CS 4110

Programming Languages & Logics

Lecture 5
IMP Properties

Command Equivalence

Intuitively, two commands are equivalent if they produce the same result under any store...

Definition (Equivalence of commands)

Two commands c and c' are equivalent (written $c \sim c'$) if, for any stores σ and σ', we have

$$\langle \sigma, c \rangle \Downarrow \sigma' \iff \langle \sigma, c' \rangle \Downarrow \sigma'.$$

Command Equivalence

For example, we can prove that every **while** command is equivalent to its "unrolling":

Theorem

For all $b \in \textbf{Bexp}$ *and* $c \in \textbf{Com}$,

$$\textbf{while}\ b\ \textbf{do}\ c \sim \textbf{if}\ b\ \textbf{then}\ (c; \textbf{while}\ b\ \textbf{do}\ c)\ \textbf{else}\ \textbf{skip}$$

Proof.

We show each implication separately... □

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- A: Then we would lose Turing completeness.
- Q: How much space do we need to represent configurations during execution of an IMP program?
- A: Can calculate a fixed bound!

Determinism

Theorem

$\forall c \in \textbf{Com}, \sigma, \sigma' \ \sigma'' \in \textbf{Store}$.
if $\langle \sigma, c \rangle \Downarrow \sigma'$ *and* $\langle \sigma, c \rangle \Downarrow \sigma''$ *then* $\sigma' = \sigma''$.

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By induction on the derivation of $\langle \sigma, c \rangle \Downarrow \sigma'$... □

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Example:

Given the derivation,

$$\dfrac{\dfrac{}{\langle \sigma, 6 \rangle \Downarrow 6} \qquad \dfrac{}{\langle \sigma, 7 \rangle \Downarrow 7}}{\dfrac{\langle \sigma, 6 \times 7 \rangle \Downarrow 42}{\langle \sigma, \mathsf{i} := 6 \times 7 \rangle \Downarrow \sigma[\mathsf{i} \mapsto 42]}}$$

we would write: $\mathcal{D} \Vdash \langle \sigma, \mathsf{i} := 42 \rangle \Downarrow \sigma[\mathsf{i} \mapsto 42]$

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In a proof by induction on derivations, for every inference rule, assume that the property P holds for all immediate subderivations, and show that it holds of the conclusion.

Large-Step Semantics

$$\textsc{Skip}\ \frac{}{\langle \sigma, \textbf{skip} \rangle \Downarrow \sigma} \qquad \textsc{Assgn}\ \frac{\langle \sigma, a \rangle \Downarrow n}{\langle \sigma, x := a \rangle \Downarrow \sigma[x \mapsto n]}$$

$$\textsc{Seq}\ \frac{\langle \sigma, c_1 \rangle \Downarrow \sigma' \qquad \langle \sigma', c_2 \rangle \Downarrow \sigma''}{\langle \sigma, c_1; c_2 \rangle \Downarrow \sigma''}$$

$$\textsc{If-T}\ \frac{\langle \sigma, b \rangle \Downarrow \textbf{true} \qquad \langle \sigma, c_1 \rangle \Downarrow \sigma'}{\langle \sigma, \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2 \rangle \Downarrow \sigma'}$$

$$\textsc{If-F}\ \frac{\langle \sigma, b \rangle \Downarrow \textbf{false} \qquad \langle \sigma, c_2 \rangle \Downarrow \sigma'}{\langle \sigma, \textbf{if}\ b\ \textbf{then}\ c_1\ \textbf{else}\ c_2 \rangle \Downarrow \sigma'}$$

$$\textsc{While-T}\ \frac{\langle \sigma, b \rangle \Downarrow \textbf{true} \qquad \langle \sigma, c \rangle \Downarrow \sigma' \qquad \langle \sigma', \textbf{while}\ b\ \textbf{do}\ c \rangle \Downarrow \sigma''}{\langle \sigma, \textbf{while}\ b\ \textbf{do}\ c \rangle \Downarrow \sigma''}$$

$$\textsc{While-F}\ \frac{\langle \sigma, b \rangle \Downarrow \textbf{false}}{\langle \sigma, \textbf{while}\ b\ \textbf{do}\ c \rangle \Downarrow \sigma}$$