CS 4110

Programming Languages & Logics

Lecture 2
Introduction to Semantics

Semantics

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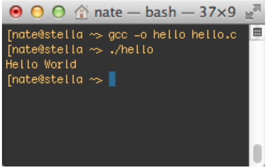
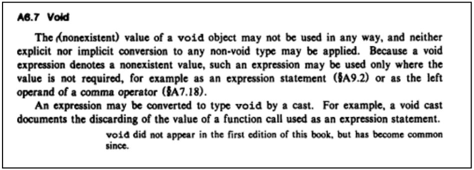
Answer: We could execute the program using an interpreter or a compiler, or we could consult a manual...





A6.7 Void

The (nonexistent) value of a `void` object may not be used in any way, and neither explicit nor implicit conversion to any non-void type may be applied. Because a void expression denotes a nonexistent value, such an expression may be used only where the value is not required, for example as an expression statement (§A9.2) or as the left operand of a comma operator (§A7.18).

An expression may be converted to type `void` by a cast. For example, a void cast documents the discarding of the value of a function call used as an expression statement.

`void` did not appear in the first edition of this book, but has become common since.

...but none of these is a satisfactory solution.

Formal Semantics

Three Approaches

- Operational $\langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle$
  - Model program by execution on abstract machine
  - Useful for implementing compilers and interpreters
- Denotational: $[\![e]\!]$
  - Model program as mathematical objects
  - Useful for theoretical foundations
- Axiomatic $\vdash \{\phi\}\ e\ \{\psi\}$
  - Model program by the logical formulas it obeys
  - Useful for proving program correctness

Arithmetic Expressions

Syntax

A language of integer arithmetic expressions with assignment.

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Metavariables:

$$
\begin{aligned}
x, y, z &\in \textbf{Var} \\
n, m &\in \textbf{Int} \\
e &\in \textbf{Exp}
\end{aligned}
$$

$\textbf{Var} = \{a, b, c, \ldots\}$

$\textbf{Int} = \mathbb{Z} = \{\ldots, -1, 0, 1, 2, \ldots\}$

$e_1 \quad e_2$

$\text{Var} \cap \text{Int} = \emptyset$

n

$x + y \qquad x + x$

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= {a, b, c, ..., 1, 2, 3, ..., -1, -2, -3, ..., a + 5, 6+z, 12 + 6 + a, ...}

Backus-Naur Form

BNF Grammar:

$$
\begin{aligned}
e ::= {} & x \\
\mid {} & n \\
\mid {} & e_1 + e_2 \\
\mid {} & e_1 * e_2 \\
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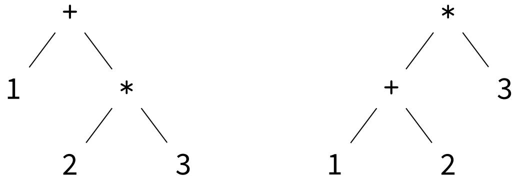

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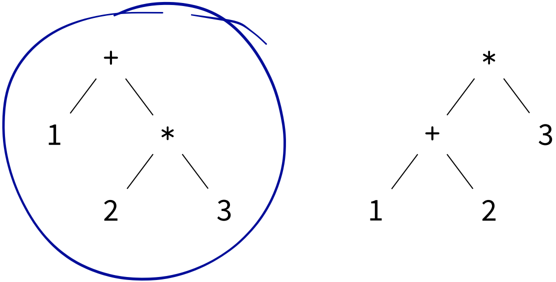

In this course, we will distinguish abstract syntax from concrete syntax, and focus primarily on abstract syntax (using conventions or parentheses at the concrete level to disambiguate as needed).

Representing Expressions

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OCaml:

```
type exp = Var of string
         | Int of int
         | Add of exp * exp
         | Mul of exp * exp
         | Assgn of string * exp * exp
```

Example: Mul(Int 2, Add(Var ”foo”, Int 1))

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Java:

```
abstract class Expr { }
class Var extends Expr { String name; ... }
class Int extends Expr { int val; ... }
class Add extends Expr { Expr exp1, exp2; ... }
class Mul extends Expr { Expr exp1, exp2; ... }
class Assgn extends Expr { String var, Expr exp1, exp2; ... }
```

Example: new Mul(new Int(2), new Add(new Var("foo"), new Int(1)))

Quiz

- $7 + (4 * 2)$ evaluates to ...?

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The rest of this lecture will make these intuitions precise...

Mathematical Preliminaries

Binary Relations

The *product* of two sets A and B, written $A \times B$, contains all ordered pairs (a, b) with $a \in A$ and $b \in B$.

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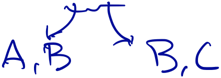
Some Important Relations

- empty – $\emptyset$
- total – $A \times B$
- identity on A – $\{(a, a) \mid a \in A\}$.   A, A
- composition $R; S$ – $\{(a, c) \mid \exists b. \, (a, b) \in R \wedge (b, c) \in S\}$



Functions

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The *image* of f is the set of elements $b \in B$ that are mapped to by at least one $a \in A$. More formally: $\text{image}(f) \triangleq \{f(a) \mid a \in A\}$

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A function $f : A \to B$ is said to be *surjective* (or *onto*) if and only if the image of f is B.

Operational Semantics

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More formally,

$$\begin{array}{rcl} \textbf{Store} & \triangleq & \textbf{Var} \rightharpoonup \textbf{Int} \\ \textbf{Config} & \triangleq & \textbf{Store} \times \textbf{Exp} \end{array}$$

Note that a store is a *partial* function from variables to integers.

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Intuitively, if facts above the line hold, then facts below the line hold. More formally, "$\to$" is the smallest relation "closed" under the inference rules.

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle} \textsc{Var}$$

Addition

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e_1' + e_2 \rangle} \ \textsc{LAdd}$$

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Multiplication

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 * e_2 \rangle \rightarrow \langle \sigma', e_1' * e_2 \rangle} \text{ LMUL}$$

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$$\frac{p = m \times n}{\langle \sigma, m * n \rangle \rightarrow \langle \sigma, p \rangle} \text{ MUL}$$

Assignment

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, x := e_1 \, ; \; e_2 \rangle \rightarrow \langle \sigma', x := e_1' \, ; \; e_2 \rangle} \; \textsc{Assgn1}$$

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$$\frac{\langle \sigma, e_1 \rangle \to \langle \sigma', e_1' \rangle}{\langle \sigma, x := e_1 \,;\; e_2 \rangle \to \langle \sigma', x := e_1' \,;\; e_2 \rangle} \textsc{ Assgn1}$$

$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n \,;\; e_2 \rangle \to \langle \sigma', e_2 \rangle} \textsc{ Assgn}$$

Notation: $\sigma[x \mapsto n]$ maps x to n and otherwise behaves like σ

Operational Semantics

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle}\ \textsc{Var} \qquad \frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e_1' \rangle}{\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e_1' + e_2 \rangle}\ \textsc{LAdd}$$

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