CS 4110

Programming Languages & Logics

Lecture 21
Normalization

22 October 2014

Announcements

- PS 5 due tonight
- PS 6 out today

Normalization

The simply-typed lambda calculus enjoys a remarkable property...

...every well-typed program terminates.

Normalization

The simply-typed lambda calculus enjoys a remarkable property...

...every well-typed program terminates.

We'll spend this lecture proving this fact.

Simply-Typed Lambda Calculus

Syntax

expressions	$e ::= x \mid \lambda x\!:\!\tau.\, e \mid e_1\ e_2 \mid ()$
values	$v ::= \lambda x\!:\!\tau.\, e \mid ()$
types	$\tau ::= \textbf{unit} \mid \tau_1 \rightarrow \tau_2$

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Dynamic Semantics

$$E ::= [\cdot] \mid E\; e \mid v\; E$$

$$\frac{e \rightarrow e'}{E[e] \rightarrow E[e']} \qquad \frac{}{(\lambda x\!:\!\tau.\, e)\; v \rightarrow e\{v/x\}}$$

Simply-Typed Lambda Calculus

Static Semantics

$$\frac{}{\Gamma \vdash () : \textbf{unit}} \text{ T-Unit}$$

$$\frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \text{ T-Var}$$

$$\frac{\Gamma, x : \tau \vdash e : \tau'}{\Gamma \vdash \lambda x : \tau .\, e : \tau \rightarrow \tau'} \text{ T-Abs}$$

$$\frac{\Gamma \vdash e_1 : \tau \rightarrow \tau' \quad \Gamma \vdash e_2 : \tau}{\Gamma \vdash e_1\ e_2 : \tau'} \text{ T-App}$$

Supporting Lemmas

Lemma (Inversion)

- *If* $\Gamma \vdash x : \tau$ *then* $\Gamma(x) = \tau$
- *If* $\Gamma \vdash \lambda x : \tau_1 . e : \tau$ *then* $\tau = \tau_1 \rightarrow \tau_2$ *and* $\Gamma, x : \tau_1 \vdash e : \tau_2$.
- *If* $\Gamma \vdash e_1\ e_2 : \tau$ *then* $\Gamma \vdash e_1 : \tau' \rightarrow \tau$ *and* $\Gamma \vdash e_2 ty \tau'$.

Supporting Lemmas

Lemma (Inversion)

- *If* $\Gamma \vdash x : \tau$ *then* $\Gamma(x) = \tau$
- *If* $\Gamma \vdash \lambda x : \tau_1 . e : \tau$ *then* $\tau = \tau_1 \to \tau_2$ *and* $\Gamma, x : \tau_1 \vdash e : \tau_2$.
- *If* $\Gamma \vdash e_1\, e_2 : \tau$ *then* $\Gamma \vdash e_1 : \tau' \to \tau$ *and* $\Gamma \vdash e_2 ty \tau'$.

Lemma (Canonical Forms)

- *If* $\Gamma \vdash v : \textbf{unit}$ *then* $v = ()$
- *If* $\Gamma \vdash v : \tau_1 \to \tau_2$ *then* $v = \lambda x : \tau_1 . e$ *and* $\Gamma, x : \tau_1 \vdash e : \tau_2$.

First Attempt

Theorem (Normalization)

If $\vdash e : \tau$ *then there exists a value* v *such that* $e \longrightarrow^* v$.

(Proof attempt on board)

Logical Relations

Idea: define a set with the following properties:

- At base types the set contains all expressions satisfying some property.
- At function types, the set contains all expressions such that the property is preserved whenever we apply the function to an argument of appropriate type that is also in the set.

Logical Relations

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- At base types the set contains all expressions satisfying some property.
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In our setting, the property will concern normalization...

Logical Relation

Definition (Logical Relation)

- $R_{\textbf{unit}}(e)$ iff $\vdash e : \textbf{unit}$ and e halts.
- $R_{\tau_1 \to \tau_2}(e)$ iff $\vdash e : \tau_1 \to \tau_2$ and e halts, and for every e' such that $R_{\tau_1}(e')$ we have $R_{\tau_2}(e\, e')$.

Supporting Lemmas

Lemma

If $R_\tau(e)$ *then* e *halts.*

Supporting Lemmas

Lemma

If $R_\tau(e)$ then e halts.

Lemma

If $\vdash e : \tau$ and $e \to e'$ then $R_\tau(e)$ iff $R_\tau(e')$.

Supporting Lemmas

Lemma

If $R_\tau(e)$ *then* e *halts.*

Lemma

If $\vdash e : \tau$ *and* $e \rightarrow e'$ *then* $R_\tau(e)$ *iff* $R_\tau(e')$.

Lemma

If $\vdash e : \tau$ *then* $R_\tau(e)$

Main Lemma

Lemma

If

- $x_1 : \tau_1 \ldots x_k : \tau_k \vdash e : \tau$,
- v_1 *to* v_k *are values such that* $\vdash v_1 : \tau_1$ *to* $\vdash v_k : \tau_k$, *and*
- $R_{\tau_1}(v_1)$ *to* $R_{\tau_k}(v_k)$,

then $R_\tau(e\{v_1/x_1\} \ldots \{v_k/x_k\})$.

(Proof on board)