{wlp(c, Q)\}\; c\; \{Q\} \;\; \textit{and} \\
&\qquad \forall R \in \textbf{Assn}.\;\; \models \{R\}\; c\; \{Q\}\; \textit{implies}\; (R \implies wlp(c, Q))
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\end{aligned}$$

Lemma (Provability of Weakest Preconditions)

$$\forall c \in \textbf{Com}, Q \in \textbf{Assn}.\; \vdash \{wlp(c, Q)\}\; c\; \{Q\}$$

$wlp(x := a, Q) = Q[a/x]$

Soundness and Completeness

Soundness: If we can prove it, then it's actually true.

$$\vdash \;\Rightarrow\; \models$$

Completeness: If it's true, then a proof exists.

$$\models \;\Rightarrow\; \vdash$$

Soundness and Completeness

Soundness: If we can prove it, then it's actually true.

Definition (Soundness)

If $\vdash \{P\}\ c\ \{Q\}$ then $\models \{P\}\ c\ \{Q\}$.

Completeness: If it's true, then a proof exists.

Definition (Completeness)

If $\models \{P\}\ c\ \{Q\}$ then $\vdash \{P\}\ c\ \{Q\}$.

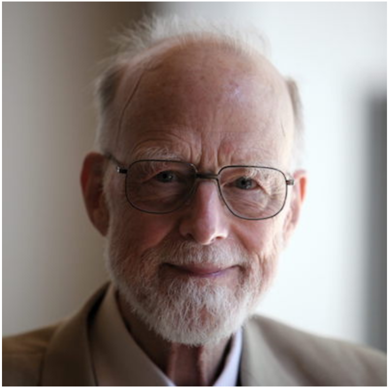
SKIP CONSEQUENCE

$\vdash \{P\}\ skip\ \{P\}$

CONSEQUENCE SEQ

$\vdash \{true\}\ skip\ \{P\}$ $\vdash \{P\}\ c\ \{Q\}$

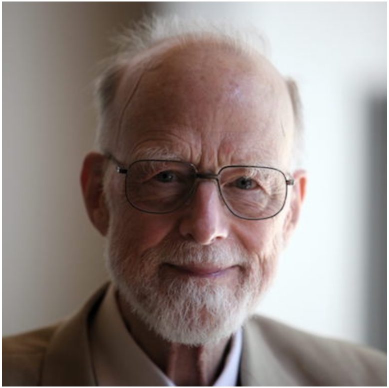
vs.



$\vDash \{true\}\ \text{skip}\ \{P\}$

Kurt Gödel

vs.



Sir Tony Hoare

Relative Completeness

$\models \{P\}\, c\, \{Q\}$

$\forall R.\quad R \Rightarrow wlp(c, Q)$

Theorem (Cook (1974))

$\forall P, Q \in \textbf{Assn}, c \in \textbf{Com}.\ \models \{P\}\, c\, \{Q\}$ *implies* $\vdash \{P\}\, c\, \{Q\}$.

LEMMA 1

$\models \;\Rightarrow\; \vdash$

LEMMA 2

$$\dfrac{\dfrac{\vdots}{\vdash \{wlp(c,Q)\}\ c\ \{Q\}}}{\vdash \{P\}\ c\ \{Q\}}\ \text{CONSEQ}$$

Relative Completeness

Theorem (Cook (1974))

$\forall P, Q \in \textbf{Assn}, c \in \textbf{Com}.\ \vDash \{P\}\, c\, \{Q\}$ *implies* $\vdash \{P\}\, c\, \{Q\}$.

Proof Sketch.

Let $\{P\}\, c\, \{Q\}$ be a valid partial correctness specification.

By the first Lemma we have $\vDash P \implies wlp(c, Q)$.

By the second Lemma we have $\vdash \{wlp(c, Q)\}\, c\, \{Q\}$.

We conclude $\vdash \{P\}\, c\, \{Q\}$ using the CONSEQUENCE rule. $\square$