CS 3220: Homework 3

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Due: October 7, 2020

Policies

You may discuss the homework problems freely with other students, but please refrain from looking at their code or writeups (or sharing your own). Ultimately, you must implement your own code and write up your own solution to be turned in. Your solution, including plots and requested output from your code should be typset and submitted via the CMS as a pdf file. Additionally, please submit any code written for the assignment via the CMS as well.

Question 1:

In class we made the observation that when we run the power method, the kth iterate can be written as

$$v^{(k)} = c_k A^k v^{(0)}$$

for some constant c_k, where our chosen starting vector is $v^{(0)}$ (you should assume $v_1^T v^{(0)} \neq 0$). Let V be the matrix whose columns are the eigenvectors (*i.e.*, the columns of V denoted v_i are the eigenvectors) of a symmetric matrix A. (Assume they are ordered with respect to the eigenvalues $\lambda_1, \lambda_2, \ldots$ and assuming $|\lambda_1| > |\lambda_2| \geq |\lambda_3| \geq \ldots |\lambda_n| \geq 0$). We showed that

$$v^{(k)} = c_k \lambda_1^k \left(\alpha_1 v_1 + \sum_{i=2}^{n} \alpha_i \left(\frac{\lambda_i}{\lambda_1} \right)^k v_i \right)$$

where $\alpha_i = v_i^T v^{(0)}$.

(a) Argue that $|c_k \alpha_1 \lambda_1^k| \to 1$ as $k \to \infty$. Are the absolute value signs necessary? Hint: consider how to write $\|A^k v^{(0)}\|_2$ in terms of α_i's and λ_i's.

Question 2:

Here, we will geometrically explore the behavior of the power method. First, implement the power method in your language of choice. Now, consider the following sequence of steps.

(a) For $n = 2$ construct a "random" orthogonal matrix. This may be accomplished by generating a 2×2 random matrix (say with $\mathcal{N}(0,1)$ entries) using built in functions in your language of choice, call it B, computing the SVD of B and using either of the resulting orthogonal matrices returned—call this orthogonal matrix V. Outline the procedure you used.

(b) Now, let $A = V \Lambda V^T$ where

$$\Lambda = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}.$$

Using a built in routine in your language of choice, compute the eigenvalues of A and verify that they are correct.

(c) Now, pick a random vector $v^{(0)}$ and normalize it. Plot a picture of the unit sphere along with points representing the vectors $v^{(0)}/\|v^{(0)}\|_2$, v_1, and v_2.

(d) Finally, for $k = 1, 2, \ldots$ plot the point on the sphere representing the iterate $v^{(k)}$, what do you observe? You need not provide plots for all k, choose some to make your point and you can also stop when it is no longer possible to visually distinguish $v^{(k)}$ and v_1.

Now, just to be sure your implementation of the power method is working do the following:

(e) Generate a random 100×100 symmetric matrix A. (To do this, generate a random 100×100 matrix B using built in functions—the specific distribution is not so important—and form $A = (B + B^T)/2$.) Using a built in function compute the eigenvalues of A, then run your version of the power method and plot $\log|\lambda^{(k)} - \lambda_1|$ vs k, you may stop when this quantity is less than 10^{-8}. Does the slope of the line agree with our assessment in class that $|\lambda^{(k)} - \lambda_1| = \mathcal{O}(|\lambda_2/\lambda_1|^{2k})$? Think about how to argue this based on the slope of the line and recall that you have computed λ_2 as well via the built in routine.

QUESTION 3:

Let $D \in \mathbb{R}^{n \times n}$ be a diagonal matrix with $D_{ii} = (1/10)^{(i-1)}$.

(a) What is $\kappa_2(D)$ as a function of n?

(b) If $Dx = b$ for a given b and I want to maximize the change in the solution when I solve $D(x + \hat{x}) = b + \hat{b}$ (recall we are just writing the new solution with respect to the point x) how do I do this? More specifically, if I fix the norm of $\|\hat{b}\|_2$ which vector(s) maximize $\|\hat{x}\|_2$?

Now, we will use that to understand what changes to b result in worst case changes in x.

(c) Let $A \in \mathbb{R}^{n \times n}$ be full rank. If $Ax = b$ and I want to maximize the change in the solution by perturbing b in what direction should $\hat{b}$ lie? Again, specifically, this means that you get to pick $\hat{b}$ and would like $\hat{x}$ to be as big as possible when $A(x + \hat{x}) = b + \hat{b}$. Note that while nominally we can think of this as having a "budget" for the perturbation size $\|\hat{b}\|_2$ the answer does not depend on this—irrespective of the size of the perturbation the direction it should lie in remains unchanged.

(d) Consider the same setup as the previous part, except now you would like to know which perturbations $\hat{b}$ result in the smallest $\hat{x}$. To have the smallest change in $\hat{x}$, in what direction should $\hat{b}$ lie?