CS 4110

Programming Languages & Logics

Lecture 11
Weakest Preconditions

Review: Decorating Programs

$\{\textbf{true}\}$

$\mathsf{x} := \mathsf{m};$
$\mathsf{y} := 0;$
$\textbf{while}\ (\mathsf{n} < \mathsf{x})\ \textbf{do}\ ($
$\quad \mathsf{x} := \mathsf{x} - \mathsf{n};$
$\quad \mathsf{y} := \mathsf{y} + 1$
$)$

$\{ \qquad\qquad\qquad \}$

Review: Decorating Programs

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)
\end{array}
$$

$\{\mathsf{n} \times \mathsf{y} + \mathsf{x} = \mathsf{m}\}$

In other words, the program divides m by n, so y is the quotient and x is the remainder.

Generating Preconditions

To fill in a precondition:

$$\{\quad\}\ c\ \{Q\}$$

there are many possible preconditions—and some are more useful than others.

Weakest Preconditions

Intuition: The weakest liberal precondition for c and Q is the weakest assertion P such that $\{P\}\ c\ \{Q\}$ is valid.

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More formally...

Definition (Weakest Liberal Precondition)

P is a weakest liberal precondition of c and Q written $wlp(c, Q)$ if:

$$\forall \sigma, I.\ \sigma \vDash_I P \iff (\mathcal{C}[\![c]\!]\ \sigma) \text{ undefined } \vee (\mathcal{C}[\![c]\!]\sigma) \vDash_I Q$$

Weakest Preconditions

POST

$$wlp(\textbf{skip}, P) \triangleq P$$

PRE

Weakest Preconditions

$$\begin{aligned} wlp(\textbf{skip}, P) &= P \\ wlp(\mathsf{x} := a, P) &= P[a/\mathsf{x}] \end{aligned}$$

$$(x < y + 1 \wedge 2 < x + 2)[x+2/x]$$

$$= x+2 < y+1 \wedge 2 < (x+2)+2$$

Weakest Preconditions

$$
\begin{array}{rcl}
wlp(\textbf{skip}, P) & = & P \\
wlp(\mathsf{x} := a, P) & = & P[a/\mathsf{x}] \\
wlp((c_1; c_2), P) & = & wlp(c_1, wlp(c_2, P))
\end{array}
$$

Weakest Preconditions

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wlp(\textbf{skip}, P) &= P \\
wlp(\mathrm{x} := a, P) &= P[a/\mathrm{x}] \\
wlp((c_1; c_2), P) &= wlp(c_1, wlp(c_2, P)) \\
wlp(\textbf{if } b \textbf{ then } c_1 \textbf{ else } c_2, P) &= (b \implies wlp(c_1, P)) \wedge \\
&\quad\ (\neg b \implies wlp(c_2, P))
\end{aligned}
$$

$P := true \mid false \mid P_1 \wedge P_2$

$\mid P_1 \Rightarrow P_2$

Weakest Preconditions

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wlp(\textbf{while } b \textbf{ do } c, P) & = & \bigwedge_i F_i(P)
\end{array}
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wlp(\mathbf{while}\ b\ \mathbf{do}\ c, P) & = & \bigwedge_i F_i(P)
\end{array}
$$

where

$$
\begin{array}{rcl}
F_0(P) & = & \mathbf{true} \\
F_{i+1}(P) & = & (\neg b \implies P) \wedge (b \implies wlp(c, F_i(P)))
\end{array}
$$

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

p := getPacket();
processPacket(p);
assert P_{safe}

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

$$\begin{array}{l} \text{p} := \text{getPacket}(); \\ \text{processPacket}(\text{p}); \\ \{P_{\text{safe}}\} \end{array}$$

Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

p := getPacket();
$\{P_{\text{filter}}(p)\}$;
processPacket(p);
$\{P_{\text{safe}}\}$

Applications of Weakest Preconditions

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Applications of Weakest Preconditions

Failing fast: avoid wasting work on bad inputs.

p := getPacket();
assert $P_{\text{filter}}(p)$;
processPacket(p);

P_{filter} should be the *weakest* precondition to avoid ruling out legitimate inputs.

David Brumley, Hao Wang, Somesh Jha, and Dawn Song. "Creating Vulnerability Signatures Using Weakest Preconditions." In *Computer Security Foundations* (CSF), 2007.

Properties of Weakest Preconditions

Lemma (Correctness of Weakest Preconditions)

$$
\begin{aligned}
&\forall c \in \mathbf{Com}, Q \in \mathbf{Assn}.\\
&\qquad \models \{\mathit{wlp}(c, Q)\}\, c\, \{Q\} \;\; \mathit{and}\\
&\qquad \forall R \in \mathbf{Assn}.\; \models \{R\}\, c\, \{Q\}\ \mathit{implies}\ (R \implies \mathit{wlp}(c, Q))
\end{aligned}
$$

Properties of Weakest Preconditions

Lemma (Correctness of Weakest Preconditions)

$$\begin{aligned}&\forall c \in \textbf{Com}, Q \in \textbf{Assn}.\\ &\qquad \models \{wlp(c, Q)\}\; c\; \{Q\} \;\; \textit{and}\\ &\qquad \forall R \in \textbf{Assn}.\; \models \{R\}\; c\; \{Q\}\; \textit{implies}\; (R \implies wlp(c, Q))\end{aligned}$$

Lemma (Provability of Weakest Preconditions)

$$\forall c \in \textbf{Com}, Q \in \textbf{Assn}.\; \vdash \{wlp(c, Q)\}\; c\; \{Q\}$$

Soundness and Completeness

Soundness: If we can prove it, then it's actually true.

Completeness: If it's true, then a proof exists.

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Definition (Soundness)

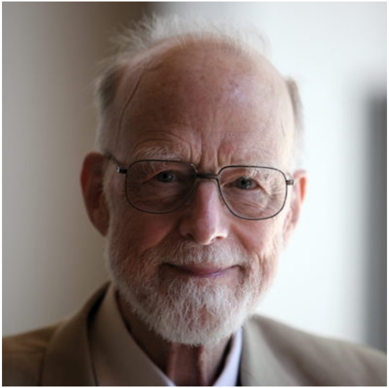
If $\vdash \{P\}\ c\ \{Q\}$ then $\models \{P\}\ c\ \{Q\}$.

Completeness: If it's true, then a proof exists.

Definition (Completeness)

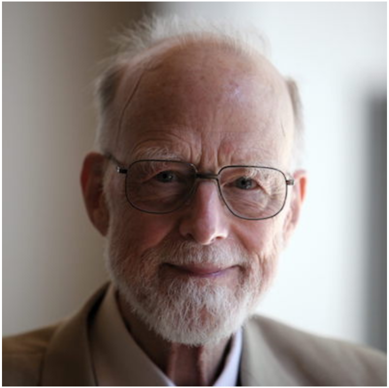
If $\models \{P\}\ c\ \{Q\}$ then $\vdash \{P\}\ c\ \{Q\}$.

vs.



Kurt Gödel

vs.



Sir Tony Hoare

Relative Completeness

Theorem (Cook (1974))

$\forall P, Q \in \textbf{Assn}, c \in \textbf{Com}.\ \vDash \{P\}\, c\, \{Q\}\ \textit{implies}\ \vdash \{P\}\, c\, \{Q\}.$

Relative Completeness

Theorem (Cook (1974))

$\forall P, Q \in \textbf{Assn}, c \in \textbf{Com}.\ \models \{P\}\, c\, \{Q\}$ *implies* $\vdash \{P\}\, c\, \{Q\}$.

Proof Sketch.

Let $\{P\}\, c\, \{Q\}$ be a valid partial correctness specification.

By the first Lemma we have $\models P \implies wlp(c, Q)$.

By the second Lemma we have $\vdash \{wlp(c, Q)\}\, c\, \{Q\}$.

We conclude $\vdash \{P\}\, c\, \{Q\}$ using the CONSEQUENCE rule. $\square$