CS 3220: Orthogonal projectors

Instructor: Anil Damle

Disclaimer

These notes are intended to highlight key points from some of the lectures. However, they are not written nor intended to be substitutes for the lectures. Furthermore, the resources section of the website contains extensive written material covering the topics in class, material that was developed with the explicit intention of being a written presentation of this material. Many of those books are several editions into their existence and, therefore, have been refined in a way these notes are not. In addition, the textbooks given are invariably far more comprehensive and thorough in their treatment of the topics. Lastly, I take full responsibility for any typos included herein; nevertheless, these notes are a work in progress and if you find anything amiss please let me know.

Orthogonal projectors

This note primarily discusses orthogonal projectors, their properties, and their construction. We start with a general definition.

Definition 1 (Projector)**.** *A matrix $P \in \mathbb{R}^{n\times n}$ is a projector $P^2 = P$.*

However, for the purposes of this class we will restrict our attention to so-called orthogonal projectors (not to be confused with orthogonal matrices—the only orthogonal projector that is an orthogonal matrix is the identity).

Definition 2 (Projector)**.** *A matrix $P \in \mathbb{R}^{n\times n}$ is an orthogonal projector if $P^2 = P$ and $P = P^T$.*

Henceforth, in these notes any time we refer to a projector we are assuming it is an orthogonal projector. Non-orthogonal projectors are interesting, they are just out of scope here. Colloquially, our starting point for a discussion of projectors is to think of them via their action on vectors: a projector maps vectors onto its range. What is interesting is the manner in which this is accomplished and the properties of this operation.

First, we observe that any matrix P satisfying $P^2 = P$ has a rather interesting property: it maps vectors in its range to themselves. To see this observe that given any $z \in \operatorname{range}(P)$ there exists w such that $z = Pw$ and

$$Pz = P(Pw) = P^2w = Pw = z.$$

This fits with our colloquial viewpoint, given a vector in the range of P it seems sensible that P maps it to itself.

Projecting to the closest point

Since for any vector x Px is in the range of P by definition, a fair question to ask is which vector P maps x to. It turns out that orthogonal projectors perform a rather remarkable operation, given any vector x they map it to the closest point, as measured by $\|\cdot\|_2$, in the range of P.

Theorem 1. *Given an orthogonal projector P and vector x, Px uniquely solves the optimization problem*

$$\min_{z \in \operatorname{range}(P)} \|z - x\|_2.$$

In other words, Px is the unique closest point to x in the range of P.

Proof. We will prove this by showing that for any $z \in \operatorname{range}(P)$ with $z \neq Px$

$$\|z - x\|_2 > \|Px - x\|_2.$$

To accomplish this observe that

$$z - x = z - Px + Px - x,$$

where taking squared norms of each side yields

$$\|z - x\|_2^2 = \|z - Px + Px - x\|_2^2. \tag{1}$$

Now, a key observation is that $z - Px \perp Px - x$; we can show this because $z = Pw$ for some w and $P = P^T$. Therefore,

$$\begin{aligned}
(z - Px)^T(Px - x) &= (Pw - Px)^T(Px - x) \\
&= (w - x)^T P(Px - x) \\
&= (w - x)^T(P^2x - Px) \\
&= (w - x)^T(Px - Px) \\
&= 0.
\end{aligned}$$

Finally, returning to (1) we may use this observation to conclude that

$$\begin{aligned}
\|z - x\|_2^2 &= \|z - Px + Px - x\|_2^2 \\
&= (z - Px)^T(z - Px) + 2(z - Px)^T(Px - x) + (Px - x)^T(Px - x) \\
&= (z - Px)^T(z - Px) + (Px - x)^T(Px - x) \\
&= \|z - Px\|_2^2 + \|Px - x\|_2^2 \\
&> \|Px - x\|_2^2,
\end{aligned}$$

where the last inequality follows from the fact that $z \neq Px$ and therefore $\|z - Px\|_2^2 > 0$. $\square$

Two subspaces

A more comprehensive view on what we can accomplish with projectors is captured by a discussion of how an orthogonal projector decomposes all of $\mathbb{R}^n$. To discuss this we need to introduce the notion of a complementary projector.

Definition 3 (Complementary projector)**.** *Given an orthogonal projector $P \in \mathbb{R}^{n \times n}$, the associated complementary projector is defined as*

$$I - P.$$

It is not hard to check that if $P^2 = P$ and $P = P^T$ then $(I-P)^2 = (I-P)$ and $(I-P)^T = (I-P)$, therefore ensuring that $I - P$ is an orthogonal projector itself.

The more interesting aspect of a complementary projector is the space it projects onto. In particular, we have that $\text{range}\,(I-P) = \text{range}\,(P)^{\perp}$, *i.e.,* the range of $I - P$ is the orthogonal complement of the range of P. This is easily verified as for any vectors y and w

$$\begin{aligned}((I-P)y)^T(Pw) &= y^T(I-P)^TPw \\ &= y^T(P-P^2)w \qquad = 0.\end{aligned}$$

Given any subspace $\mathcal{V}$ we can uniquely decompose any vector x into a component in $\mathcal{V}$ and a component in $\mathcal{V}^{\perp}$. Given the orthogonal projector onto $\mathcal{V}$ we see this immediately as for any x

$$x = Px + (I-P)x,$$

where the first component is the part in $\mathcal{V}$ and the second component is the part in $\mathcal{V}^{\perp}$. (There is a nice pictorial representation for this, these notes may be updated at some point to include it.) A consequence of this we will see later is that provided we have an orthonormal basis for $\mathcal{V}$ we can project vectors onto $\mathcal{V}^{\perp}$ without explicitly constructing a basis for it.

Constructing orthogonal projectors

Now that we have seen all of the nice properties of orthogonal projectors (albeit their many uses have mostly been omitted) we will explore how to construct them. To do this, we first consider an illustrative example where the range of the projector we want to construct has dimension one.

Given some vector u with $\|u\|_2 = 1$, say we would like to construct the orthogonal projector onto the span u. While not discussed above and a proof is omitted here, it turns out that the orthogonal projector onto a subspace is unique. Therefore, we seek the matrix P_u such that P_u is symmetric, $P_u^2 = P_u$ and $P_u x$ is the closest point in the span of u to x. We will solve for the last property explicitly and get the first two for free.

Consider solving

$$\min_{z \in \text{span}\,\{u\}} \|z - x\|_2.$$

Since any $z \in \text{span}\,\{u\}$ can be parametrized as $z = \alpha u$ we can instead solve

$$\min_{\alpha \in \mathbb{R}} \|\alpha u - x\|_2^2$$

for alpha. (Squaring the norm does not change the optimal point since norms are non-negative.) Explicitly writing out the norm squared we get that

$$\|\alpha u - x\|_2 = \alpha^2 u^T u - 2\alpha u^T x + x^T x.$$

Using that $u^T u = 1$ and the fact that $x^T x$ does not depend on α we have that the optimal alpha solves

$$\min_{\alpha \in \mathbb{R}} \alpha^2 - 2\alpha u^T x.$$

This is a convex quadratic function in α, so taking the derivative and setting it equal to zero yields the minimizer

$$\alpha = u^T x.$$

Given the preceding we observe that for any vector x

$$u(u^T x)$$

is the closest point in span $\{u\}$ to x. Arranging the parentheses differently we see that

$$P_u = uu^T,$$

where one may verify that P_u is symmetric and $P_u^2 = P_u$. Therefore, we have explicitly produced a formula for the orthogonal projector onto a one dimensional subspace represented by a unit vector. It turns out that this idea generalizes nicely to arbitrary dimensional linear subspaces given an orthonormal basis. Specifically, given a matrix $V \in \mathbb{R}^{n\times k}$ with orthonormal columns

$$P = VV^T$$

is the orthogonal projector onto its column space. An alternative way of saying this is that given any linear subspace $\mathcal{V}$ of dimension k, if V is an $n \times k$ matrix whose columns form an orthonormal basis for $\mathcal{V}$ then the orthogonal projector onto $\mathcal{V}$ is $P = VV^T$.

We close with a quick aside on actually applying orthogonal projectors written this way to vectors. While we have thought of everything as a projection matrix, if we are working with an orthonormal basis for a subspace it is inefficient to work with it as a matrix. In particular, simply storing V, representing the basis, is far more efficient (as it is nk numbers as opposed to $\mathcal{O}(n^2)$). Similarly, when applying projectors (or the associated complementary projector) careful grouping of operations as

$$VV^Tx = V(V^Tx)$$

and

$$(I - VV^T)x = x - V(V^Tx)$$

shows they can be applied in $\mathcal{O}(nk)$ as opposed to the naïve matrix vector multiplication cost of $\mathcal{O}(n^2)$.