Theorem: $\forall n$, n is even or odd.

Proof: Induct on structure of n.

$P(n) = n$ is even $\vee$ n is odd

Case

$$\frac{}{0 \in Nat}\text{ZERO}$$

$n = 0$. Zero is even.

Case

$$\frac{n \in Nat}{succ(n) \in Nat}\text{SUCC}$$

IHOP: $P(n)$, i.e., n is even or odd.

WTS: $P(succ(n))$, i.e., $n+1$ is even or odd.

Cases:

n even

$n+1$ is odd.

n is odd

$n+1$ is even. $\square$

TH: PROGRESS.

PROOF. INDUCTION ON STRUCTURE OF e.

$$P(e) = \forall \sigma.\ fvs(e) \subseteq dom(\sigma) \Longrightarrow e \in Int \ \vee \ \exists \sigma', e'.\ \langle \sigma, e \rangle \to \langle \sigma', e' \rangle$$

CASE $e = x$.

ASSUME $fvs(e) \subseteq dom(\sigma)$.

BY DEF OF fvs,

$$fvs(e) = fvs(x) = \{x\}$$

SO $x \in dom(\sigma)$.

LET $n = \sigma(x)$.

BY RULE VAR:

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \to \langle \sigma, n \rangle}\ \text{VAR}$$

CASE $e = n$. HERE, $e \in Int$.

CASE $e = e_1 + e_2$

BY IHOP, WE ASSUME $P(e_1)$

AND $P(e_2')$.

CASES:

$e_1 = n_1 \wedge e_2 = n_2$

BY ADD:

$$\frac{}{\langle \sigma, n_1 + n_2 \rangle \to \langle \sigma, p \rangle}$$

WHERE $p = n_1 + n_2$.

$e_1 \notin lat$

$$fvs(e) = fvs(e_1) \cup fvs(e_2) \quad \text{BY DEF OF } fvs$$

$$\subseteq dom(\sigma) \quad \text{BY ASSUMPTION}$$

SO: $fvs(e_1) \subseteq dom(\sigma)$.

BY IH OF $P(e_1)$ AND ,

WE HAVE $\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle$.

BY LADD:

$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \to \langle \sigma', e_1' + e_2 \rangle}$$

$e_1 = n_1 \quad \wedge \quad e_2 \notin \text{Int}$

SIMILARLY:

$$\text{fvs}(e_2) \subseteq \text{dom}(\sigma)$$

USING IHOP $P(e_2)$

$$\langle \sigma, e_2 \rangle \to \langle \sigma', e_2' \rangle$$

BY RADD:

$$\langle \sigma, e_1 + e_2 \rangle \to \langle \sigma', e_1 + e_2' \rangle$$