CS 4110

Programming Languages & Logics

Lecture 20
Normalization

Type "Completeness"?

Are all well-behaved programs well-typed?

Normalization

The simply-typed lambda calculus enjoys a remarkable property:

Every well-typed program terminates.

Simply-Typed Lambda Calculus

Syntax

expressions	$e ::= x \mid \lambda x\!:\!\tau.\, e \mid e_1\, e_2 \mid ()$
values	$v ::= \lambda x\!:\!\tau.\, e \mid ()$
types	$\tau ::= \mathbf{unit} \mid \tau_1 \rightarrow \tau_2$

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Dynamic Semantics

$$E ::= [\cdot] \mid E\, e \mid v\, E$$

$$\frac{e \to e'}{E[e] \to E[e']} \qquad \frac{}{(\lambda x\!:\!\tau.\, e)\, v \to e\{v/x\}}$$

Simply-Typed Lambda Calculus

Static Semantics

$$\frac{}{\Gamma \vdash () : \textbf{unit}}\ \textsc{T-Unit}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau}\ \textsc{T-Var}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau.\, e : \tau \to \tau'}\ \textsc{T-Abs}$$

$$\frac{\Gamma \vdash e_1 : \tau \to \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\, e_2 : \tau'}\ \textsc{T-App}$$

Supporting Lemmas

Lemma (Inversion)

- *If* $\Gamma \vdash x : \tau$ *then* $\Gamma(x) = \tau$
- *If* $\Gamma \vdash \lambda x : \tau_1 .\, e : \tau$ *then* $\tau = \tau_1 \to \tau_2$ *and* $\Gamma, x : \tau_1 \vdash e : \tau_2$*.*
- *If* $\Gamma \vdash e_1\, e_2 : \tau$ *then* $\Gamma \vdash e_1 : \tau' \to \tau$ *and* $\Gamma \vdash e_2 : \tau'$*.*

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Lemma (Canonical Forms)

- *If* $\Gamma \vdash v : \textbf{unit}$ *then* $v = ()$
- *If* $\Gamma \vdash v : \tau_1 \to \tau_2$ *then* $v = \lambda x : \tau_1 . e$ *and* $\Gamma, x : \tau_1 \vdash e : \tau_2$*.*

First Attempt

Theorem (Normalization)

If $\vdash e : \tau$ *then there exists a value* v *such that* $e \rightarrow^{*} v$.

Logical Relations

Idea: define a set with the following properties:

- At base types the set contains all expressions satisfying some property.
- At function types, the set contains all expressions such that the property is preserved whenever we apply the function to an argument of appropriate type that is also in the set.

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In our setting, the property will concern normalization...

Logical Relation

Definition (Logical Relation)

- $R_{\textbf{unit}}(e)$ iff $\vdash e : \textbf{unit}$ and e halts.
- $R_{\tau_1 \to \tau_2}(e)$ iff $\vdash e : \tau_1 \to \tau_2$ and e halts, and for every e' such that $R_{\tau_1}(e')$ we have $R_{\tau_2}(e\, e')$.

Supporting Lemmas

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Lemma (Goal)

If $\vdash e : \tau$ then $R_\tau(e)$

Main Lemma

Lemma (Goal – Strengthened)

If

- $x_1 : \tau_1, \ldots, x_k : \tau_k \vdash e : \tau$,
- v_1 *through* v_k *are values such that* $\vdash v_1 : \tau_1$ *through* $\vdash v_k : \tau_k$, *and*
- $R_{\tau_1}(v_1)$ *through* $R_{\tau_k}(v_k)$,

then $R_\tau(e\{v_1/x_1\} \ldots \{v_k/x_k\})$.