$$X = \bigcup_i F^i(\emptyset)$$

1. X is a F.P. of F

$$F(X) = F\left(\bigcup_i F^i(\emptyset)\right)$$
$$= \bigcup_i F\left(F^i(\emptyset)\right)$$
$$= \bigcup_i F^{i+1}(\emptyset)$$
$$= \emptyset \cup \bigcup_i F^{i+1}(\emptyset)$$
$$= F^0(\emptyset) \cup \bigcup_i F^{i+1}(\emptyset)$$
$$= \bigcup_i F^i(\emptyset)$$
$$= X$$

2. $\nexists\, Y \subset X$ s.t. $F(Y) = Y$

ASSUME B.W.O.C. $\exists Y$.

SHOW $F^i(\emptyset) \subseteq Y \quad \forall i$.

INDUCT ON i.

B.C. $i = 0$

W.T.S. $F^0(\emptyset) \subseteq Y$

$\emptyset \subseteq Y$ ✓

F IS MONOTONIC

$A \subseteq B$

$\Rightarrow F(A) \subseteq F(B)$

INDUCTIVE STEP Assume (IHOP)

$F^i(\emptyset) \subseteq Y$; show $F^{i+1}(\emptyset) \subseteq Y$.

$$F^{i+1}(\emptyset) = F(F^i(\emptyset))$$

By monotonicity,

$$F^{i+1}(\emptyset) \subseteq \underbrace{F(Y) = Y}_{\text{because it's a F.P.}}$$

$$\bigcup_i F^i(\emptyset) = \underset{\subseteq Y}{F^0(\emptyset)} \cup \underset{\subseteq Y}{F^1(\emptyset)} \cup \underset{\subseteq Y}{\cdots}$$

$$\subseteq Y \cup Y \cup Y \cup \cdots$$

$$= Y$$

$$\Rightarrow \Leftarrow \qquad \square$$

$C[\![c_1 ; c_2]\!] \triangleq C[\![c_2]\!] \circ C[\![c_1]\!]$

$$C[\![\text{skip} ; c]\!] =$$

$$C[\![c]\!] \circ C[\![\text{skip}]\!]$$

$$= C[\![c]\!] \circ \{(\sigma, \sigma)\}$$

$$= C[\![c]\!]$$

($f(x) = x$
$g(x) = x^2 + 2$
$g \circ f = g$)

$= \{(\sigma, \sigma)\} \circ C[\![c]\!]$

$= C[\![\text{skip}]\!] \circ C[\![c]\!]$

$= C[\![c; \text{skip}]\!]$

$C[\![\overline{\text{while false do } c}]\!] = \ldots$

$\text{fix}(F)$

WHERE

$F(f) = \{(\sigma, \sigma) \mid (\sigma, \text{false}) \in B[\![\text{false}]\!]\} \cup$

$\{(\sigma, \sigma'') \mid f(\ldots) \ldots\}$

$= F^0(\emptyset) \cup F^1(\emptyset) \cup \ldots$

BY INDUCTION ON i,

$F^i(\emptyset) = \{(\sigma, \sigma)\}$

$= ID \cup ID \cup ID \cup \ldots$

$= ID$

$= C[\![\text{skip}]\!]$