CS 4110

Programming Languages & Logics

Lecture 23
Type Inference

Review: Polymorphic λ-Calculus

Syntax

$$e ::= n \mid x \mid \lambda x\!:\!\tau.\, e \mid e_1\, e_2 \mid \Lambda X.\, e \mid e\, [\tau]$$
$$v ::= n \mid \lambda x\!:\!\tau.\, e \mid \Lambda X.\, e$$

Dynamic Semantics

$$E ::= [\cdot] \mid E\, e \mid v\, E \mid E\, [\tau]$$

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']} \qquad \frac{}{(\lambda x\!:\!\tau.\, e)\, v \rightarrow e\{v/x\}} \qquad \frac{}{(\Lambda X.\, e)\, [\tau] \rightarrow e\{\tau/X\}}$$

Review: Polymorphic λ-Calculus

$$\frac{}{\Delta, \Gamma \vdash n : \mathbf{int}} \qquad \frac{\Gamma(x) = \tau}{\Delta, \Gamma \vdash x : \tau}$$

$$\frac{\Delta, \Gamma, x : \tau \vdash e : \tau' \quad \Delta \vdash \tau \text{ ok}}{\Delta, \Gamma \vdash \lambda x : \tau.\, e : \tau \to \tau'} \qquad \frac{\Delta, \Gamma \vdash e_1 : \tau \to \tau' \quad \Delta, \Gamma \vdash e_2 : \tau}{\Delta, \Gamma \vdash e_1\, e_2 : \tau'}$$

$$\frac{\Delta \cup \{X\}, \Gamma \vdash e : \tau}{\Delta, \Gamma \vdash \Lambda X.\, e : \forall X.\, \tau} \qquad \frac{\Delta, \Gamma \vdash e : \forall X.\, \tau' \quad \Delta \vdash \tau \text{ ok}}{\Delta, \Gamma \vdash e\,[\tau] : \tau'\{\tau / X\}}$$

Review: Polymorphic λ-Calculus

Polymorphism let us write a doubling function that works for *any* type of function:

$$\text{double} \triangleq \Lambda X.\, \lambda f{:}X \to X.\, \lambda x{:}X.\, f\,(f\,x).$$

The type of this expression is:

$$\forall X.\, (X \to X) \to X \to X$$

You can use the polymorphic function by providing a type:

$$\text{double}\,[\mathbf{int}]\,(\lambda n{:}\mathbf{int}.\, n + 1)\; 7$$

Type Inference

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For example, we can write:

```
let double f x = f (f x)
```

and OCaml will figure out that the type is:

```
('a → 'a) → 'a → 'a
```

which is equivalent to the same System F type:

$$\forall A.\ (A \to A) \to A \to A$$

Type Inference

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We can also write

```
double (fun x → x+1) 7
```

and OCaml will infer that the polymorphic function `double` is instantiated at the type `int`.

ML Polymorphism

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- Not prenex: $(\forall \alpha.\, \alpha \to \alpha) \to$ **int**

ML Polymorphism

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These restrictions, called *prenex polymorphism*, stipulate that $\forall$s may only appear in the “outermost” position.

Examples

- Prenex: $\forall \alpha.\ \alpha \rightarrow \alpha$
- Not prenex: $(\forall \alpha.\ \alpha \rightarrow \alpha) \rightarrow \textbf{int}$

These restrictions have the following practical ramifications:

- Can’t instantiate type variables with polymorphic types
- Can’t put a polymorphic type on the left of an arrow

Example

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```
        OCaml version 4.01.0

# fun x -> x x;;
Error: This expression has type 'a -> 'b
    but an expression was expected of type 'a
    The type variable 'a occurs inside 'a -> 'b
```

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Type inference for the STLC means guessing a τ in every abstraction in an *untyped* version:

$$\lambda x.\, e$$

to produce a *typed* program:

$$\lambda x : \tau.\, e$$

that we can use in the typing rule for functions.

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Putting all these pieces together:

$\lambda a : \textbf{int} \rightarrow \textbf{bool}.\ \lambda b : \textbf{int}.\ \lambda c : \textbf{int}.\ \text{if } a\ (b+1) \text{ then } b \text{ else } c$

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$$\Gamma \vdash e : \tau \mid C$$

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Given a typing context Γ and an expression e, it generates a set of *constraints*—equations between types.

If these constraints are solvable, then e can be well-typed in Γ.

A solution to a set of constraints is a *type substitution* σ that, for each equation, makes both sides syntactically equal.

STLC for Type Inference

Let's define the type inference judgment for this STLC language:

$$e ::= x \mid \lambda x\!:\!\tau.\, e \mid e_1\, e_2 \mid n \mid e_1 + e_2$$
$$\tau ::= \textbf{int} \mid X \mid \tau_1 \to \tau_2$$

You can use a type variable *X* wherever you want to have a type inferred.

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$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau \mid \emptyset}\ \text{CT-VAR}$$

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$$\frac{\Gamma \vdash e_1 : \tau_1 \mid C_1 \quad \Gamma \vdash e_2 : \tau_2 \mid C_2}{\Gamma \vdash e_1 + e_2 : \textbf{int} \mid C_1 \cup C_2 \cup \{\tau_1 = \textbf{int}, \tau_2 = \textbf{int}\}}\ \textsc{CT-Add}$$

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$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \mid C}{\Gamma \vdash \lambda x : \tau_1 .\, e : \tau_1 \rightarrow \tau_2 \mid C}\ \textsc{CT-Abs}$$

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$$\frac{\Gamma, x : \tau_1 \vdash e : \tau_2 \mid C}{\Gamma \vdash \lambda x : \tau_1 .\, e : \tau_1 \to \tau_2 \mid C}\ \text{CT-ABS}$$

$$\frac{\begin{array}{c}\Gamma \vdash e_1 : \tau_1 \mid C_1 \quad \Gamma \vdash e_2 : \tau_2 \mid C_2 \\ X \text{ fresh} \quad C' = C_1 \cup C_2 \cup \{\tau_1 = \tau_2 \to X\}\end{array}}{\Gamma \vdash e_1\, e_2 : X \mid C'}\ \text{CT-APP}$$

Solving Constraints

A *type substitution* is a finite map from type variables to types.

Example: The substitution

$$[X \mapsto \textbf{int}, Y \mapsto \textbf{int} \rightarrow \textbf{int}]$$

maps type variable X to **int** and Y to $\textbf{int} \rightarrow \textbf{int}$.

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Given two substitutions σ_1 and σ_2, we write $\sigma_1 \circ \sigma_2$ for their composition: $(\sigma_1 \circ \sigma_2)(\tau) = \sigma_1(\sigma_2(\tau))$.

Unification

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We say that a substitution σ *unifies* constraint $\tau = \tau'$ if $\sigma(\tau) = \sigma(\tau')$.

We say that substitution σ *satisfies* (or *unifies*) set of constraints *C* if σ unifies every constraint in *C*.

Unification

If:

- $\Gamma \vdash e : \tau \mid C$, and
- σ satisfies C,

then e has type τ' under Γ,
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If there are no substitutions that satisfy C, then e is not typeable.

So let's find a substitution σ that unifies a set of constraints C!

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else if $\tau' = X$ and X not a free variable of τ then
    $unify(C'\{\tau/X\}) \circ [X \mapsto \tau]$
else if $\tau = \tau_o \to \tau_1$ and $\tau' = \tau'_o \to \tau'_1$ then
    $unify(C' \cup \{\tau_0 = \tau'_0, \tau_1 = \tau'_1\})$

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else
  fail

Unification Properties

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The solution, if it exists, is the most general solution: if $\sigma = \mathit{unify}(C)$ and σ' is a solution to C, then there is some σ'' such that $\sigma' = (\sigma'' \circ \sigma)$.