



We have now seen two operational models for programming languages: small-step and large-step. In this lecture, we consider a different semantic model, called *denotational semantics*.

The idea in denotational semantics is to express the meaning of a program as the mathematical function that expresses what the program computes. We can think of an IMP program  $c$  as a function from stores to stores: given an initial store, the program produces a final store. For example, the program  $\text{foo} := \text{bar} + 1$  can be thought of as a function that when given an input store  $\sigma$ , produces a final store  $\sigma'$  that is identical to  $\sigma$  except that it maps  $\text{foo}$  to the integer  $\sigma(\text{bar}) + 1$ ; that is,  $\sigma' = \sigma[\text{foo} \mapsto \sigma(\text{bar}) + 1]$ . We will model programs as functions from input stores to output stores. As opposed to operational models, which tell us *how* programs execute, the denotational model shows us *what* programs compute.

## 1 A Denotational Semantics for IMP

For each program  $c$ , we write  $\mathcal{C}[[c]]$  for the *denotation* of  $c$ , that is, the mathematical function that  $c$  represents:

$$\mathcal{C}[[c]] : \text{Store} \rightarrow \text{Store}.$$

Note that  $\mathcal{C}[[c]]$  is actually a partial function (as opposed to a total function), both because the store may not be defined on the free variables of the program and because program may not terminate for certain input stores. The function  $\mathcal{C}[[c]]$  is not defined for non-terminating programs as they have no corresponding output stores.

We will write  $\mathcal{C}[[c]]\sigma$  for the result of applying the function  $\mathcal{C}[[c]]$  to the store  $\sigma$ . That is, if  $f$  is the function that  $\mathcal{C}[[c]]$  denotes, then we write  $\mathcal{C}[[c]]\sigma$  to mean the same thing as  $f(\sigma)$ .

We must also model expressions as functions, this time from stores to the values they represent. We will write  $\mathcal{A}[[a]]$  for the denotation of arithmetic expression  $a$ , and  $\mathcal{B}[[b]]$  for the denotation of boolean expression  $b$ .

$$\mathcal{A}[[a]] : \text{Store} \rightarrow \text{Int}$$

$$\mathcal{B}[[b]] : \text{Store} \rightarrow \{\text{true}, \text{false}\}$$

Now we want to define these functions. To make it easier to write down these definitions, we will describe (partial) functions using sets of pairs. More precisely, we will represent a partial map  $f : A \rightarrow B$  as a set of pairs  $F = \{(a, b) \mid a \in A \text{ and } b = f(a) \in B\}$  such that, for each  $a \in A$ , there is at most one pair of the form  $(a, -)$  in the set. Hence  $(a, b) \in F$  is the same as  $b = f(a)$ .

We can now define denotations for IMP. We start with the denotations of expressions:

$$\begin{aligned}\mathcal{A}[[n]] &= \{(\sigma, n)\} \\ \mathcal{A}[[x]] &= \{(\sigma, \sigma(x))\} \\ \mathcal{A}[[a_1 + a_2]] &= \{(\sigma, n) \mid (\sigma, n_1) \in \mathcal{A}[[a_1]] \wedge (\sigma, n_2) \in \mathcal{A}[[a_2]] \wedge n = n_1 + n_2\}\end{aligned}$$

$$\begin{aligned}\mathcal{B}[[\mathbf{true}]] &= \{(\sigma, \mathbf{true})\} \\ \mathcal{B}[[\mathbf{false}]] &= \{(\sigma, \mathbf{false})\} \\ \mathcal{B}[[a_1 < a_2]] &= \{(\sigma, \mathbf{true}) \mid (\sigma, n_1) \in \mathcal{A}[[a_1]] \wedge (\sigma, n_2) \in \mathcal{A}[[a_2]] \wedge n_1 < n_2\} \cup \\ &\quad \{(\sigma, \mathbf{false}) \mid (\sigma, n_1) \in \mathcal{A}[[a_1]] \wedge (\sigma, n_2) \in \mathcal{A}[[a_2]] \wedge n_1 \geq n_2\}\end{aligned}$$

The denotations for commands are as follows:

$$\begin{aligned}\mathcal{C}[[\mathbf{skip}]] &= \{(\sigma, \sigma)\} \\ \mathcal{C}[[x := a]] &= \{(\sigma, \sigma[x \mapsto n]) \mid (\sigma, n) \in \mathcal{A}[[a]]\} \\ \mathcal{C}[[c_1; c_2]] &= \{(\sigma, \sigma') \mid \exists \sigma''. ((\sigma, \sigma'') \in \mathcal{C}[[c_1]] \wedge (\sigma'', \sigma') \in \mathcal{C}[[c_2]])\}\end{aligned}$$

Note that  $\mathcal{C}[[c_1; c_2]] = \mathcal{C}[[c_2]] \circ \mathcal{C}[[c_1]]$ , where  $\circ$  is the composition of relations, defined as follows: if  $R_1 \subseteq A \times B$  and  $R_2 \subseteq B \times C$  then  $R_2 \circ R_1 \subseteq A \times C$  is  $R_2 \circ R_1 = \{(a, c) \mid \exists b \in B. (a, b) \in R_1 \wedge (b, c) \in R_2\}$ . If  $\mathcal{C}[[c_1]]$  and  $\mathcal{C}[[c_2]]$  are total functions, then  $\circ$  is function composition.

$$\begin{aligned}\mathcal{C}[[\mathbf{if } b \mathbf{ then } c_1 \mathbf{ else } c_2]] &= \{(\sigma, \sigma') \mid (\sigma, \mathbf{true}) \in \mathcal{B}[[b]] \wedge (\sigma, \sigma') \in \mathcal{C}[[c_1]]\} \cup \\ &\quad \{(\sigma, \sigma') \mid (\sigma, \mathbf{false}) \in \mathcal{B}[[b]] \wedge (\sigma, \sigma') \in \mathcal{C}[[c_2]]\} \\ \mathcal{C}[[\mathbf{while } b \mathbf{ do } c]] &= \{(\sigma, \sigma) \mid (\sigma, \mathbf{false}) \in \mathcal{B}[[b]]\} \cup \\ &\quad \{(\sigma, \sigma') \mid (\sigma, \mathbf{true}) \in \mathcal{B}[[b]] \wedge \exists \sigma''. ((\sigma, \sigma'') \in \mathcal{C}[[c]] \wedge (\sigma'', \sigma') \in \mathcal{C}[[\mathbf{while } b \mathbf{ do } c]])\}\end{aligned}$$

But now we have a problem: the last “definition” is not really a definition because it expresses  $\mathcal{C}[[\mathbf{while } b \mathbf{ do } c]]$  in terms of itself! This is not a definition but a recursive equation. What we want is the solution to this equation.

## 2 Fixed points

We gave a recursive equation that the function  $\mathcal{C}[[\mathbf{while } b \mathbf{ do } c]]$  must satisfy. To understand some of the issues involved, let’s consider a simpler example. Consider the following equation for a function  $f : \mathbb{N} \rightarrow \mathbb{N}$ .

$$f(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x-1) + 2x - 1 & \text{otherwise} \end{cases} \quad (1)$$

This is not a definition for  $f$ , but rather an equation that we want  $f$  to satisfy. What function, or functions, satisfy this equation for  $f$ ? The only solution to this equation is the function  $f(x) = x^2$ .

In general, there may be no solutions for a recursive equation (e.g., there are no functions  $g : \mathbb{N} \rightarrow \mathbb{N}$  that satisfy the recursive equation  $g(x) = g(x) + 1$ ), or multiple solutions (e.g., find two functions  $g : \mathbb{R} \rightarrow \mathbb{R}$  that satisfy  $g(x) = 4 \times g(\frac{x}{2})$ ).

We can compute solutions to such equations by building successive approximations. Each approximation is closer and closer to the solution. To solve the recursive equation for  $f$ , we start with the partial function  $f_0 = \emptyset$  (i.e.,  $f_0$  is the empty relation; it is a partial function with the empty set for its domain). We compute successive approximations using the recursive equation.

$$\begin{aligned}
 f_0 &= \emptyset \\
 f_1 &= \begin{cases} 0 & \text{if } x = 0 \\ f_0(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
 &= \{(0, 0)\} \\
 f_2 &= \begin{cases} 0 & \text{if } x = 0 \\ f_1(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
 &= \{(0, 0), (1, 1)\} \\
 f_3 &= \begin{cases} 0 & \text{if } x = 0 \\ f_2(x-1) + 2x - 1 & \text{otherwise} \end{cases} \\
 &= \{(0, 0), (1, 1), (2, 4)\} \\
 &\vdots
 \end{aligned}$$

This sequence of successive approximations  $f_i$  gradually builds the function  $f(x) = x^2$ .

We can model this process of successive approximations using a higher-order function  $F$  that takes one approximation  $f_k$  and returns the next approximation  $f_{k+1}$ :

$$F : (\mathbb{N} \rightarrow \mathbb{N}) \rightarrow (\mathbb{N} \rightarrow \mathbb{N})$$

where

$$(F(f))(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x-1) + 2x - 1 & \text{otherwise} \end{cases}$$

A solution to the recursive equation 1 is a function  $f$  such that  $f = F(f)$ . In general, given a function  $F : A \rightarrow A$ , we have that  $a \in A$  is a *fixed point* of  $F$  if  $F(a) = a$ . We also write  $a = \text{fix}(F)$  to indicate that  $a$  is a fixed point of  $F$ .

So the solution to the recursive equation 1 is a fixed-point of the higher-order function  $F$ . We can compute this fixed point iteratively, starting with  $f_0 = \emptyset$  and at each iteration computing  $f_{k+1} = F(f_k)$ . The fixed point is the limit of this process:

$$\begin{aligned}
 f &= \text{fix}(F) \\
 &= f_0 \cup f_1 \cup f_2 \cup f_3 \cup \dots \\
 &= \emptyset \cup F(\emptyset) \cup F(F(\emptyset)) \cup F(F(F(\emptyset))) \cup \dots \\
 &= \bigcup_{i \geq 0} F^i(\emptyset)
 \end{aligned}$$