

CS 4110

Programming Languages & Logics

Lecture 4 Large-Step Semantics



Review

So far we've:

- Defined a simple language of arithmetic expressions
- Formalized its semantics as a “small-step” relation:
 $\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle$ and $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$

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So far we've:

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Today we'll:

- Develop an alternate semantics based on a “large-step” relation
- Prove the equivalence of the two semantics

Large-Step Semantics

Idea: Define a new relation that captures the *complete* evaluation of an expression.

Formally: Define a relation $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$. Our new \Downarrow binary relation has this type:

$$\Downarrow \subseteq (\mathbf{Store} \times \mathbf{Exp}) \times (\mathbf{Store} \times \mathbf{Int})$$

Intuition: Completely evaluating the expression e in store σ produces the number n while updating the store to σ' .

Variables

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \Downarrow \langle \sigma, n \rangle} \text{VAR}$$

Integers

$$\overline{\langle \sigma, n \rangle} \Downarrow \langle \sigma, n \rangle \text{ INT}$$

Addition

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \quad n = n_1 + n_2}{\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \text{ADD}$$

Multiplication

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \quad n = n_1 \times n_2}{\langle \sigma, e_1 * e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \text{MUL}$$

Assignment

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma'[x \mapsto n_1], e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle} \text{ASSGN}$$

Large-Step Semantics

$$\frac{}{\langle \sigma, n \rangle \Downarrow \langle \sigma, n \rangle} \text{INT}$$

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \Downarrow \langle \sigma, n \rangle} \text{VAR}$$

$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma', e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle \quad n = n_1 + n_2}{\langle \sigma, e_1 + e_2 \rangle \Downarrow \langle \sigma'', n \rangle} \text{ADD}$$

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$$\frac{\langle \sigma, e_1 \rangle \Downarrow \langle \sigma', n_1 \rangle \quad \langle \sigma' [x \mapsto n_1], e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \Downarrow \langle \sigma'', n_2 \rangle} \text{ASSGN}$$

Example

Assume that $\sigma(\text{bar}) = 7$. Let $\sigma' = \sigma[\text{foo} \mapsto 3]$.

$$\frac{\frac{\frac{\overline{\langle \sigma, 3 \rangle \Downarrow \langle \sigma, 3 \rangle}^{\text{INT}}}{\langle \sigma', \text{foo} \rangle \Downarrow \langle \sigma', 3 \rangle}^{\text{VAR}} \quad \frac{\overline{\langle \sigma', \text{bar} \rangle \Downarrow \langle \sigma', 7 \rangle}^{\text{VAR}}}{\langle \sigma', \text{foo} * \text{bar} \rangle \Downarrow \langle \sigma', 21 \rangle}^{\text{MUL}}}{\langle \sigma, \text{foo} := 3; \text{foo} * \text{bar} \rangle \Downarrow \langle \sigma', 21 \rangle}^{\text{ASSGN}}$$

Equivalence

Theorem (Equivalence of small-step and large-step)

$\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$ if and only if $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$

Equivalence

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$$\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle \text{ if and only if } \langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$$

To streamline the proof, we'll use the following multi-step relation:

$$\frac{}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma, e \rangle} \text{REFL}$$
$$\frac{\langle \sigma, e \rangle \rightarrow \langle \sigma', e' \rangle \quad \langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle} \text{TRANS}$$

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Lemma

1. If $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', n \rangle$, then:

- ▶ $\langle \sigma, e + e_2 \rangle \rightarrow^* \langle \sigma', n + e_2 \rangle$
- ▶ $\langle \sigma, n_1 + e \rangle \rightarrow^* \langle \sigma', n_1 + n \rangle$
- ▶ $\langle \sigma, e * e_2 \rangle \rightarrow^* \langle \sigma', n * e_2 \rangle$
- ▶ $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$
- ▶ $\langle \sigma, x := e; e_2 \rangle \rightarrow^* \langle \sigma', x := n; e_2 \rangle$

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- ▶ $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$
- ▶ $\langle \sigma, x := e; e_2 \rangle \rightarrow^* \langle \sigma', x := n; e_2 \rangle$

2. If $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$ and $\langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle$, then $\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle$

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 - ▶ $\langle \sigma, e * e_2 \rangle \rightarrow^* \langle \sigma', n * e_2 \rangle$
 - ▶ $\langle \sigma, n_1 * e \rangle \rightarrow^* \langle \sigma', n_1 * n \rangle$
 - ▶ $\langle \sigma, x := e; e_2 \rangle \rightarrow^* \langle \sigma', x := n; e_2 \rangle$
2. If $\langle \sigma, e \rangle \rightarrow^* \langle \sigma', e' \rangle$ and $\langle \sigma', e' \rangle \rightarrow^* \langle \sigma'', e'' \rangle$, then $\langle \sigma, e \rangle \rightarrow^* \langle \sigma'', e'' \rangle$
3. If $\langle \sigma, e \rangle \rightarrow \langle \sigma'', e'' \rangle$ and $\langle \sigma'', e'' \rangle \Downarrow \langle \sigma', n \rangle$, then $\langle \sigma, e \rangle \Downarrow \langle \sigma', n \rangle$