

CS 4110

# Programming Languages & Logics

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Lecture 3

Inductive Definitions and Proofs

3 September 2014



# Announcements

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## Office Hours

- Fran: Thursday at 2-3pm
- Nate: Friday at 11-12pm
- Nitesh: TBA

## Piazza and CMS

- Sign up!

## Homework #1

- Out: today
- Due: in a week

# Arithmetic Expressions

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Last time we defined a simple language of arithmetic expressions,

$$e ::= x \mid n \mid e_1 + e_2 \mid e_1 * e_2 \mid x := e_1 ; e_2$$

as well as a small-step operational semantics

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Example:

Assuming  $\sigma$  is a store that maps *foo* to 4...

$$\frac{\frac{\frac{\sigma(\text{foo}) = 4}{\langle \sigma, \text{foo} \rangle \rightarrow \langle \sigma, 4 \rangle} \text{Var}}{\langle \sigma, \text{foo} + 2 \rangle \rightarrow \langle \sigma, 4 + 2 \rangle} \text{LAdd}}{\langle \sigma, (\text{foo} + 2) * (\text{bar} + 1) \rangle \rightarrow \langle \sigma, (4 + 2) * (\text{bar} + 1) \rangle} \text{LMul}$$

# Properties

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- **Determinism:** every configuration has at most one successor

$\forall e \in \mathbf{Exp}. \forall \sigma, \sigma', \sigma'' \in \mathbf{Store}. \forall e', e'' \in \mathbf{Exp}.$   
if  $\langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle$  and  $\langle \sigma, e \rangle \longrightarrow \langle \sigma'', e'' \rangle$   
then  $e' = e''$  and  $\sigma' = \sigma''$ .

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- **Termination:** evaluation of every expression terminates,

$$\forall e \in \mathbf{Exp}. \forall \sigma \in \mathbf{Store}. \exists \sigma' \in \mathbf{Store}. \exists e' \in \mathbf{Exp}. \\ \langle \sigma, e \rangle \longrightarrow^* \langle \sigma', e' \rangle \text{ and } \langle \sigma', e' \rangle \not\longrightarrow,$$

where  $\langle \sigma', e' \rangle \not\longrightarrow$  is shorthand for

$$\neg (\exists \sigma'' \in \mathbf{Store}. \exists e'' \in \mathbf{Exp}. \langle \sigma', e' \rangle \longrightarrow \langle \sigma'', e'' \rangle).$$

# Soundness

It is tempting to try to prove the following property.

- **Soundness**: evaluation of every expression yields an integer,

$$\forall e \in \mathbf{Exp}. \forall \sigma \in \mathbf{Store}. \exists \sigma' \in \mathbf{store}. \exists n' \in \mathbf{Int}. \\ \langle \sigma, e \rangle \longrightarrow^* \langle \sigma', n' \rangle,$$

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## Counterexample

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In generally, evaluation of an expression can “get stuck”...

# Well-Formedness

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Well-Formedness:

A configuration  $\langle \sigma, e \rangle$  is *well-formed* if and only if  $fvs(e) \subseteq dom(\sigma)$ .

# Progress and Preservation

Now we can formulate two properties that imply soundness:

- Progress:

$\forall e \in \mathbf{Exp}. \forall \sigma \in \mathbf{Store}.$

$\langle \sigma, e \rangle$  well-formed  $\implies$

$e \in \mathbf{Int}$  or  $(\exists e' \in \mathbf{Exp}. \exists \sigma' \in \mathbf{Store}. \langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle)$

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$\forall e, e' \in \mathbf{Exp}. \forall \sigma, \sigma' \in \mathbf{Store}.$

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 $\langle \sigma', e' \rangle$  well-formed.

How are we going to prove these properties? Induction!

# Inductive Sets

# Inductive Sets

An *inductively-defined set*  $A$  is one that can be described using a finite collection of inference rules:

$$\frac{a_1 \in A \quad \dots \quad a_n \in A}{a \in A}$$

This rule states that if  $a_1$  through  $a_n$  are elements of  $A$ , then  $a$  is also an element of  $A$ .

An inference rule with no premises is often called an *axiom*.

The set  $A$  is the smallest set “closed” under these axioms and rules.

# Inductive Set Examples

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The natural numbers are an inductive set.

$$\overline{0 \in \mathbb{N}} \quad \frac{n \in \mathbb{N}}{\overline{\text{succ}(n) \in \mathbb{N}}}$$

# Inductive Set Examples

Every BNF grammar defines an inductive set.

$$e ::= x \mid n \mid e_1 + e_2 \mid e_1 * e_2 \mid x := e_1 ; e_2$$

can be equivalently defined as:

$$\begin{array}{c} \frac{}{x \in \mathbf{Exp}} \qquad \frac{}{n \in \mathbf{Exp}} \\ \\ \frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{e_1 + e_2 \in \mathbf{Exp}} \qquad \frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{e_1 * e_2 \in \mathbf{Exp}} \\ \\ \frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{x := e_1 ; e_2 \in \mathbf{Exp}} \end{array}$$



# Inductive Set Examples

The small-step evaluation relation  $\longrightarrow$  is an inductive set.

$$\frac{n = \sigma(x)}{\langle \sigma, x \rangle \rightarrow \langle \sigma, n \rangle} \text{Var}$$

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, e_1 + e_2 \rangle \rightarrow \langle \sigma', e'_1 + e_2 \rangle} \text{LAdd}$$

$$\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle}{\langle \sigma, n + e_2 \rangle \rightarrow \langle \sigma', n + e'_2 \rangle} \text{RAdd}$$

$$\frac{p = m + n}{\langle \sigma, n + m \rangle \rightarrow \langle \sigma, p \rangle} \text{Add}$$

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, e_1 * e_2 \rangle \rightarrow \langle \sigma', e'_1 * e_2 \rangle} \text{LMul}$$

$$\frac{\langle \sigma, e_2 \rangle \rightarrow \langle \sigma', e'_2 \rangle}{\langle \sigma, n * e_2 \rangle \rightarrow \langle \sigma', n * e'_2 \rangle} \text{RMul}$$

$$\frac{p = m \times n}{\langle \sigma, m * n \rangle \rightarrow \langle \sigma, p \rangle} \text{Mul}$$

$$\frac{\langle \sigma, e_1 \rangle \rightarrow \langle \sigma', e'_1 \rangle}{\langle \sigma, x := e_1 ; e_2 \rangle \rightarrow \langle \sigma', x := e'_1 ; e_2 \rangle} \text{Assgn1}$$

$$\frac{\sigma' = \sigma[x \mapsto n]}{\langle \sigma, x := n ; e_2 \rangle \rightarrow \langle \sigma', e_2 \rangle} \text{Assgn}$$

# Inductive Set Examples

The multi-step evaluation relation is an inductive set.

$$\frac{}{\langle \sigma, e \rangle \longrightarrow^* \langle \sigma, e \rangle} \text{Refl}$$

$$\frac{\langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle}{\langle \sigma, e \rangle \longrightarrow^* \langle \sigma', e' \rangle} \text{Step}$$

$$\frac{\langle \sigma, e \rangle \longrightarrow^* \langle \sigma', e' \rangle \quad \langle \sigma', e' \rangle \longrightarrow^* \langle \sigma'', e'' \rangle}{\langle \sigma, e \rangle \longrightarrow^* \langle \sigma'', e'' \rangle} \text{Trans}$$

# Inductive Set Examples

The set of free variables of an expression is an inductive set.

$$\frac{}{y \in fvs(y)}$$

$$\frac{y \in fvs(e_1)}{y \in fvs(e_1 + e_2)}$$

$$\frac{y \in fvs(e_2)}{y \in fvs(e_1 + e_2)}$$

$$\frac{y \in fvs(e_1)}{y \in fvs(e_1 * e_2)}$$

$$\frac{y \in fvs(e_2)}{y \in fvs(e_1 * e_2)}$$

$$\frac{y \in fvs(e_1)}{y \in fvs(x := e_1 ; e_2)}$$

$$\frac{y \neq x \quad y \in fvs(e_2)}{y \in fvs(x := e_1 ; e_2)}$$

# Induction Principle

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Recall the principle of mathematical induction.

To prove  $\forall n. P(n)$ , we must establish several cases.

- Base case:  $P(0)$
- Inductive case:  $P(m) \Rightarrow P(m + 1)$

# Induction Principle

Every inductive set has an analogous principle.

To prove  $\forall a. P(a)$  we must establish several cases.

- **Base cases:**  $P(a)$  holds for each axiom

$$\overline{a \in A}$$

- **Inductive cases:** For each inference rule

$$\frac{a_1 \in A \quad \dots \quad a_n \in A}{a \in A}$$

if  $P(a_1)$  and ... and  $P(a_n)$  then  $P(a)$

# Example: Progress

Recall the progress property.

$\forall e \in \mathbf{Exp}. \forall \sigma \in \mathbf{Store}.$

$\langle \sigma, e \rangle$  well-formed  $\implies$

$e \in \mathbf{Int}$  or  $(\exists e' \in \mathbf{Exp}. \exists \sigma' \in \mathbf{Store}. \langle \sigma, e \rangle \longrightarrow \langle \sigma', e' \rangle)$

We'll prove this by structural induction on  $e$ .

$\frac{}{x \in \mathbf{Exp}}$

$\frac{}{n \in \mathbf{Exp}}$

$\frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{e_1 + e_2 \in \mathbf{Exp}}$

$\frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{e_1 * e_2 \in \mathbf{Exp}}$

$\frac{e_1 \in \mathbf{Exp} \quad e_2 \in \mathbf{Exp}}{x := e_1 ; e_2 \in \mathbf{Exp}}$