

CS 4110

Programming Languages & Logics

Lecture 3 Inductive Definitions and Proofs



Arithmetic Expressions

Last time we defined a simple language of arithmetic expressions,

$$e ::= x \mid n \mid e_1 + e_2 \mid e_1 * e_2 \mid x := e_1 ; e_2$$

and a small-step operational semantics, $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$.

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Example:

Assuming σ is a store that maps *foo* to 4...

$$\frac{\frac{\frac{\sigma(\text{foo}) = 4}{\langle \sigma, \text{foo} \rangle \rightarrow \langle \sigma, 4 \rangle} \text{VAR}}{\langle \sigma, \text{foo} + 2 \rangle \rightarrow \langle \sigma, 4 + 2 \rangle} \text{LADD}}{\langle \sigma, (\text{foo} + 2) * (\text{bar} + 1) \rangle \rightarrow \langle \sigma, (4 + 2) * (\text{bar} + 1) \rangle} \text{LMUL}$$

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- **Determinism:** Every configuration has at most one successor.

$\forall e \in \mathbf{Exp}. \forall \sigma, \sigma', \sigma'' \in \mathbf{Store}. \forall e', e'' \in \mathbf{Exp}.$
if $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$ and $\langle \sigma, e \rangle \rightarrow \langle \sigma'', e'' \rangle$
then $e' = e''$ and $\sigma' = \sigma''$.

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- **Termination:** Evaluation of every expression terminates.

$\forall e \in \mathbf{Exp}. \forall \sigma \in \mathbf{Store}. \exists \sigma' \in \mathbf{Store}. \exists e' \in \mathbf{Exp}.$
 $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$ and $\langle \sigma', e' \rangle \not\rightarrow,$

Where $\langle \sigma', e' \rangle \not\rightarrow$ is shorthand for:

$\neg (\exists \sigma'' \in \mathbf{Store}. \exists e'' \in \mathbf{Exp}. \langle \sigma', e' \rangle \rightarrow \langle \sigma'', e'' \rangle)$

Soundness

- **Soundness:** Evaluation of every expression yields an integer.

$$\forall e \in \mathbf{Exp}. \forall \sigma \in \mathbf{Store}. \exists \sigma' \in \mathbf{store}. \exists n' \in \mathbf{Int}. \\ \langle \sigma, e \rangle \rightarrow^* \langle \sigma', n' \rangle$$

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It is tempting to try to prove this property.

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Counterexample

If $\sigma = \emptyset$, then $\langle \sigma, x \rangle \not\rightarrow$.

In general, evaluation of an expression can *get stuck*...

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$$fvs(x) = \{x\}$$

$$fvs(x + y) = \{x, y\}$$

$$fvs(x := 5; x) = \{\}$$

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$x + (x := 5; x)$  "REMOVE"

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Well-Formedness:

A configuration $\langle \sigma, e \rangle$ is *well-formed* if and only if $fvs(e) \subseteq dom(\sigma)$.

Progress and Preservation

Now we can formulate two properties that imply soundness:

- Progress:

$\forall e \in \mathbf{Exp}. \forall \sigma \in \mathbf{Store}.$

$\langle \sigma, e \rangle$ well-formed \implies

$e \in \mathbf{Int}$ or $(\exists e' \in \mathbf{Exp}. \exists \sigma' \in \mathbf{Store}. \langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle)$

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- Preservation:

$\forall e, e' \in \mathbf{Exp}. \forall \sigma, \sigma' \in \mathbf{Store}.$

$\langle \sigma, e \rangle$ well-formed and $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \implies$

$\langle \sigma', e' \rangle$ well-formed.

$\langle \emptyset, x \rangle$

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$\langle \sigma', e' \rangle$ well-formed.

How are we going to prove these properties? Induction!

Inductive Sets

Inductive Sets

An *inductively-defined set* A is one that can be described using a finite collection of inference rules:

$$\frac{a_1 \in A \quad \dots \quad a_n \in A}{a \in A}$$

This rule states that if a_1 through a_n are elements of A , then a is also an element of A .


An inference rule with no premises is often called an *axiom*.

The set A is the smallest set “closed” under these axioms and rules.

Inductive Set Examples

The natural numbers are an inductive set.

$$\frac{}{0 \in \mathbb{N}} \quad \frac{n \in \mathbb{N}}{\text{succ}(n) \in \mathbb{N}}$$

$\text{succ}(\text{succ}(0))$ 

\mathbb{N}

Inductive Set Examples

Every BNF grammar defines an inductive set.

$$e ::= x \mid n \mid e_1 + e_2 \mid e_1 * e_2 \mid x := e_1 ; e_2$$

can be equivalently defined as:

$$\begin{array}{c} \frac{}{x \in \mathbf{Exp}} \qquad \frac{}{n \in \mathbf{Exp}} \\ \\ \frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{e_1 + e_2 \in \mathbf{Exp}} \qquad \frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{e_1 * e_2 \in \mathbf{Exp}} \\ \\ \frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{x := e_1 ; e_2 \in \mathbf{Exp}} \end{array}$$

Inductive Set Examples

The small-step evaluation relation \rightarrow is an inductive set.

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle} \text{VAR}$$

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e'_1 + e_2 \rangle} \text{LADD}$$

$$\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle}{\langle \sigma, n + e_2 \rangle \rightarrow \langle \sigma', n + e'_2 \rangle} \text{RADD}$$

$$\frac{p = m + n}{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle} \text{ADD}$$

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, e_1 * e_2 \rangle \rightarrow \langle \sigma', e'_1 * e_2 \rangle} \text{LMUL}$$

$$\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle}{\langle \sigma, n * e_2 \rangle \rightarrow \langle \sigma', n * e'_2 \rangle} \text{RMUL}$$

$$\frac{p = m \times n}{\langle \sigma, m * n \rangle \rightarrow \langle \sigma, p \rangle} \text{MUL}$$

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \rightarrow \langle \sigma', x := e'_1 ; e_2 \rangle} \text{ASSGN1}$$

$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle} \text{ASSGN}$$

Inductive Set Examples

The multi-step evaluation relation is an inductive set.

$$\frac{}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle} \text{ REFL}$$

$$\frac{\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle} \text{ STEP}$$

$$\frac{\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle \quad \langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle} \text{ TRANS}$$

Inductive Set Examples

The set of free variables of an expression is an inductive set.

$$\frac{}{y \in fvs(y)}$$

$$\frac{y \in fvs(e_1)}{y \in fvs(e_1 + e_2)}$$

$$\frac{y \in fvs(e_2)}{y \in fvs(e_1 + e_2)}$$

$$\frac{y \in fvs(e_1)}{y \in fvs(e_1 * e_2)}$$

$$\frac{y \in fvs(e_2)}{y \in fvs(e_1 * e_2)}$$

$$\frac{y \in fvs(e_1)}{y \in fvs(x := e_1 ; e_2)}$$

$$\frac{y \neq x \quad y \in fvs(e_2)}{y \in fvs(x := e_1 ; e_2)}$$

Induction Principle

Recall the principle of mathematical induction.

To prove $\forall n. P(n)$, we must establish several cases.

- Base case: $P(0)$
- Inductive case: $P(m) \Rightarrow P(m + 1)$

$$P(\text{succ}(m))$$

Induction Principle

Every inductive set has an analogous principle.

To prove $\forall a. P(a)$ we must establish several cases.

- **Base cases:** $P(a)$ holds for each axiom

$$\frac{}{a \in A}$$

- **Inductive cases:** For each inference rule

$$\frac{a_1 \in A \quad \dots \quad a_n \in A}{a \in A}$$

if $P(a_1)$ and ... and $P(a_n)$ then $P(a)$

Example: Progress

Recall the progress property.

$\forall e \in \mathbf{Exp}. \forall \sigma \in \mathbf{Store}.$

$\langle \sigma, e \rangle$ well-formed \implies

$e \in \mathbf{Int}$ or $(\exists e' \in \mathbf{Exp}. \exists \sigma' \in \mathbf{Store}. \langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle)$

We'll prove this by structural induction on e .

$\frac{}{x \in \mathbf{Exp}}$

$\frac{}{n \in \mathbf{Exp}}$

$\frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{e_1 + e_2 \in \mathbf{Exp}}$

$\frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{e_1 * e_2 \in \mathbf{Exp}}$

$\frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{x := e_1 ; e_2 \in \mathbf{Exp}}$